9-1-1997

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Publication Information

Shader, Bryan L. (1997). "A Simple Proof of Fiedler's Conjecture Concerning Orthogonal Matrices." *Rocky Mountain Journal of Mathematics* 27.4, 1239-1243.

ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 4, Fall 1997

A SIMPLE PROOF OF FIEDLER'S CONJECTURE CONCERNING ORTHOGONAL MATRICES

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ABSTRACT. We give a simple proof that an $n \times n$ orthogonal matrix with $n \geq 2$ which cannot be written as a direct sum has at least $4n - 4$ nonzero entries.

1. The result. What is the least number of nonzero entries in a real orthogonal matrix of order n? Since the identity matrix I_n is orthogonal the answer is clearly n. A more interesting question is: what is the least number of nonzero entries in a real orthogonal matrix which, no matter how its rows and columns are permuted, cannot be written as a direct sum of (orthogonal) matrices? Examples of orthogonal matrices of each order $n \geq 2$ which cannot be written as a direct sum and which have $4n - 4$ nonzero entries are given in [**1**]. M. Fiedler conjectured that an orthogonal matrix of order $n \geq 2$ which cannot be written as a direct sum has at least $4n - 4$ nonzero entries.

Using a combinatorial property of orthogonal matrices, Fiedler's conjecture was proven in [**1**]. A $(0, 1)$-matrix A of order n is *combinatorially orthogonal* provided no pair of rows of A has inner product 1 and no pair of columns of A has inner product 1. Clearly, if Q is an orthogonal matrix of order n, then the $(0, 1)$-matrix obtained from Q by replacing each of its nonzero entries by a 1 is combinatorially orthogonal. A quite lengthy and complex combinatorial argument is used in [**1**] to show that if A is a combinatorially orthogonal matrix of order $n \geq 2$ and A cannot be written as a direct sum, then A has at least $4n - 4$ nonzero entries. Clearly this result implies Fiedler's conjecture. In this note we give a simple matrix theoretic proof of Fiedler's conjecture.

Received by the editors on November 10, 1994, and in revised form on November 7, 1995.
This research is supported in part by the National Security Agency under grant MDA904-94-H-2051.

Theorem 1.1. *Let Q be an orthogonal matrix of order n of the form*

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

where U is a $k \times k+l$ matrix, and W is an $m+l \times m$ matrix for some positive integers k and m and nonnegative integer l with $k+l+m=n$. Then the rank, $r(V)$, of V equals l.

Proof. Since the rows of U are linearly independent $r(U) = k$. Similarly, $r(W) = m$. Since the rank of a sum of matrices is less than or equal to the sum of the ranks of the matrices,

$$r(Q) \leq r(U) + r(W) + r(V).$$

Thus $l \leq r(V)$, since $r(Q) = k + l + m$. Because Q is orthogonal, the rows of V belong to the orthogonal complement in R^{k+l} of the space spanned by the rows of U. Since $r(U) = k$, this implies that $r(V) \leq l$. Therefore $r(V) = l$. □

We note that, by taking $l = 0$ in Theorem 1.2, we have $V = O$; and hence an orthogonal matrix Q of order n can be written as a direct sum of matrices (after possibly permuting its rows and columns) if and only if Q contains a zero submatrix whose dimensions sum to n.

Corollary 1.2. *Let*

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

be an $n \times n$ orthogonal matrix where U is $k \times k+1$ and W is $l+1 \times l$, $k+l = n-1$ and $k, l \geq 1$. Then there exist nonzero vectors x and y such that $V = xy^T$, and both

$$U' = \begin{bmatrix} U \\ y^T \end{bmatrix} \quad \text{and} \quad W' = [\, x \quad W \,] \tag{1}$$

are orthogonal matrices.

Proof. By Theorem 1.1, V has rank one. Hence, there exist vectors x and y such that $V = xy^T$. Since Q is orthogonal, the sum of the squares of the entries in its first k rows equals k, and the sum of the squares of the entries in its first $k+1$ columns equals $k+1$. Hence, the sum of the squares of the entries in V equals 1. It follows that $(x^T x)(y^T y) = 1$. Thus, by replacing x by $(1/\sqrt{x^T x})x$ and y by $\sqrt{x^T x}y$, we may assume that $x^T x = 1$ and $y^T y = 1$. Since Q is orthogonal, y^T is orthogonal to each row of U, and x is orthogonal to each column of W. The corollary now follows. $\square$

We now prove Fiedler's conjecture. We let $\#(A)$ denote the number of nonzero entries in the matrix A.

Theorem 1.3. *Let Q be an orthogonal matrix of order $n \geq 2$ which cannot be written as a direct sum of matrices (no matter how its rows and columns are permuted). Then Q has at least $4n-4$ nonzero entries.*

Proof. The proof is by induction on n. First suppose that Q contains a $k \times l$ zero submatrix for some positive integers k and l with $k+l = n-1$. Without loss of generality we may assume that

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

where U is $k \times k+1$, and W is $l+1 \times l$. By Corollary 1.2, there exist x and y such that $V = xy^T$ and the matrices U' and W' in (1) are orthogonal matrices.

Suppose that U' can be written as a direct sum of two matrices. Then U' contains an $r \times s$ zero submatrix which does not intersect the last row of U' for some positive integers r and s with $r+s = k+1$. It follows that Q contains an $r \times s + (n-k-1)$ zero submatrix. Hence, by the observation immediately after Theorem 1.1, Q can be written as a direct sum of matrices. This contradicts our assumptions. Thus, U' cannot be written as a direct sum of matrices. A similar argument shows that W' cannot be written as a direct sum of matrices.

Clearly,

$$\#(Q) = \#(U') + \#(W') - 1 + (\#(y) - 1)(\#(x) - 1).$$

By induction U' has at least k nonzero entries, and W' has at least $4l$ nonzero entries. Thus,

$$\begin{aligned}\#(Q) &\geq 4k + 4l - 1 + (\#(y) - 1)(\#(x) - 1)\\ &= (4n - 4) - 1 + (\#(y) - 1)(\#(x) - 1).\end{aligned}$$

Since Q has no $r \times s$ zero submatrix with $r + s \geq n$, $\#(y) \geq 2$ and $\#(x) \geq 2$. Therefore, $\#(Q) \geq 4n - 4$.

Now suppose that Q does not contain a $k \times l$ zero submatrix for any positive integers k and l with $k + l = n - 1$. If $n = 2$, then each entry of Q is nonzero and hence $\#(Q) \geq 4(n - 1)$. Assume that $n \geq 3$. Then each row and column of Q has at least 3 nonzero entries. Thus, if $n = 3$, then $\#(Q) > 4(n - 1)$.

Assume that $n \geq 4$. If each row and column of Q has at least 4 nonzero entries, then $\#(Q) \geq 4n > 4(n - 1)$. Suppose that some row or column of Q has exactly 3 nonzero entries. We may assume without loss of generality that row 1 of Q has exactly 3 nonzero entries, and that these occur in columns 1, 2 and 3. Let

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & 0 & \cdots & 0 \\ u & v & w & & X & \end{bmatrix},$$

where X is $n - 1 \times n - 3$.

By Theorem 1.1, the rank of $[u\, v\, w]$ is 2. Without loss of generality we may assume that u and v are linearly independent.

Since each of u, v and w is orthogonal to each column of X,

$$Q' = [\, u' \quad v' \quad X\,]$$

is an orthogonal matrix of order $n - 1$, where u' and v' are the vectors obtained from u and v by applying the Gram-Schmidt process.

Suppose that Q' can be written as a direct sum of two matrices. Then there exist positive integers r and s with $r + s \geq n - 2$ such that X contains an $r \times s$ zero submatrix. It follows that Q contains an $r + 1 \times s$ zero submatrix, which contradicts our assumptions. Hence Q' cannot be written as a direct sum of matrices.

By the induction hypothesis, $\#(Q') \geq 4n - 8$. Clearly $\#(u') = \#(u)$, and

$$\#(Q) = \#(Q') - \#(v') + 3 + \#(v) + \#(w).$$

Thus it follows that

$$\#(Q) \geq 4n - 5 + \#(v) + \#(w) - \#(v').$$

Since rows $2, 3, \ldots, n$ of Q are orthogonal to the first row of Q, no row of $[u\,v\,w]$ contains exactly one nonzero entry. Thus, each row of $[v\,w]$ contains at least as many nonzero entries as the corresponding row of v'. Since the second and third columns of Q are orthogonal, some row of $[v\,w]$ has no zero entries. Thus, for some i, row i of $[v\,w]$ has more nonzero entries than row i of v'. It follows that $\#(Q) \geq 4n - 4$. $\square$

The techniques used in the proof of Theorem 1.3 can be used to classify, as was done in [**1**], the orthogonal matrices of order n which cannot be written as a direct sum and which have exactly $4n-4$ nonzero entries.

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