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ON MATRICES WITH SIGNED NULL-SPACES*

SI-JU KIM[†], BRYAN L. SHADER[‡], AND SUK-GEUN HWANG[§]

Abstract. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A . A matrix A has *signed null-space* provided there exists a set \mathcal{S} of sign patterns such that the set of sign patterns of vectors in the null-space of \tilde{A} is \mathcal{S} for each $\tilde{A} \in \mathcal{Q}(A)$. Some properties of matrices with signed null-spaces are investigated.

Key words. totally L -matrices, signed compounds, signed null-spaces

AMS subject classification. 05C50

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1. Introduction. The *sign* of a real number a is defined by

$$\text{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \text{ and} \\ 1 & \text{if } a > 0. \end{cases}$$

A *sign pattern* is a $(0, 1, -1)$ -matrix. The *sign pattern of a matrix* A is the matrix obtained from A by replacing each entry with its sign. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A .

Let A be an m by n matrix and b an m by 1 vector. The linear system $Ax = b$ has *signed solutions* provided there exists a collection \mathcal{S} of n by 1 sign patterns such that the set of sign patterns of the solutions to $\tilde{A}x = \tilde{b}$ is \mathcal{S} for each $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. This notion generalizes that of a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is *sign-solvable* provided each linear system $\tilde{A}x = \tilde{b}$ ($\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set \mathcal{S} has cardinality 1.

The matrix A has *signed null-space* provided $Ax = 0$ has signed solutions. Thus A has signed null-space if and only if there exists a set \mathcal{S} of sign patterns such that the set of sign patterns of vectors in the null-space of \tilde{A} is \mathcal{S} for each $\tilde{A} \in \mathcal{Q}(A)$. An *L-matrix* is a matrix A , with the property that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. A square L -matrix is a *sign-nonsingular (SNS)*-matrix. A *totally L-matrix* is an m by n matrix such that each m by m submatrix is an SNS-matrix. It is known that totally L -matrices have signed null-spaces [3]. We also have the fact as a corollary of some results in this paper. Thus matrices with signed null-spaces generalize totally L -matrices.

A vector is *mixed* if it has a positive entry and a negative entry. A matrix is *row-mixed* if each of its rows is mixed. A *signing* is a nonzero diagonal $(0, 1, -1)$ -matrix.

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A signing is *strict* if each of its diagonal entries is nonzero. A matrix B is *strictly row-mixable* provided there exists a strict signing D such that BD is row-mixed.

In this paper, some properties of matrices with signed null-spaces are investigated, and we show that there exists an m by n matrix A with signed null-space such that A contains a totally L -matrix with m rows as its submatrix and the columns of A are distinct up to multiplication by -1 for any $n \in \{m, m + 1, \dots, 2m\}$.

We use the following standard notation throughout the paper. If k is a positive integer, then $\langle k \rangle$ denotes the set $\{1, 2, \dots, k\}$. Let A be an m by n matrix. If α is a subset of $\{1, 2, \dots, m\}$ and β is a subset of $\{1, 2, \dots, n\}$, then $A[\alpha|\beta]$ denotes the submatrix of A determined by the rows whose indices are in α and the columns whose indices are in β . We sometimes use $A[*|\beta]$ instead of $A[\langle m \rangle|\beta]$. The submatrix complementary to $A[\alpha|\beta]$ is denoted by $A(\alpha|\beta)$. In particular, $A(-|\beta)$ denotes the submatrix obtained from A by deleting columns whose indices are in β . We write $\text{diag}(d_1, d_2, \dots, d_n)$ for the n by n diagonal matrix whose (i, i) -entry is d_i . Let $J_{m,n}$ denote the m by n matrix, all of whose entries are 1, and let e_i denote the column vector, all of whose entries are 0 except for the i th entry, which is 1.

2. Matrices with signed null-space. We say that an m by n matrix $A = [a_{ij}]$ contains a mixed cycle provided there exist distinct i_1, i_2, \dots, i_k and distinct j_1, j_2, \dots, j_k such that

$$a_{i_t, j_t} a_{i_t, j_{t+1}} < 0 \text{ for } t = 1, \dots, k - 1 \text{ and } a_{i_k, j_k} a_{i_k, j_1} < 0.$$

An m by n $(0, 1, -1)$ -matrix has *signed m th compound* provided each of its m by m submatrices having term rank m is an *SNS*-matrix.

We make use of the following results of matrices with signed null-spaces.

THEOREM 2.1 (see [3]). *Let A be a strictly row-mixable m by n matrix. Then the following three conditions are equivalent.*

- (a) A has signed null-space.
- (b) A has term rank m , and its m th compound is signed.
- (c) AD has no mixed cycle for each strict signing such that AD is row-mixed.

THEOREM 2.2 (see [2], [3]). *If a strictly row-mixable matrix A has signed null-space, then there exist matrices B and C (possibly with no rows) and nonzero vectors b and c such that B and C are strictly row-mixable matrices with signed null-spaces,*

$$\begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

have signed null-spaces, and, up to permutation of rows and columns,

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix}.$$

The converse also holds.

Let A be an m by n $(0, 1, -1)$ -matrix. The matrix B is *conformally contractible* to A provided there exists an index k such that the rows and columns of B can be permuted so that B has the form

$$\left[\begin{array}{ccc|c|c} A[\langle m \rangle|\langle n \rangle \setminus \{k\}] & x & y \\ \hline 0 & \dots & 0 & 1 & -1 \end{array} \right],$$

where $x = [x_1, \dots, x_m]^T$ and $y = [y_1, \dots, y_m]^T$ are $(0, 1, -1)$ -vectors such that $x_i y_i \geq 0$ for $i = 1, 2, \dots, m$, and the sign pattern of $x + y$ is the k th column of A .

Let B be conformally contractible to A . It is known that A has signed null-space if and only if B has signed null-space [3]. All matrices we consider from now on are assumed to be $(0, 1, -1)$ -matrices.

THEOREM 2.3 (see [4]). *Let an m by n matrix A have a k by $k + 1$ submatrix B whose complementary submatrix in A has term rank $m - k$. If there is a matrix B^* obtained from B by replacing some nonzero entries with 0's if necessary such that $J_{2,3}$ is the zero pattern of a matrix obtained from B^* by a sequence of conformal contractions, then A does not have signed null-space.*

Let M be an m by n strictly row-mixable matrix of the form

$$(2.1) \quad M = \begin{bmatrix} & & 0 \\ & * & \vdots \\ & & 0 \\ & 1 & 1 \end{bmatrix}.$$

PROPOSITION 2.4. *M has signed null-space if and only if*

$$A = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & M & & 0 \\ \hline 0 & \cdots & 0 & 1 & -1 \\ & & & & 1 \end{array} \right]$$

has signed null-space.

Proof. Let M have signed null-space, and let C be any $m + 1$ by $m + 1$ submatrix of A . If C contains the last column of A , then $C(m + 1|m + 1)$ is an m by m submatrix of M . Hence $C(m + 1|m + 1)$ is an *SNS*-matrix, or $C(m + 1|m + 1)$ has identically zero determinant by Theorem 2.1. Thus C is an *SNS*-matrix, or C has identically zero determinant. Hence we may assume that C does not contain the last column of A . If C contains neither the $n - 1$ th column nor the n th column, then clearly C has identically zero determinant. If C contains only one of the $(n - 1)$ th column or the n th column, then $C(m + 1|m + 1)$ is an m by m submatrix of M . Hence $C(m + 1|m + 1)$ is an *SNS*-matrix, or $C(m + 1|m + 1)$ has identically zero determinant. Therefore, C is an *SNS*-matrix, or C has identically zero determinant. Let C contain both the $(n - 1)$ th column and the n th column of A . Then $C(m + 1|m + 1)$ is an *SNS*-matrix, or $C(m + 1|m + 1)$ has identically zero determinant. If $C(m + 1|m + 1)$ has identically zero determinant, then there exists an s by t zero submatrix of $C(m + 1|m + 1)$ such that $s + t = m + 1$. From this, it is easy to show that C has a p by q zero submatrix such that $p + q = m + 2$; i.e., C has identically zero determinant. Let $C(m + 1|m + 1)$ be an *SNS*-matrix. Since C is conformally contractible to $C(m + 1|m + 1)$, C is also an *SNS*-matrix. Thus the $(m + 1)$ th compound of A is signed. Since M has signed null-space, the term rank of M is m , and hence the term rank of A is $m + 1$. Thus A has signed null-space by Theorem 2.1. The converse is trivial. \square

We say that A is a *single extension* of M in Proposition 2.4. Proposition 2.4 means that a strictly row-mixable matrix has signed null-space if and only if its single extension has signed null-space.

Let

$$G = \begin{bmatrix} & & & & 0 \\ & & & & \vdots \\ & & * & & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 \end{bmatrix}$$

condition. We prove the result by induction on m . Trivially, we have the result for $m = 2$. Let $m \geq 3$. By Theorem 2.2, A can be rearranged as

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix},$$

where matrices B and C (possibly with no rows) are strictly row-mixable matrices which have signed null-spaces, and vectors b and c are nonzero.

$$\begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

also have signed null-spaces. Let $A[\alpha|\beta] = \begin{bmatrix} B \\ b \end{bmatrix}$ and $A[\gamma|\delta] = \begin{bmatrix} c \\ C \end{bmatrix}$ such that $|\alpha| = k$, $|\beta| = s$, $|\gamma| = l$, and $|\delta| = t$. Then $k + l - 1 = m$ and $s + t = n$.

Let $k > 1$ and $l > 1$. If $A[\alpha|\beta]$ has one of the unit vectors $\pm e_k$ as a column, then we can assume that $A[\alpha|\beta]$ is of the form

$$\begin{bmatrix} B' & O \\ b' & 1 \end{bmatrix}.$$

Let B' have no duplicate columns up to multiplication by -1 . By induction, B' and hence A have at least two rows with exactly three nonzero entries. Thus we are done. Therefore, we assume that B' has duplicate columns up to multiplication by -1 . Then $b' \neq 0$. If b' has at least two nonzero entries, then $A[\alpha|\beta]$ is a strictly row-mixable matrix with no duplicate columns up to multiplication by -1 . Since $A[\alpha|\beta]$ is not conformally contractible to a matrix, B has at least one row with exactly three nonzero entries. Let b' have exactly one nonzero entry. Let the columns 1, 2 of B' be a pair of duplicate columns up to multiplication by -1 , and let p be the number of nonzero entries in the column 1 of B' . Let D be a strict signing such that $M = B'D$ is row-mixed. Since B' has signed null-space, M has no mixed cycle, and hence the columns 1 and 2 of M must be identical or $p = 1$. If $p \geq 2$, then the matrix M' obtained from M by multiplying the column 2 by -1 has a mixed cycle. Thus M' is a row-mixed matrix with signed null space, which is impossible by Theorem 2.1. Hence $p = 1$. Therefore, every duplicate column of B' is of the form e_i or $-e_i$ for some i . Hence B' has only one pair of duplicate columns, which are e_i or $-e_i$ for some $i (< k)$. The matrix obtained from B' by deleting one of the duplicate columns, which are e_i or $-e_i$, satisfies the conditions of the hypothesis if its i th row has at least three nonzero entries. This implies that B has at least one row with exactly three nonzero entries. Let $C' = A[\gamma|\{s\} \cup \delta]$. Similarly, C' has a row i with exactly three nonzero entries for some $i (\neq 1)$. Hence C has at least one row with exactly three nonzero entries. Therefore, A has at least two rows with exactly three nonzero entries. Similarly, in the case in which $A[\gamma|\delta]$ has one of the unit vectors $\pm e_1$ as a column, we have the result. Assume that $A[\alpha|\beta]$ and $A[\gamma|\delta]$ do not have the unit vectors $\pm e_k$ and $\pm e_1$, respectively, as columns. Since b is nonzero, the k by $s + 1$ matrix B^* obtained from $A[\alpha|\beta]$ by adding e_k as a column is a strictly row-mixable matrix with no duplicate columns up to multiplication by -1 . Since B has signed null-space, B^* also has signed null-space. Applying the similar method above to B^* , we can derive that B has at least one row with exactly three nonzero entries. Similarly, C also has at least one row with exactly three nonzero entries. Hence we have the result when $k > 1$ and $l > 1$.

Let $k = 1$. Then $s = 1$ since the columns of A are distinct up to multiplication by -1 . Hence we may assume that $A = [a_{ij}]$ is of the form

$$\begin{bmatrix} 1 & c \\ O & C \end{bmatrix}.$$

If C has no duplicate columns up to multiplication by -1 , then we have the result for C by induction, and hence we have the result for A . Let C have duplicate columns up to multiplication by -1 . Then the duplicate columns of C are of the form e_i or $-e_i$ for some i , as we have shown before. This implies that the number of identical columns of C up to multiplication by -1 is at most 3. Therefore, we may assume that the zero pattern of A is of the form

$$\left[\begin{array}{c|ccc|c} 1 & u & \cdots & u & v & \cdots & v & w & \cdots & w & 0 & \text{or} & 1 \\ \hline & x & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & \\ & & & x & & & & & & & & & \\ \hline & & & & v & & & & & & & & \\ & & & & & \ddots & & & & & & & \\ & & & & & & v & & & & & & \\ \hline & & & & & & & v & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & v & & & \\ \hline & & & & & & & & & & & & \\ & & & & & & & & & & & T & \end{array} \right],$$

where $u = (1, 1, 0)$, $v = (1, 1)$, $w = (1, 0)$, and $x = (1, 1, 1)$, and the unspecified entries are zero. Let ϵ be the set of indices of columns in A corresponding to $\begin{bmatrix} S \\ T \end{bmatrix}$. Then we may also assume that $A[\gamma \setminus \{1\}|\epsilon]$ has no duplicate columns up to multiplication by -1 , and the columns are also different from the ones of $A(1|\epsilon)$ up to multiplication by -1 . If $\begin{bmatrix} S \\ T \end{bmatrix}$ is vacuous, we are done since $l \geq 3$ and every row but the first row of A has at least three nonzero entries. Let only T be vacuous. Notice that each column of S has at least two nonzero entries. Hence each row of S has at most one nonzero entry. For, suppose that a row of S has two nonzero entries. Since the columns of $A[\gamma \setminus \{1\}|\epsilon]$ are distinct up to multiplication by -1 , we may assume that there exists a submatrix of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & * & * \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & * & * \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & * \\ 0 & 0 & 1 & * & 1 \end{bmatrix},$$

where $*$ is 0 or 1. By Theorem 2.3, A does not have signed null-space. This is a contradiction. Next, suppose that a row r of $A[\gamma \setminus \{1\}|\langle n \rangle]$ has four nonzero entries. Since each row of S has at most one nonzero entry and each column of S has at least two nonzero entries, we have a submatrix of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & 1 & * \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

which is also impossible by Theorem 2.3. Hence each row of $A[\gamma \setminus \{1\} | \langle n \rangle]$ has exactly three nonzero entries. Thus we have the result when T is vacuous. Let T be nonvacuous. Notice that the submatrix of A corresponding to T is a strictly row-mixable matrix with signed null-space. Let T' be the matrix obtained from T by deleting zero columns. Then we may assume that T is of the form $[O \ T']$. If the submatrix A' of A corresponding to T' has no duplicate columns up to multiplication by -1 , then A' has at least two rows with exactly three nonzero entries by induction. Hence we have the result. Suppose that A' has duplicate columns up to multiplication by -1 . It is easy to show that such columns of A' have exactly one nonzero entry as we have shown above. We want to show that the number of identical columns of A' is at most three. Suppose that there are four identical columns in A' up to multiplication by -1 . We may assume that the zero pattern of the submatrix consisting of such duplicate columns of A' is of the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ & O & & \end{bmatrix}.$$

Since $A[\gamma \setminus \{1\} | \epsilon]$ has no duplicate columns up to multiplication by -1 , we may assume that $A[\gamma \setminus \{1\} | \epsilon]$ has a submatrix whose zero pattern is

$$\begin{bmatrix} 1 & * & * \\ * & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where $*$ is 0 or 1. Hence we can have a submatrix N of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & * & * & * \\ 1 & 0 & 1 & * & * \\ 0 & 1 & * & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 & * & * & * \\ 1 & 0 & 0 & 1 & * & * \\ 0 & 1 & 0 & * & 1 & * \\ 0 & 0 & 1 & * & * & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

where $*$ is 0 or 1. By Theorem 2.3, A does not have signed null-space. This is a contradiction. Thus we can assume that T' is of the form

$$\begin{bmatrix} T'_1 & T'_2 \\ O & T'_3 \end{bmatrix},$$

where T'_1 is a block diagonal matrix whose diagonal blocks are either $[1 \ 1]$ or $[1 \ 1 \ 1]$, and the submatrix of A corresponding to $\begin{bmatrix} T'_2 \\ T'_3 \end{bmatrix}$ has no duplicate columns up to multiplication by -1 . Continuing this process, we can assume that T is of the form

$$\begin{bmatrix} T_1 & & * \\ O & \ddots & \\ & & T_q \end{bmatrix},$$

where $T_i = [O \ T'_i]$ for $i = 1, 2, \dots, q$ and T'_i are block diagonal matrices whose diagonal blocks are either $[1 \ 1]$ or $[1 \ 1 \ 1]$ for $i = 1, 2, \dots, q - 1$.

Let λ_i be the set of indices of rows in A corresponding to T_i . Let ϵ_i and δ_i be the set of indices of nonzero columns in A and zero columns in A corresponding

to T_i , respectively. It is easy to show that each row of $A[\lambda_i|\epsilon_i \cup \delta_{i+1}]$ has at most three nonzero entries for $i = 1, 2, \dots, q - 1$ by a method similar to that used in the case in which only T is vacuous. If the submatrix A'_q of A corresponding to T'_q has no duplicate columns up to multiplication by -1 , then A'_q satisfies the hypothesis. Hence we have the result. If A'_q has duplicate columns up to multiplication by -1 , then we may assume that $T'_q = [T''_q \ T'''_q]$, where T''_q is a block diagonal matrix whose diagonal blocks are $[1 \ 1]$ or $[1 \ 1 \ 1]$. As we have shown above in the case in which T is vacuous, each row of T'_q has exactly three nonzero entries. If T'_q has at least two rows, then we are done.

Thus we may assume that $T'_q = [1 \ 1 \ 1]$. Then $A[\langle m - 1 \rangle | n - 2, n - 1, n]$ cannot have a row whose zero pattern is equal to $(1, 1, 1)$ because, if so, then A has $J_{2,3}$ as a submatrix, and this is impossible by Theorem 2.3. If $A[m - 1 | n - 2, n - 1, n] = O$, then we are done. Hence we may assume that the zero pattern of $A[m - 1 | n - 2, n - 1, n]$ is either $[1 \ 1 \ 0]$ or $[1 \ 0 \ 0]$.

Let the zero pattern of $A[m - 1 | n - 2, n - 1, n]$ be $[1 \ 1 \ 0]$. If the r th row of $A[\langle m - 2 \rangle | n - 2, n - 1, n]$ has the zero pattern $(1, 1, 0)$ for some r , then there exist distinct i_1, i_2, \dots, i_k and distinct j_1, j_2, \dots, j_k such that $a_{i_1, j_1}, a_{i_2, j_1}, \dots, a_{i_k, j_k}$ are nonzero, where $i_1 = 1, i_k = r$, and $j_k = n - 2$. There also exist distinct p_1, p_2, \dots, p_t and distinct q_1, q_2, \dots, q_t such that $a_{p_1, q_1}, a_{p_2, q_1}, \dots, a_{p_t, q_t}$ are nonzero, where $p_1 = 1, p_t = m - 1$, and $q_t = n - 2$. Choosing some entries from these entries, we have a matrix which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$. This is impossible by Theorem 2.3. We can apply a method similar to that used above to show that $A[\langle m - 2 \rangle | n] = O$. Hence each row of $A[\langle m - 2 \rangle | n - 2, n - 1, n]$ has a zero pattern of the forms $(0, 0, 0)$, $(1, 0, 0)$, or $(0, 1, 0)$. Let T'_{q-1} have at least two rows. It is easy to show that, if each row of $A[\lambda_{q-1}|\epsilon_{q-1} \cup \delta_q \cup \epsilon_q]$ has at least four nonzero entries, we have a submatrix of A which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$ by the method just used above. By Theorem 2.3, it is impossible. Hence some row of $A[\lambda_{q-1}|\epsilon_{q-1} \cup \delta_q \cup \epsilon_q]$ has exactly three nonzero entries. Thus we have the result when T'_{q-1} has at least two rows. Therefore, we may assume that T'_{q-1} is either $[1 \ 1]$ or $[1 \ 1 \ 1]$. Notice that $T_q = T'_q = [1 \ 1 \ 1]$.

Let $T'_{q-1} = [1 \ 1 \ 1]$. If $A[\langle m - 2 \rangle | n - 2, n - 1, n] \neq O$, then we can show that there exists a submatrix of A which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$. This is impossible. Hence we may assume that $A[\langle m - 2 \rangle | n - 2, n - 1, n] = O$. Then $A[\langle m - 1 \rangle | \langle n - 3 \rangle]$ has at least two rows with exactly three nonzero entries by induction. Hence we are done. Let $T'_{q-1} = [1 \ 1]$. Notice that $A[\langle m - 2 \rangle | n - 4, n - 3]$ has no submatrix whose zero pattern is $J_{2,2}$ by Theorem 2.1. That is, all rows of $A[\langle m - 2 \rangle | n - 4, n - 3]$ except for one row have at least one zero entry. Since the conformal contraction of $A[\langle m - 1 \rangle | \langle m - 3 \rangle]$ on the last row has signed null-space, $A[\langle m - 1 \rangle | \langle n - 3 \rangle]$ has at least one row with exactly three nonzero entries. Thus we have the result if $A[\langle m - 2 \rangle | n - 2, n - 1, n] = O$. Let $A[\langle m - 2 \rangle | n - 2, n - 1, n] \neq O$. Since we are done if the $(m - 2)$ nd row of A has exactly three nonzero entries, we may assume that the $(m - 2)$ nd row of A has at least four nonzero entries. Deleting the cases in which a contradiction occurs, we may assume that the zero pattern of $A[m - 2, m - 1, m | n - 6, n - 5, n - 4, n - 3, n - 2, n - 1, n]$ is

$$\left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right].$$

It is easy to show that $A[\langle m-2 \rangle | n-1, n] = O$ by using a method similar to that used above. If the columns of $A[m-2, m-1 | n-4, n-2]$ are identical up to multiplication by -1 , then it is easy to find a strict signing D such that AD is a row-mixed matrix and $A[m-2, m-1 | n-4, n-2]D$ contains a mixed cycle. This is impossible by Theorem 2.1. Hence the columns of $A[\langle m-1 \rangle | \langle n \rangle]$ are not identical up to multiplication by -1 . Therefore, $A[\langle m-1 \rangle | \langle n-2 \rangle]$ satisfies the hypothesis. Thus we have the result when the zero pattern of $A[m-1 | n-2, n-1, n]$ is $[1 \ 1 \ 0]$.

In the case in which the zero pattern of $A[m-1 | n-2, n-1, n]$ is $[1 \ 0 \ 0]$, the last row of $A[\langle m-1 \rangle | \langle n-3 \rangle]$ must have two or three nonzero entries. If it has two nonzero entries, then we are done. Let it have three nonzero entries. Then we have a submatrix of A which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$ if $A[\langle m-2 \rangle | n-2, n-1, n] \neq O$ by a method similar to that used above. Hence we have $A[\langle m-2 \rangle | n-2, n-1, n] = O$. Therefore, $A[\langle m-1 \rangle | \langle n-3 \rangle]$ has at least two rows with exactly three nonzero entries by induction. Thus we have the result for $k = 1$. Similarly, we have the same result for $l = 1$. \square

3. Matrices containing totally L -matrices. Let A be a matrix with signed null-space. A is a *maximal matrix with signed null-space* if any matrix obtained from A by replacing a zero entry with a nonzero entry does not have signed null-space.

LEMMA 3.1. *An m by $m+2$ totally L -matrix is a maximal matrix with signed null-space.*

Proof. Let A be an m by $m+2$ totally L -matrix. Let A^* be an m by $m+2$ matrix obtained from A by replacing a zero entry with 1 or -1 . Notice that every m by m submatrix of A^* has term rank m . Since A^* has a row with four nonzero entries, A^* is not a totally L -matrix. Therefore, there exists an m by m submatrix of A^* that is not an SNS -matrix. Hence A^* does not have signed null-space by Theorem 2.1. \square

LEMMA 3.2. *Let A be an m by $m+2$ totally L -matrix, and let \mathbf{x} be an m by 1 column vector which has at least two nonzero entries. Then $B = [A \ \mathbf{x}]$ does not have signed null-space.*

Proof. We will prove the result by induction on m . The statement is clear for $m = 2$. We may assume that

$$B = [b_{ij}] = \left[\begin{array}{c|c|c} M' & O & \mathbf{x} \\ \hline & I_2 & \end{array} \right],$$

where I_2 is the identity matrix of order 2. If $b_{m-1, m+3} = 0$ or $b_{m, m+3} = 0$, say, $b_{m, m+3} = 0$, then $B(m | m+2)$ does not have signed null-space by induction. Hence we have the result by Theorem 2.3. Therefore, we may assume that the last two positions of \mathbf{x} have nonzero entries. Since a totally L -matrix is a maximal matrix with signed null-space, $B(- | m+2)$ does not have signed null-space. Hence B does not have signed null-space. \square

We say that an m by $m+2$ totally L -matrix contains k *double-extensions* (or $m-2k-2$ *single-extensions*) if A is obtained from

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

by a sequence of $m-2k-2$ single-extensions and k double-extensions up to row and column permutations and multiplication of rows and columns by -1 .

PROPOSITION 3.3. *Let A be an m by n matrix with signed null-space whose*

columns are nonzero and distinct up to multiplication by -1 . If A contains an m by $m + 2$ totally L -matrix with k double-extensions, then $n \leq 2m - 2k$.

Proof. We will prove the result by induction on k . Let T_k be an m by $m + 2$ totally L -matrix with k double-extensions contained in A . Notice that each column of A which does not correspond to T_k has exactly one zero entry by Lemma 3.2. If $k = 0$, then it is known [1] that T_k has a signed r th compound for each $r = 1, 2, \dots, m$. Hence we can have the identity matrix I_m as a submatrix of A . Since T_0 has exactly two columns with exactly one nonzero entry, $n \leq m + 2 + (m - 2) = 2m = 2m - 2k$. Let $k \neq 0$. By Proposition 2.4 and Lemma 3.2, we may assume that A is of the form

$$\left[\begin{array}{c|cc|ccc} A_1 & & & & & & \\ \hline & A_2 & & & O & & \\ \hline A_3 & 1 & -1 & & 0 & 0 & 0 \\ & 1 & 1 & & -1 & 0 & 0 \\ \hline O & 0 & 1 & & 1 & 1 & 0 \\ & 1 & 0 & & 1 & 0 & 1 \end{array} \right].$$

Then $A(m - 1, m|n - 1, n)$ has signed null-space, and it contains an $m - 2$ by m totally L -matrix with $k - 1$ double-extensions. The columns of $A(m - 1, m|n - 1, n)$ are distinct up to multiplication by -1 because, if not, then A_3 has a column of the forms $(0, 1)^T$ or $(0, -1)^T$, say, $(0, 1)^T$. Then A has a submatrix

$$(3.1) \quad B = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

which is not an SNS -matrix. Since A contains an m by $m + 2$ totally L -matrix, the complementary submatrix to B in A has term rank $m - 4$. Hence A does not have signed null-space by Theorem 2.1. This is a contradiction. Therefore, $n - 2 \leq 2(m - 2) - 2(k - 1) = 2m - 2k - 2$ by induction. Thus we have $n \leq 2m - 2k$. \square

Let l be the number of single-extensions contained in A . Then we have $l = m - 2k - 2$. Hence we can restate the result of Proposition 3.3 in terms of l : $n \leq m + l + 2$.

COROLLARY 3.4. *Let T be an m by $m + 2$ totally L -matrix which contains no single-extensions. Then there is no m by n matrix A with signed null-space such that A contains T properly, and the columns of A are nonzero and distinct up to multiplication by -1 .*

Proof. Let A be an m by n matrix with signed null-space, and let A contain T . Since T contains no single-extensions, $l = 0$. Hence $n \leq m + l + 2 = m + 2$. Hence $A = T$. \square

Let M be an m by n matrix of the form in (2.1) with signed null-space, and let A be the $m + 1$ by $n + 2$ matrix such that

$$A = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & 1 & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 & -1 & 1 \end{array} \right].$$

Since $A(-|n + 2)$ is conformally contractible to M , $A(-|n + 2)$ has signed null-space. Since M has signed null-space, A has signed null-space by Theorem 2.1. Let T_k be an m by $m + 2$ totally L -matrix with k double-extensions. Let $\{i_1, i_2, \dots, i_l\}$ with

$i_1 < i_2 < \cdots < i_l$ be the set of indices of rows used when single-extensions are constructed in T_k . Notice that T_k does not contain any e_{i_j} , $j = 1, 2, \dots, l$. The remark above and Proposition 2.5 imply that

$$(3.2) \quad T = [T_k \ e_{i_1} \ e_{i_2} \ \cdots \ e_{i_l}]$$

is an m by $2m - 2k$ matrix whose columns are distinct up to multiplication by -1 , and it has signed null-space.

Let j be the index of a row of T_k used when a double-extension is done, and suppose that T_k does not have e_j as a column. $[T_k \ e_j]$ has a submatrix of the form in (3.1), and hence it does not have signed null-space, as we have shown in the proof of Proposition 3.3.

Let \mathcal{T}_k be the set of all matrices of the form in (3.2). Notice that columns of $A \in \mathcal{T}_k$ are nonzero and distinct up to multiplication by -1 . We can express the m by n matrices A with $n = 2m - 2k$ in Proposition 3.3 in terms of elements of \mathcal{T}_k .

PROPOSITION 3.5. *In Proposition 3.3, $n = 2m - 2k$ if and only if there exists a permutation matrix Q such that A is equal to TQ up to multiplication of rows and columns by -1 for some $T \in \mathcal{T}_k$.*

Proof. Let A be an m by n matrix such that $A = TQ$ for some permutation matrix Q and $T \in \mathcal{T}_k$. Then $m = 2k + l + 2$, and hence $n = m + 2 + l = m + 2 + (m - 2 - 2k) = 2m - 2k$. Conversely, let A be an m by $2m - 2k$ matrix satisfying the conditions in Proposition 3.3. Let T_k be an m by $m + 2$ totally L -matrix with k double-extensions contained in A . Then there exists a permutation matrix Q and strict signings D, E such that $DAQE$ is a submatrix of matrix T of the form in (3.2) by Lemma 3.2 and the remark above. Since T is an m by $2m - 2k$ matrix, $A = DTQ^{-1}E$. Since $T \in \mathcal{T}_k$, we have the result. \square

COROLLARY 3.6. *Let m be a positive integer with $m \geq 2$, and let n be any integer in $\{m, m + 1, \dots, 2m\}$. Then there exists an m by n matrix A with signed null-space such that A contains a totally L -matrix with m rows as its submatrix and the columns of A are nonzero and distinct up to multiplication by -1 .*

Proof. Let n be any integer in $\{m, m + 1, \dots, 2m\}$. If $n \leq m + 2$, then we can take an m by n totally L -matrix as such a matrix A . If $n > m + 2$, there exists an m by $m + 2$ totally L -matrix T_{n-m-2} with $n - m - 2$ single-extensions. Hence there exists an m by n matrix $A \in \mathcal{T}_{n-m-2}$ which contains T_{n-m-2} by the remark above. \square

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