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SIGN PATTERNS THAT ALLOW A POSITIVE OR NONNEGATIVE LEFT INVERSE*

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Abstract. An m by n sign pattern $\mathcal{S}$ is an m by n matrix with entries in $\{+, -, 0\}$. Such a sign pattern allows a positive (resp., nonnegative) left inverse, provided that there exist an m by n matrix A with the sign pattern $\mathcal{S}$ and an n by m matrix B with only positive (resp., nonnegative) entries satisfying $BA = I_n$, where I_n is the n by n identity matrix. For $m > n \geq 2$, a characterization of m by n sign patterns with no rows of zeros that allow a positive left inverse is given. This leads to a characterization of all m by n sign patterns with $m \geq n \geq 2$ that allow a positive left inverse, giving a generalization of the known result for the square case, which involves a related bipartite digraph. For $m \geq n$, m by n sign patterns with all entries in $\{+, 0\}$ and m by 2 sign patterns with $m \geq 2$ that allow a nonnegative left inverse are characterized, and some necessary or sufficient conditions for a general m by n sign pattern to allow a nonnegative left inverse are presented.

Key words. bipartite digraph, nonnegative left inverse, positive left inverse, positive left null-vector, sign pattern, strong Hall

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1. Introduction.

An m by n *sign pattern* $\mathcal{S} = [s_{ij}]$ is an m by n matrix with entries in $\{+, -, 0\}$. If a sign pattern $\mathcal{S}$ has all entries in $\{+, 0\}$, then $\mathcal{S}$ is a *nonnegative* sign pattern. A *subpattern* of $\mathcal{S}$ is an m by n sign pattern $\mathcal{U} = [u_{ij}]$ such that $u_{ij} = 0$ whenever $s_{ij} = 0$. If $\mathcal{U}$ is a subpattern of $\mathcal{S}$, then $\mathcal{S}$ is a *superpattern* of $\mathcal{U}$. The *sign pattern class* $Q(\mathcal{S})$ of an m by n sign pattern $\mathcal{S}$ is the set of m by n matrices $A = [a_{ij}]$ such that $\mathrm{sgn}(a_{ij}) = s_{ij}$ for all i, j. If $A \in Q(\mathcal{S})$, then A is a *realization* of $\mathcal{S}$.

Let $A = [a_{ij}]$ be an m by n matrix. If each entry of A is positive (resp., nonnegative), then A is *positive* (resp., *nonnegative*), written $A > 0$ (resp., $A \geq 0$). A *left inverse* of an m by n matrix A is an n by m matrix B such that $BA = I_n$, where I_n denotes the n by n identity matrix. If $B > 0$, then B is a *positive* left inverse (abbreviated as *PLI*) of A. If $B \geq 0$, then B is a *nonnegative* left inverse (abbreviated as *NLI*) of A. In general, neither a PLI nor an NLI of A is unique. It is easily verified that A has a left inverse if and only if $\mathrm{rank}\, A = n$; thus, if A has a left inverse, then necessarily $m \geq n$. An m by n sign pattern $\mathcal{S}$ *allows a positive (resp., nonnegative) left inverse*, provided there exist $A \in Q(\mathcal{S})$ and a matrix $B > 0$ (resp., $B \geq 0$) such that $BA = I_n$. Note that if P_1 and P_2 are permutation matrices, then $\mathcal{S}$ allows a PLI (resp., an NLI) if and only if $P_1 \mathcal{S} P_2$ allows a PLI (resp., an NLI).

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A motivation for studying PLIs and NLIs comes from determining the qualitative behavior of solutions of $A^T x = b$ with A an m by n matrix; see, for example, [2, Chapter 1] and [5] for applications in economics. Specifically, A has a PLI (resp., an NLI) if and only if for each n by 1 nonzero vector $b \geq 0$ there exists an m by 1 vector $x > 0$ (resp., $x \geq 0$) satisfying $A^T x = b$ or equivalently $x^T A = b^T$; see Proposition 4.1 for a proof.

Square sign patterns with entries in $\{+, -\}$ that allow a positive (left) inverse are characterized in [6], and this characterization is extended to arbitrary square sign patterns in [4]. These results are summarized in [2, section 9.2]. In section 2, we characterize nonsquare sign patterns that allow a PLI, and combine the square and nonsquare characterizations. In section 3, we discuss sign patterns that allow an NLI. More specifically, we characterize nonnegative sign patterns and m by 2 sign patterns with $m \geq 2$ that allow an NLI, and present some necessary or sufficient conditions for general m by n sign patterns with $m \geq n$ to allow an NLI. We conclude with some remarks in section 4.

2. Positive left inverses. We begin this section with a necessary and sufficient condition for a column sign pattern to allow a PLI or an NLI.

PROPOSITION 2.1. *Let $\mathcal{S} = (s_1, s_2, \ldots, s_m)^T$ be an m by 1 sign pattern. Then the following are equivalent:*

(i) *$\mathcal{S}$ has a $+$ entry.*

(ii) *$\mathcal{S}$ allows a PLI.*

(iii) *$\mathcal{S}$ allows an NLI.*

Proof. Suppose there is an index $i \in \{1, 2, \ldots, m\}$ with $s_i = +$. For $j \in \{1, \ldots, m\}$, set

$$a_j = \begin{cases} 1 & \text{if } j \neq i \text{ and } s_j = +, \\ -1 & \text{if } j \neq i \text{ and } s_j = -, \\ 0 & \text{if } j \neq i \text{ and } s_j = 0, \\ 1 + \sum_{k \neq i} |a_k| & \text{if } j = i. \end{cases}$$

Then $A = (a_1, \ldots, a_m)^T \in Q(\mathcal{S})$, and $(1, 1, \ldots, 1)A = 1 + \sum_{k \neq i} (|a_k| + a_k) = c > 0$. This implies that $\frac{1}{c}(1, 1, \ldots, 1)$ is a PLI of A. Thus, $\mathcal{S}$ allows a PLI.

It is clear that (ii) implies (iii). Next, suppose that the sign pattern $\mathcal{S}$ allows an NLI. Then there exist $A = (a_1, \ldots, a_m)^T \in Q(\mathcal{S})$ and $B = (b_1, \ldots, b_m) \geq 0$ such that $BA = 1$, i.e., $\sum_{j=1}^{m} b_j a_j = 1 > 0$. This implies that there exists an i with $b_i a_i > 0$. Since $b_i \geq 0$, it follows that $b_i > 0$; hence $a_i > 0$ and thus $s_i = +$. ∎

We now consider $m \geq n \geq 2$. The following two lemmas give necessary conditions for a sign pattern to allow a PLI.

LEMMA 2.2. *Let $\mathcal{S}$ be an m by n sign pattern with $n \geq 2$. If $\mathcal{S}$ allows a PLI, then each column of $\mathcal{S}$ has a $+$ and a $-$ entry.*

Proof. Suppose that there exist $A \in Q(\mathcal{S})$ and an n by m positive matrix B such that $BA = I_n$. Let $i \in \{1, 2, \ldots, n\}$. Since the (i, i)-entry of BA is 1 and each entry of B is positive, it follows that some entry in column i of A is positive. Hence, column i of $\mathcal{S}$ has a $+$ entry.

Since $n \geq 2$, there exists $j \in \{1, \ldots, n\}$ with $j \neq i$. The (j, i)-entry of BA is 0, so since $B > 0$ and at least one entry of column i of A is positive, it follows that at least one entry of column i of A must be negative. Thus, column i of $\mathcal{S}$ has a $-$ entry. ∎

An m by n sign pattern $\mathcal{S}$ with $n \geq 2$ is *strong Hall*, provided that for every nonempty proper subset γ of $\{1, 2, \ldots, n\}$ the submatrix of $\mathcal{S}$ consisting of the columns indexed by γ has nonzero entries in at least $|\gamma|+1$ rows (see [3]). Note that if $\mathcal{S}$ is strong Hall, then necessarily $m \geq n$. Also, for $m \geq n$, $\mathcal{S}$ is not strong Hall if and only if there exist permutation matrices P_1 and P_2 such that

$$P_1 \mathcal{S} P_2 = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ O & \mathcal{S}_{22} \end{bmatrix}, \tag{2.1}$$

where $\mathcal{S}_{11}$ is a k by ℓ sign pattern for some integers k, ℓ with $n > \ell \geq 1$ and $k \leq \ell$.

LEMMA 2.3. *Let $\mathcal{S}$ be an m by n sign pattern with $n \geq 2$. If $\mathcal{S}$ allows a PLI, then $\mathcal{S}$ is strong Hall.*

Proof. To prove the contrapositive, assume that $\mathcal{S}$ is not strong Hall. If $m < n$, then it is clear that $\mathcal{S}$ does not allow a PLI. Otherwise, without loss of generality, we may assume that $\mathcal{S}$ has the form (2.1). If $k < \ell$, then the first ℓ columns of each realization of $\mathcal{S}$ are linearly dependent, and hence $\mathcal{S}$ does not allow a PLI.

Otherwise, $k = \ell < n$. Suppose that there exists a matrix $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right] \in Q(\mathcal{S})$ with a left inverse $B = \left[\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{smallmatrix}\right]$, where B_{11} is an ℓ by ℓ matrix. Clearly, the ℓ by ℓ matrix A_{11} is invertible, and by $BA = I_n$, it follows that $\left[\begin{smallmatrix} B_{21} & B_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} A_{11} \\ O \end{smallmatrix}\right] = O$. Thus, $B_{21}A_{11} = O$, and since A_{11} is invertible, the $(n-\ell)$ by ℓ matrix $B_{21} = O$. Since $n - \ell \geq 1$ and $\ell \geq 1$, every left inverse of a matrix in $Q(\mathcal{S})$ has a zero entry, and hence $\mathcal{S}$ does not allow a PLI. ∎

Note that if $\mathcal{S}$ is a square sign pattern of order $n \geq 2$, then $\mathcal{S}$ is strong Hall if and only if $\mathcal{S}$ is fully indecomposable (see [3]), and $\mathcal{S}$ allows a PLI if and only if $\mathcal{S}$ allows a positive inverse. The next theorem, first proved in [4], provides a characterization of square sign patterns that allow a positive inverse. In order to recall this characterization, we use the following definition as in [1] and [2]. Let $\mathcal{S} = [s_{ij}]$ be an m by n sign pattern. The *bipartite digraph* $D(\mathcal{S})$ of $\mathcal{S}$ is the digraph with row vertices $u_1, \ldots, u_m$, column vertices $v_1, \ldots, v_n$, an arc $u_i \to v_j$ if $s_{ij} = +$, and an arc $v_j \to u_i$ if $s_{ij} = -$. Note that there exists at most one arc between u_i and v_j.

THEOREM 2.4 (see [2, Theorem 9.2.1]). *An n by n square sign pattern $\mathcal{S}$ with $n \geq 2$ allows a positive (left) inverse if and only if $\mathcal{S}$ is strong Hall and the bipartite digraph $D(\mathcal{S})$ of $\mathcal{S}$ is strongly connected.*

Let $\mathcal{S}$ be an m by n sign pattern and let $D(\mathcal{S})$ be its bipartite digraph. A *strong component* of $D(\mathcal{S})$ is a maximal strongly connected subdigraph of $D(\mathcal{S})$. If α is a strong component of $D(\mathcal{S})$, then $|\alpha|$ denotes the number of vertices in α.

Remark 2.5. Let α be a strong component of $D(\mathcal{S})$. Since $D(\mathcal{S})$ is a bipartite digraph with no cycles of length 2, it follows that if $|\alpha| \geq 2$, then α has at least two row vertices and at least two column vertices.

Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be the strong components of $D(\mathcal{S})$. The *condensed* digraph $CD(\mathcal{S})$ of $\mathcal{S}$ has vertices $\alpha_1, \alpha_2, \ldots, \alpha_t$ and an arc $\alpha_i \to \alpha_j$ if and only if $i \neq j$ and $D(\mathcal{S})$ has at least one arc from a vertex in α_i to a vertex in α_j. A strong component α_i of $D(\mathcal{S})$ is a *source* if there is no arc coming into α_i in $CD(\mathcal{S})$ and there is at least one arc coming out of α_i in $CD(\mathcal{S})$; α_i is a *sink* if there is no arc coming out of α_i in $CD(\mathcal{S})$ and there is at least one arc coming into α_i in $CD(\mathcal{S})$; and α_i is *isolated* if there are no arcs coming into or out of α_i in $CD(\mathcal{S})$.

LEMMA 2.6. *Let $\mathcal{S}$ be an m by n sign pattern which has a $+$ and a $-$ entry in each column and no rows of zeros. Then the following hold for $D(\mathcal{S})$:*

(i) *Each sink and source strong component of $D(\mathcal{S})$ has at least one row vertex.*

(ii) *Each isolated strong component has at least two row vertices.*

Proof. (i) Let α be a sink or source strong component. If $|\alpha| = 1$, then since each column of $\mathcal{S}$ has a $+$ and a $-$ entry, it follows that no sink or source strong component consists of exactly one column vertex. Hence, α is a row vertex. If $|\alpha| \geq 2$, then Remark 2.5 implies that α has at least one row vertex.

(ii) By the assumptions on the rows and columns of $\mathcal{S}$, there is no isolated strong component with exactly one vertex. Hence, by Remark 2.5, each isolated strong component has at least two row vertices. ∎

Let A be an m by n matrix with $m \geq n$. If there exists an m by 1 vector $y > 0$ satisfying $y^T A = 0$, then y^T is a *positive left* nullvector of A. The following theorem gives a characterization of nonsquare sign patterns with no rows of zeros that allow a PLI. Note that conditions for such a sign pattern to allow a PLI are weaker than those for square sign patterns (Theorem 2.4), although the bipartite digraph is used in our proof for a nonsquare sign pattern.

THEOREM 2.7. *For $m > n \geq 2$, let $\mathcal{S}$ be an m by n sign pattern with no rows of zeros. Then the following are equivalent:*

(i) *There exists a matrix $A \in Q(\mathcal{S})$ with a PLI and a positive left nullvector.*

(ii) *$\mathcal{S}$ allows a PLI.*

(iii) *Each column of $\mathcal{S}$ has a $+$ and a $-$ entry, and $\mathcal{S}$ is strong Hall.*

Proof. Clearly, (i) implies (ii). By Lemmas 2.2 and 2.3, (ii) implies (iii).

To prove that (iii) implies (i), assume that $\mathcal{S}$ is strong Hall and that $\mathcal{S}$ has a $+$ and a $-$ entry in each column. We claim that it suffices to show that there exists an m by $(m-n)$ sign pattern $\mathcal{C}$ so that the m by m sign pattern $[\mathcal{S} \mid \mathcal{C}]$ allows a positive (left) inverse. To prove this claim, suppose there exists an m by m matrix $[A \mid C] \in Q([\mathcal{S} \mid \mathcal{C}])$ with a positive (left) inverse $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$ where B_1 is an n by m positive matrix and B_2 is an $(m-n)$ by m positive matrix. Then $B_1 A = I_n$ and hence B_1 is a PLI of A, implying that $\mathcal{S}$ allows a PLI. In addition, since $B_2 A = O$ and B_2 has at least one positive row, A has a positive left nullvector. Therefore, by Theorem 2.4, it suffices to find an m by $(m-n)$ sign pattern $\mathcal{C}$ such that the m by m sign pattern $[\mathcal{S} \mid \mathcal{C}]$ is strong Hall and its bipartite digraph $D([\mathcal{S} \mid \mathcal{C}])$ is strongly connected.

Consider the bipartite digraph $D(\mathcal{S})$ of $\mathcal{S}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be its strong components, where $\alpha_1, \ldots, \alpha_k$ are sinks, $\alpha_{k+1}, \ldots, \alpha_{k+\ell}$ are sources, $\alpha_{k+\ell+1}, \ldots, \alpha_{k+\ell+r}$ are isolated, and $\alpha_{k+\ell+r+1}, \ldots, \alpha_t$ are neither sinks, sources, nor isolated strong components. By Lemma 2.6 (i), each sink and source strong component has a row vertex. Let r_i be a fixed row vertex of α_i for each $i \in \{1, \ldots, k+\ell\}$. Also, by Lemma 2.6 (ii), each isolated strong component has at least two row vertices. Let r_i^+, r_i^- be distinct fixed row vertices of α_i for each $i \in \{k+\ell+1, \ldots, k+\ell+r\}$. Let $\mathcal{C}_{n+1}$ be the m by 1 column sign pattern with nonzero jth coordinate:

$$\left\{ \begin{array}{ll} + & \text{if } u_j \in \{r_1, \ldots, r_k\} \cup \{r^-_{k+\ell+1}, \ldots, r^-_{k+\ell+r}\}, \\ - & \text{if } u_j \in \{r_{k+1}, \ldots, r_{k+\ell}\} \cup \{r^+_{k+\ell+1}, \ldots, r^+_{k+\ell+r}\}, \\ + & \text{otherwise.} \end{array} \right. \tag{2.2}$$

Then $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$ is obtained from $D(\mathcal{S})$ by appending a new column vertex c_{n+1}, and arcs $r_j \to c_{n+1}$ if r_j is in a sink component; $c_{n+1} \to r_j$ if r_j is in a source component; $r_j^- \to c_{n+1}$ and $c_{n+1} \to r_j^+$ if r_j^- and r_j^+ are in the same isolated component; as well as some additional arcs coming into vertex c_{n+1}.

To prove that $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$ is strongly connected, we show that for each vertex w of $D(\mathcal{S})$ there exists in $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$ a walk from c_{n+1} to w and a walk from w to c_{n+1}. Note that if w is not in an isolated strong component of $D(\mathcal{S})$, then there is a walk from w to a vertex in a sink strong component α_i of $D(\mathcal{S})$ $(i \in \{1, \ldots, k\})$. Since α_i is strongly connected, this walk from w can be extended to the fixed row vertex r_i of α_i. By (2.2), there is an arc $r_i \to c_{n+1}$ in $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$. Hence, there is a walk from w to c_{n+1}. Similarly, there is a walk from c_{n+1} to w.

Next, suppose that w is a vertex in an isolated strong component α_i in $D(\mathcal{S})$ ($i \in \{k+\ell+1, \ldots, k+\ell+r\}$). Since α_i is strongly connected, there is a walk from w to the fixed row vertex r_i^- of α_i. By (2.2), there are arcs $r_i^- \to c_{n+1}$ and $c_{n+1} \to r_i^+$ in $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$. Since α_i is strongly connected, there is a walk from r_i^+ to w. Thus, there exist a walk from w to c_{n+1} and a walk from c_{n+1} to w.

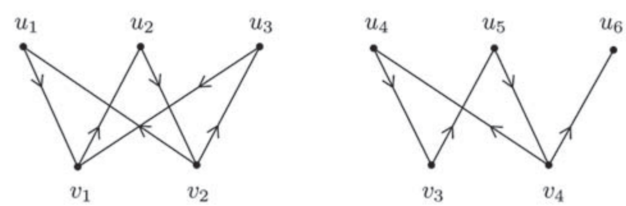
Finally, define $\mathcal{C}_{n+2}, \ldots, \mathcal{C}_m$ to be m by 1 column sign patterns, each having no zeros, at least one $+$, and at least one $-$ entry. Then it is easily verified that $D([\mathcal{S} \mid \mathcal{C}_{n+1} \mid \ldots \mid \mathcal{C}_m])$ is strongly connected. Since $\mathcal{S}$ is strong Hall and $[\mathcal{C}_{n+1} \mid \ldots \mid \mathcal{C}_m]$ has no zeros, it is clear that $[\mathcal{S} \mid \mathcal{C}_{n+1} \mid \ldots \mid \mathcal{C}_m]$ is strong Hall, completing the proof. $\square$

Example 2.8. Consider the 6 by 4 sign pattern

$$\mathcal{S} = \begin{bmatrix} + & - & 0 & 0 \\ - & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & - \\ 0 & 0 & - & + \\ 0 & 0 & 0 & - \end{bmatrix}$$

with

$D(\mathcal{S})$



Each column of $\mathcal{S}$ has a $+$ and a $-$ entry, and $\mathcal{S}$ is strong Hall. Thus, by Theorem 2.7, $\mathcal{S}$ allows a PLI. However, $D(\mathcal{S})$ is not strongly connected, illustrating a distinction between the nonsquare and square cases (see Theorem 2.4). In fact, $D(\mathcal{S})$ has one sink strong component α_1 that consists of vertex u_6, one source strong component α_2 with vertices u_4, u_5, v_3, v_4, and one isolated strong component α_3 with vertices u_1, u_2, u_3, v_1, v_2. Taking $r_1 = u_6$, $r_2 = u_5$, $r_3^+ = u_1$, and $r_3^- = u_2$ in the proof of Theorem 2.7, it follows that

$$\mathcal{C}_5 = \begin{bmatrix} - \\ + \\ + \\ + \\ - \\ + \end{bmatrix}.$$

The last column $\mathcal{C}_6$ can be taken to be any 6 by 1 column having a $+$ and a $-$ entry, and no zeros. Let $\mathcal{C} = [\mathcal{C}_5 \mid \mathcal{C}_6]$. In order to determine a matrix $[A \mid C] \in Q([\mathcal{S} \mid \mathcal{C}])$ with a positive (left) inverse $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$, the algorithm described in the proof of [2, Theorem 9.2.1] can be used. Then B_1 is a PLI of A, and the rows of B_2 are positive left nullvectors of A.

The next lemma is used to prove Theorem 2.10, in which square and nonsquare cases are combined.

LEMMA 2.9. *Let $\mathcal{S}$ be an m by n sign pattern with $n \geq 2$ and let $\mathcal{T}$ be the sign pattern obtained from $\mathcal{S}$ by deleting the rows of zeros in $\mathcal{S}$. Then*

(i) *$\mathcal{S}$ is strong Hall if and only if $\mathcal{T}$ is strong Hall, and*

(ii) *$\mathcal{S}$ allows a positive (nonnegative) left inverse if and only if $\mathcal{T}$ allows a positive (nonnegative) left inverse.*

Proof. Without loss of generality, assume that $\mathcal{S} = \left[\begin{smallmatrix} \mathcal{T} \\ O \end{smallmatrix}\right]$. The proof of (i) follows immediately from the definition of strong Hall.

To prove (ii), suppose first that $\mathcal{S}$ allows a PLI. Let $A_1 \in Q(\mathcal{T})$ and $A = \left[\begin{smallmatrix} A_1 \\ O \end{smallmatrix}\right] \in Q(\mathcal{S})$ have $B = [B_1\ B_2]$ as a PLI. Then $B_1 A_1 = I_n$ and hence $\mathcal{T}$ allows a PLI. Next, suppose that $\mathcal{T}$ allows a PLI. Let $A_1 \in Q(\mathcal{T})$ have B_1 as a PLI. With J denoting the all 1's matrix, it is easily verified that $B = [B_1\ J]$ is a PLI for $A = \left[\begin{smallmatrix} A_1 \\ O \end{smallmatrix}\right] \in Q(\mathcal{S})$. Hence, $\mathcal{S}$ allows a PLI. The nonnegative case can be shown by a similar argument to that above. ☐

THEOREM 2.10. *Let $m \geq n \geq 2$. The m by n sign pattern $\mathcal{S}$ allows a PLI if and only if*

(i) *each column of $\mathcal{S}$ has a $+$ and a $-$ entry;*

(ii) *$\mathcal{S}$ is strong Hall; and*

(iii) *the bipartite digraph $D(\mathcal{S}_1)$ of $\mathcal{S}_1$ is strongly connected whenever $\mathcal{S}$ is permutationally equivalent to $\left[\begin{smallmatrix} \mathcal{S}_1 \\ O \end{smallmatrix}\right]$ and $\mathcal{S}_1$ is an n by n sign pattern.*

Proof. For the necessity, suppose that $\mathcal{S}$ allows a PLI. Then (i) and (ii) follow from Lemmas 2.2 and 2.3, and (iii) follows from Theorem 2.4 and Lemma 2.9 (ii).

For the sufficiency, first assume $m = n$. Then $\mathcal{S}$ is permutationally equivalent to $\mathcal{S}_1$, and by Theorem 2.4 the result follows from (ii) and (iii). Next, suppose that $m > n$. If $\mathcal{S}$ has no rows of zeros, then, by Theorem 2.7, the result follows from (i) and (ii). Otherwise, without loss of generality, assume that $\mathcal{S} = \left[\begin{smallmatrix} \mathcal{T} \\ O \end{smallmatrix}\right]$, where $\mathcal{T}$ has no rows of zeros. By Lemma 2.9 (i), it follows from (ii) that $\mathcal{T}$ is strong Hall. Thus, if $\mathcal{T}$ is an n by n sign pattern, then (iii) and Theorem 2.4 imply that $\mathcal{T}$ allows a PLI. By Lemma 2.9 (ii), this implies that $\mathcal{S}$ allows a PLI. Otherwise, since it follows from (i) that each column of $\mathcal{T}$ has a $+$ and a $-$ entry, Theorem 2.7 implies that $\mathcal{T}$ allows a PLI. Therefore, by Lemma 2.9 (ii), $\mathcal{S}$ allows a PLI. ☐

Remark 2.11. For $m \geq n \geq 2$, let $\mathcal{S}$ be an m by n sign pattern. Then the following hold:

(i) If $\mathcal{S}$ satisfies (i), (ii), and (iii) in Theorem 2.10, then so does every superpattern of $\mathcal{S}$. Hence, if $\mathcal{S}$ allows a PLI, then every superpattern of $\mathcal{S}$ allows a PLI.

(ii) Suppose that $\mathcal{S} = \left[\begin{smallmatrix} \mathcal{S}_1 \\ O \end{smallmatrix}\right]$, where $\mathcal{S}_1$ is a square sign pattern, satisfies (iii) in Theorem 2.10. Then, in contrast with Theorem 2.7 (i), there is no matrix $A = \left[\begin{smallmatrix} A_1 \\ O \end{smallmatrix}\right] \in Q(\mathcal{S})$ with a PLI that also has a positive left nullvector, since the equation $[y^T\ z^T]\left[\begin{smallmatrix} A_1 \\ O \end{smallmatrix}\right] = 0$ and the fact that A_1 is nonsingular together imply that $y = 0$.

The following theorem gives sufficient conditions for an m by n sign pattern with $m > n \geq 1$ to have a realization with a PLI and a positive left nullvector.

THEOREM 2.12. *Let $\mathcal{S}$ be an m by n sign pattern with $m > n$ and let $\mathcal{T}$ be the t by n sign pattern obtained from $\mathcal{S}$ by deleting the rows of zeros in $\mathcal{S}$.*

(i) *If $n = 1$ and $\mathcal{T}$ has a $+$ and a $-$ entry, then there exists a matrix in $Q(\mathcal{S})$ with a PLI and a positive left nullvector.*

(ii) *If $t > n \geq 2$ and $\mathcal{T}$ allows a PLI, then there exists a matrix in $Q(\mathcal{S})$ with a PLI and a positive left nullvector.*

Proof. (i) By Proposition 2.1, a $+$ entry implies the existence of $A \in Q(\mathcal{S})$ with a

PLI. Since A has a positive and a negative entry, it can be easily verified that A has a positive left nullvector.

(ii) When $m = t$, the result follows by Theorem 2.7. If $m > t > n$, then Theorem 2.7 implies that there exists a matrix $A \in Q(\mathcal{T})$ with a PLI B and a positive left nullvector y^T. Note that the positive matrix $[B \mid J]$ is a PLI and the vector $[y^T\ 1 \cdots 1]$ is a positive left nullvector of the matrix $\left[\begin{smallmatrix} A \\ O \end{smallmatrix}\right] \in Q(\mathcal{S})$. Hence, the result follows. □

3. Nonnegative left inverses. In this section we determine structures of nonsquare sign patterns that allow an NLI, as well as structures of NLIs.

For $m \geq n$, let $\mathcal{S}$ be an m by n sign pattern with a realization of rank n. Then, by induction, it can be shown that $\mathcal{S}$ is permutationally equivalent to

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \cdots & \mathcal{S}_{1k} \\ O & \mathcal{S}_{22} & \cdots & \mathcal{S}_{2k} \\ \vdots & & \ddots & \vdots \\ O & \cdots & O & \mathcal{S}_{kk} \end{bmatrix}, \tag{3.1}$$

where $k \geq 1$, $\mathcal{S}_{ii}$ is a square fully indecomposable sign pattern for each $i \in \{1, \ldots, k-1\}$, and $\mathcal{S}_{kk}$ is strong Hall. Note that $\mathcal{S}$ is strong Hall if and only if $k = 1$. If $\mathcal{S}$ is an n by n fully indecomposable sign pattern, then $\mathcal{S}$ allows a nonnegative (left) inverse if and only if $\mathcal{S}$ allows a positive (left) inverse; see [2, Theorems 9.2.1 and 9.2.3]. In addition, [2, Theorem 9.2.6] provides a complete characterization of n by n partly decomposable sign patterns that allow a nonnegative (left) inverse.

Remark 3.1. Suppose $m > n$. Let $\mathcal{S}'$ be the square submatrix of $\mathcal{S}$ obtained by deleting the columns and rows associated with $\mathcal{S}_{kk}$. Suppose that $\mathcal{S}$ allows an NLI. Then the square sign pattern $\mathcal{S}'$ also allows an NLI. Hence, for $k = 2$, $\mathcal{S}'$ is fully indecomposable and must satisfy one of the equivalent conditions in [2, Theorem 9.2.1] (see also Theorem 2.4), and for $k \geq 3$, $\mathcal{S}'$ is partly decomposable and must satisfy the conditions in [2, Theorem 9.2.6]. Furthermore, by an argument similar to that in the proof of Lemma 2.3, it is easily verified that an NLI B of a matrix in $Q(\mathcal{S})$ has the block form $B = [B_{ij}]$ with $1 \leq i, j \leq k$ and the (i,j)-block $B_{ij} = O$ whenever $i > j$. Thus, it follows that the strong Hall sign pattern $\mathcal{S}_{kk}$ also allows an NLI.

We now investigate various necessary and/or sufficient conditions for a strong Hall nonsquare sign pattern to allow an NLI. We first consider strong Hall sign patterns with a $+$ and a $-$ entry in each column.

PROPOSITION 3.2. *For $m > n \geq 2$, let $\mathcal{S}$ be an m by n strong Hall sign pattern with a $+$ and a $-$ entry in each column, and let $\mathcal{T}$ be the t by n sign pattern obtained from $\mathcal{S}$ by deleting the rows of zeros in $\mathcal{S}$. If $t > n$, then $\mathcal{S}$ allows an NLI. If $t = n$, then $\mathcal{S}$ allows an NLI if and only if $D(\mathcal{T})$ is strongly connected.*

Proof. The result follows directly from Theorem 2.10 and the fact that if $\mathcal{S}$ allows a PLI, then $\mathcal{S}$ allows an NLI. □

Let $\mathcal{I}_n$ denote the n by n sign pattern with I_n as a realization, i.e., $I_n \in Q(\mathcal{I}_n)$. Clearly, $\mathcal{I}_n$ allows an NLI. Thus, in order to allow an NLI, an m by n sign pattern with $m \geq n$ need not have a $-$ entry in each column as is required to allow a PLI (see Lemma 2.2), but clearly must have a $+$ entry in each column. We first consider the case that $\mathcal{S}$ has a nonnegative column having only $+$ or 0 entries. For ease of notation, we sometimes use $(M)_{ij}$ to denote the (i,j)-entry of a matrix M.

PROPOSITION 3.3. *For $m \geq n \geq 2$, let $\mathcal{S}$ be an m by n sign pattern with at least one nonnegative column. If $\mathcal{S}$ allows an NLI, then each nonnegative column has at most $m - n + 1$ positive entries.*

Proof. Let B be an NLI of $A \in Q(\mathcal{S})$, and let t be the number of positive entries in any nonnegative column of A. Without loss of generality, assume that the first column of A is a nonnegative column with its first t entries positive. Since $(BA)_{h1} = 0$ for each $h \in \{2, \dots, n\}$, it follows that B has the block form $B = [B_{ij}]$ with $1 \leq i, j \leq 2$, where the $(2, 1)$-block B_{21} is the $(n-1)$ by t zero matrix. Hence, the equality $\operatorname{rank} B = n$ implies that the rank of the $(n-1)$ by $(m-t)$ matrix B_{22} is $n-1$. Thus, $n-1 \leq m-t$ and the result follows. □

If all columns are nonnegative, then the following result gives a necessary and sufficient condition for such a sign pattern to allow an NLI.

THEOREM 3.4. *For $m \geq n \geq 1$, let $\mathcal{S}$ be an m by n nonnegative sign pattern. Then $\mathcal{S}$ allows an NLI if and only if $\mathcal{S}$ is permutationally equivalent to*

$$\begin{bmatrix} \mathcal{I}_n \\ \mathcal{T} \end{bmatrix},$$

where $\mathcal{T}$ is an $(m-n)$ by n nonnegative sign pattern.

Proof. The case $n = 1$ follows directly from Proposition 2.1. Suppose that $n \geq 2$.

For the sufficiency, assume without loss of generality that

$$\mathcal{S} = \begin{bmatrix} \mathcal{I}_n \\ \mathcal{T} \end{bmatrix}.$$

Let $T \in Q(\mathcal{T})$ and $A = \left[\begin{smallmatrix} I_n \\ T \end{smallmatrix}\right] \in Q(\mathcal{S})$. Since $[I_n \mid O]\, A = I_n$, it follows that $\mathcal{S}$ allows an NLI.

For the necessity, suppose that $\mathcal{S} = [s_{ij}]$ allows an NLI; i.e., there exist $A = [a_{ij}] \in Q(\mathcal{S})$ and an n by m nonnegative matrix $B = [b_{ij}]$ such that $BA = I_n$. Let $i \in \{1, \dots, n\}$. Since $(BA)_{ii} = 1$, there exists $j_i \in \{1, \dots, m\}$ such that $b_{ij_i} a_{j_i i} > 0$. This implies that $s_{j_i i} = +$. Also, for each $k \in \{1, \dots, n\} \setminus \{i\}$, $(BA)_{ik} = 0$ implies that $b_{ij_i} a_{j_i k} = 0$. Thus, row j_i of $\mathcal{S}$ is equal to row i of $\mathcal{I}_n$. As this holds for each $i \in \{1, \dots, n\}$, the result follows. □

Remark 3.5. Let $\mathcal{S} = \left[\begin{smallmatrix} \mathcal{I}_n \\ \mathcal{J} \end{smallmatrix}\right]$ be the m by n nonnegative sign pattern with $m \geq n \geq 2$, where $\mathcal{J}$ is the sign pattern with all entries positive. Then, by Theorem 3.4, $\mathcal{S}$ allows an NLI. However, in contrast with Remark 2.11 (i), Theorem 3.4 implies that no nonnegative superpattern of $\mathcal{S}$ (except $\mathcal{S}$ itself) allows an NLI.

Next, we consider sign patterns that have at least one nonnegative column and at least one column with a $+$ and a $-$ entry. We use e_i to denote the ith column vector of an identity matrix.

THEOREM 3.6. *For $m \geq n \geq 2$, let $\mathcal{S}$ be an m by n sign pattern that has $p \geq 1$ nonnegative columns and $n - p \geq 1$ columns with a $+$ and a $-$ entry. Suppose that $\mathcal{S}$ allows an NLI. Then $\mathcal{S}$ is permutationally equivalent to a matrix of the form*

$$\begin{bmatrix} \mathcal{I}_p & \mathcal{S}_{12} \\ \mathcal{S}_{21} & \mathcal{S}_{22} \\ O & \mathcal{S}_{32} \end{bmatrix}, \tag{3.2}$$

where $\mathcal{S}_{21}$ is an r by p nonnegative sign pattern with no rows of zeros, O is an s by p zero matrix with $s \geq 1$, and each of the last $n-p$ columns of $\mathcal{S}$ has a $+$ and a $-$ entry. Furthermore, if $\mathcal{S}$ is strong Hall, then $\mathcal{S}_{21}$ is not vacuous and has no column of zeros.

Proof. Without loss of generality, we may assume that the first p columns of $\mathcal{S}$ are nonnegative, and that each of the last $n - p$ columns of $\mathcal{S}$ has a $+$ and a $-$ entry.

Since $\mathcal{S}$ allows an NLI, so does the m by p nonnegative sign pattern consisting of the first p columns of S. Therefore, by Theorem 3.4, we may permute the rows of $\mathcal{S}$ to obtain a matrix of the form (3.2), where S_{21} is a nonnegative matrix with no row of zeros, O is an s by p zero matrix with $s \geq 0$, and each of the last $n-p$ columns has a $+$ and a $-$ entry.

Let A be a matrix in $Q(\mathcal{S})$ that has an NLI, say B. Since $BA = I_n$, each of the vectors $e_1^T, \ldots, e_n^T$ is a nontrivial, nonnegative linear combination of the rows of A. Since the first p columns of A are nonnegative and $n > p$, this requires that $s \geq 1$, and we conclude that $\mathcal{S}$ has the desired form.

If $\mathcal{S}_{21}$ is vacuous or has a column of zeros, then $\mathcal{S}$ has an $(m-1)$ by 1 zero submatrix. Hence $\mathcal{S}$ is not strong Hall, and the result follows by taking the contrapositive. ∎

PROPOSITION 3.7. *For $m \geq n \geq 2$, let $\mathcal{S}$ be an m by n sign pattern that has $p \geq 1$ nonnegative columns and $n-p \geq 1$ columns with a $+$ and a $-$ entry. Suppose that $\mathcal{S}$ allows an NLI and has the form* (3.2). *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ O & A_{32} \end{bmatrix} \in Q(\mathcal{S})$$

have an NLI $B = \left[\begin{smallmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{smallmatrix}\right]$, where each of A_{21}, A_{22}, B_{12}, and B_{22} may be vacuous if $\mathcal{S}$ is not strong Hall. Then the following hold:

(i) *B_{11} is a diagonal matrix, and B_{21} and B_{22} are zero matrices.*
(ii) *The sign pattern $\mathcal{S}_{32}$ allows an NLI.*
(iii) *If row q of $\mathcal{S}_{21}$ has at least two positive entries, then column q of B_{12} is a zero column.*
(iv) *Each column of B_{12} has at most one positive entry. Furthermore, the sign pattern of B_{12} is a subpattern of $\mathcal{S}_{21}^T$.*

Proof. Assume that $\mathcal{S}_{21}$ is not vacuous.

Since $BA = I_n$, it follows that $B_{21}A_{11} + B_{22}A_{21} = O$. Moreover, since B_{21}, B_{22}, A_{11}, and A_{21} are nonnegative, and no row of A_{11} or A_{21} is all zeros, $B_{21} = O$ and $B_{22} = O$. Also, $BA = I_n$ implies that $B_{11}A_{11} + B_{12}A_{21} = I_p$. Since B_{11}, B_{12}, A_{11}, and A_{21} are nonnegative, both $B_{11}A_{11}$ and $B_{12}A_{21}$ are diagonal matrices. Since $A_{11} \in Q(\mathcal{I}_p)$, A_{11} is an invertible diagonal matrix, and hence B_{11} is a diagonal matrix. Thus, (i) is proven.

Since B_{21} and B_{22} are zero matrices, and $BA = I_n$, B_{23} is an NLI of A_{32}, and (ii) is proven.

Since $B_{12}A_{21}$ is a diagonal matrix and B_{12} is nonnegative, the ith row of $B_{12}A_{21}$ is a nonnegative linear combination of the rows of A_{21} (weighted by the entries of the ith row of B_{12}). As the ith row of $B_{12}A_{21}$ is a nonnegative multiple of e_i^T, and A_{21} is a nonnegative matrix with no row of zeros, it follows that if the (i,j)-entry of B_{12} is nonzero, then the jth row of A_{21} is a multiple of e_i^T. In particular, this implies that each column of B_{12} has at most one nonzero entry. If the jth row of A_{21} has at least two positive entries, then column j of B_{12} is a column of zeros, proving (iii). If the (i,j)-entry of B_{12} is nonzero, then the (j,i)-entry of A_{21} is nonzero, completing the proof of (iv).

If $\mathcal{S}_{21}$ is vacuous, then A_{21}, A_{22}, B_{12}, and B_{22} are vacuous, in which case the proofs of (i) for B_{11}, B_{21} and (ii) are similar, but statements (i) for B_{22}, (iii), and (iv) are vacuous. ∎

For $m \geq 2$, Proposition 3.2, Theorem 3.4, and the following theorem completely characterize the m by 2 sign patterns that allow an NLI.

THEOREM 3.8. *For $m \geq 2$, let $\mathcal{S}$ be an m by 2 sign pattern such that the first column is nonnegative and the second column has a + and a − entry. Then $\mathcal{S}$ allows an NLI if and only if the first column of $\mathcal{S}$ has a + entry and* $[0\ +]$ *is a row of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{S}$ allows an NLI. Then the first column of $\mathcal{S}$ also allows an NLI. Hence, Theorem 3.4 implies that the first column of $\mathcal{S}$ has a + entry. By Theorem 3.6, we may assume without loss of generality that $\mathcal{S}$ is of the form (3.2). Since $\mathcal{S}_{32}$ is a column sign pattern, Propositions 3.7 (ii) and 2.1 imply that $\mathcal{S}_{32}$ has a + entry. Hence, $[0\ +]$ is a row of $\mathcal{S}$.

For the converse, suppose that the first column of $\mathcal{S}$ has a + entry and $[0\ +]$ is a row of $\mathcal{S}$. Suppose that $[+\ -]$ is also a row of $\mathcal{S}$. Then without loss of generality, $A \in \mathcal{S}$ has the form

$$\begin{bmatrix} a & -b \\ u & v \\ 0 & c \end{bmatrix},$$

where $a, b, c > 0$, and u and v are $(m-2)$ by 1 vectors. It is easy to verify that

$$\begin{bmatrix} 1/a & O & b/ac \\ 0 & O & 1/c \end{bmatrix}$$

is an NLI of A.

Next suppose that $[+\ -]$ is not a row of $\mathcal{S}$. Then without loss of generality, $A \in \mathcal{S}$ has the form

$$\begin{bmatrix} a & b \\ u & v \\ 0 & -c \\ 0 & d \end{bmatrix},$$

where $a, c, d > 0$, $b \geq 0$, and u and v are $(m-3)$ by 1 vectors. It is easy to verify that

$$\begin{bmatrix} 1/a & O & b/ac & 0 \\ 0 & O & 1/c & 2/d \end{bmatrix}$$

is an NLI of A.

Hence, $\mathcal{S}$ allows an NLI. ◻

Note that the proof of Theorem 3.8 actually shows that if $\mathcal{S}$ is an m by 2 matrix whose first column is nonnegative, second column has a + and a − entry, and $[0\ +]$ is one of its rows, then *each* matrix with sign pattern $\mathcal{S}$ has an NLI.

Example 3.9. The strong Hall sign pattern

$$\mathcal{S} = \begin{bmatrix} + & - \\ + & - \\ 0 & + \end{bmatrix}$$

does not allow a PLI (by Lemma 2.2), but does allow an NLI (by Theorem 3.8) since

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} = I_2.$$

In general (as noted in the introduction) an NLI is not unique. For instance,

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

is another NLI of the above matrix.

In the next theorem, it is shown that if a sign pattern $\mathcal{S}$ of the form (3.2) has a $(3, 2)$-block $\mathcal{S}_{32}$ that allows an NLI or PLI, then some conditions on the negative entries in $\mathcal{S}_{12}$ insure that $\mathcal{S}$ allows an NLI.

THEOREM 3.10. *For $m \geq n \geq 2$, let $\mathcal{S}$ be an m by n sign pattern of the form (3.2) with $p \geq 1$, $n - p \geq 1$, and $\mathcal{S}_{21}$, $\mathcal{S}_{22}$ arbitrary. Then the following hold:*

(i) *If $\mathcal{S}_{32}$ allows an NLI and $\mathcal{S}_{12}$ has only 0 or $-$ entries, then $\mathcal{S}$ allows an NLI.*

(ii) *If $\mathcal{S}_{32}$ allows a PLI and each row of $\mathcal{S}_{12}$ has a $-$ entry, then $\mathcal{S}$ allows an NLI.*

Proof. (i) Let

$$A = \begin{bmatrix} I_p & A_{12} \\ A_{21} & A_{22} \\ O & A_{32} \end{bmatrix} \in Q(\mathcal{S}), \tag{3.3}$$

where $-A_{12} \geq 0$ and A_{32} has B_{23} as an NLI. Let

$$B = \begin{bmatrix} I_p & O & B_{13} \\ O & O & B_{23} \end{bmatrix} \tag{3.4}$$

with $B_{13} = -A_{12}B_{23}$, which is a nonnegative matrix. Then $B \geq 0$, $BA = I_n$, and hence the result follows.

(ii) Let $A \in Q(\mathcal{S})$ be of the form (3.3) and let B be of the form (3.4). If B_{23} is a PLI of A_{32} and $B_{13} = -A_{12}B_{23}$, then $B_{13} > 0$, provided that the negative entries of A_{12} are sufficiently large in magnitude, and $BA = I_n$ as required. ∎

4. Concluding remarks. In section 3, we have characterized nonnegative sign patterns, strong Hall sign patterns with each column having a $+$ and a $-$ entry, and m by 2 sign patterns that allow an NLI. For other cases, we have given some necessary or sufficient conditions for $\mathcal{S}$ to allow an NLI. A characterization for the blocks of the last column of a sign pattern $\mathcal{S}$ of the form (3.1) with $k \geq 2$ that allows an NLI remains open. We conclude by showing (in Theorem 4.2) that some conditions on the submatrix $\mathcal{S}_{kk}$ of a sign pattern $\mathcal{S}$ of the form (3.1) with $k \geq 2$ insure that $\mathcal{S}$ allows an NLI for arbitrary $\mathcal{S}_{1k}, \ldots, \mathcal{S}_{k-1,k}$.

Let $\mathcal{S}$ allow a PLI and $A \in Q(\mathcal{S})$. The following proposition, which is used to prove Theorem 4.2, describes a relation between a PLI of A and the qualitative behavior of solutions of $x^TA = b^T$. The latter equation is given in the introduction as motivation for studying PLIs and NLIs.

PROPOSITION 4.1. *For $m \geq n$, let A be an m by n matrix. Then A has a PLI if and only if for each n by 1 nonzero vector $b \geq 0$ there exists an m by 1 vector $x > 0$ satisfying $x^TA = b^T$.*

Proof. Suppose that an n by m matrix $B > 0$ is a PLI of A. For an n by 1 nonzero vector $b \geq 0$, it is clear that $(b^TB)A = b^T$ and $b^TB > 0$. Hence, the result follows.

Next, suppose that for each n by 1 nonzero vector $b \geq 0$ there exists an m by 1 vector $x > 0$ satisfying $x^TA = b^T$. Take b to be the ith column e_i of I_n and let $x_i > 0$

be a solution of $x^T A = e_i^T$. Then the matrix

$$B = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

is a PLI of A. ∎

THEOREM 4.2. *For $m > s \geq 1$, $n > t \geq 1$, and $m > n$, let $\mathcal{S}_{11}$ be an s by t sign pattern that allows an NLI and let $\mathcal{S}_{22}$ be an $(m-s)$ by $(n-t)$ sign pattern that allows a PLI. Suppose that if $n-t=1$, then $\mathcal{S}_{22}$ has a $-$ entry, and if $n-t \geq 2$, then $\mathcal{S}_{22}$ is not permutationally equivalent to the sign pattern $\left[\begin{smallmatrix} \mathcal{T} \\ O \end{smallmatrix}\right]$ in which $\mathcal{T}$ is a square sign pattern. Then, for an arbitrary s by $(n-t)$ sign pattern $\mathcal{S}_{12}$, the sign pattern $\mathcal{S} = \left[\begin{smallmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ O & \mathcal{S}_{22} \end{smallmatrix}\right]$ allows an NLI.*

Proof. Let A_{11} be a matrix in $Q(\mathcal{S}_{11})$ with B_{11} as an NLI. By Theorem 2.12, there exists $A_{22} \in Q(\mathcal{S}_{22})$ that has a PLI B_{22} and a positive left nullvector y^T. Let $A_{12} \in Q(\mathcal{S}_{12})$. Then A_{12} can be written as $A_{12} = V_1 - V_2$, where $V_1, V_2 \geq 0$ and the entrywise (Hadamard) product $V_1 \circ V_2 = O$. Let $v_i^T \geq 0$ for $1 \leq i \leq s$ denote row i of V_1. If $v_i \neq 0$, then by Proposition 4.1 there exists an $(m-s)$ by 1 vector $x_i > 0$ such that $x_i^T A_{22} = v_i^T$. If $v_i = 0$, then $x_i^T A_{22} = v_i^T = 0$ when $x_i^T = y^T$. Thus, $K_1 = [x_1, \ldots, x_s]^T > 0$ and $K_1 A_{22} = V_1$. Similarly, there exists $K_2 > 0$ such that $K_2 A_{22} = V_2$.

Let $A_{12}(\epsilon) = \epsilon V_1 - V_2 = (\epsilon K_1 - K_2) A_{22}$ for a sufficiently small $\epsilon > 0$ such that $K_2 - \epsilon K_1 > 0$. Note that $V_1 \circ V_2 = O$ implies that $A_{12}(\epsilon) \in Q(\mathcal{S}_{12})$. Let $B_{12} = B_{11}(K_2 - \epsilon K_1)$. Since $K_2 - \epsilon K_1 > 0$ and $B_{11} \geq 0$ with no rows of zeros, it follows that $B_{12} > 0$. It can be easily verified that $\left[\begin{smallmatrix} B_{11} & B_{12} \\ O & B_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} A_{11} & A_{12}(\epsilon) \\ O & A_{22} \end{smallmatrix}\right] = I_n$. Hence, the result follows. ∎

Remark 4.3. Take $\mathcal{S}_{11}$ and $\mathcal{S}_{22}$ in Theorem 4.2 to be $\mathcal{S}'$ in Remark 3.1 and $\mathcal{S}_{kk}$ in the form (3.1) with $k \geq 2$, respectively. Then the conditions on $\mathcal{S}_{kk}$ in Theorem 4.2 insure that the sign pattern $\mathcal{S}$ of the form (3.1) with $k \geq 2$ allows an NLI for arbitrary $\mathcal{S}_{1k}, \ldots, \mathcal{S}_{k-1,k}$.

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