A Simple Proof of Fiedler's Conjecture Concerning Orthogonal Matrices

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A SIMPLE PROOF OF FIEDLER’S CONJECTURE
CONCERNING ORTHOGONAL MATRICES

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ABSTRACT. We give a simple proof that an \( n \times n \) orthogonal matrix with \( n \geq 2 \) which cannot be written as a direct sum has at least \( 4n - 4 \) nonzero entries.

1. The result. What is the least number of nonzero entries in a real orthogonal matrix of order \( n \)? Since the identity matrix \( I_n \) is orthogonal the answer is clearly \( n \). A more interesting question is: what is the least number of nonzero entries in a real orthogonal matrix which, no matter how its rows and columns are permuted, cannot be written as a direct sum of (orthogonal) matrices? Examples of orthogonal matrices of each order \( n \geq 2 \) which cannot be written as a direct sum and which have \( 4n - 4 \) nonzero entries are given in [1]. M. Fiedler conjectured that an orthogonal matrix of order \( n \geq 2 \) which cannot be written as a direct sum has at least \( 4n - 4 \) nonzero entries.

Using a combinatorial property of orthogonal matrices, Fiedler’s conjecture was proven in [1]. A \((0, 1)\)-matrix \( A \) of order \( n \) is combinatorially orthogonal provided no pair of rows of \( A \) has inner product 1 and no pair of columns of \( A \) has inner product 1. Clearly, if \( Q \) is an orthogonal matrix of order \( n \), then the \((0, 1)\)-matrix obtained from \( Q \) by replacing each of its nonzero entries by a 1 is combinatorially orthogonal. A quite lengthy and complex combinatorial argument is used in [1] to show that if \( A \) is a combinatorially orthogonal matrix of order \( n \geq 2 \) and \( A \) cannot be written as a direct sum, then \( A \) has at least \( 4n - 4 \) nonzero entries. Clearly this result implies Fiedler’s conjecture. In this note we give a simple matrix theoretic proof of Fiedler’s conjecture.

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Theorem 1.1. Let $Q$ be an orthogonal matrix of order $n$ of the form

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

where $U$ is a $k \times k + l$ matrix, and $W$ is an $m + l \times m$ matrix for some positive integers $k$ and $m$ and nonnegative integer $l$ with $k + l + m = n$.

Then the rank, $r(V)$, of $V$ equals $l$.

Proof. Since the rows of $U$ are linearly independent $r(U) = k$. Similarly, $r(W) = m$. Since the rank of a sum of matrices is less than or equal to the sum of the ranks of the matrices,

$$r(Q) \leq r(U) + r(W) + r(V).$$

Thus $l \leq r(V)$, since $r(Q) = k + l + m$. Because $Q$ is orthogonal, the rows of $V$ belong to the orthogonal complement in $R^{k+l}$ of the space spanned by the rows of $U$. Since $r(U) = k$, this implies that $r(V) \leq l$. Therefore $r(V) = l$. 

We note that, by taking $l = 0$ in Theorem 1.2, we have $V = O$; and hence an orthogonal matrix $Q$ of order $n$ can be written as a direct sum of matrices (after possibly permuting its rows and columns) if and only if $Q$ contains a zero submatrix whose dimensions sum to $n$.

Corollary 1.2. Let

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

be an $n \times n$ orthogonal matrix where $U$ is $k \times k + 1$ and $W$ is $l + 1 \times l$, $k + l = n - 1$ and $k, l \geq 1$. Then there exist nonzero vectors $x$ and $y$ such that $V = xy^T$, and both

$$(1) \quad U' = \begin{bmatrix} U \\ y^T \end{bmatrix} \quad \text{and} \quad W' = \begin{bmatrix} x & W \end{bmatrix}$$

are orthogonal matrices.

Proof. By Theorem 1.1, $V$ has rank one. Hence, there exist vectors $x$ and $y$ such that $V = xy^T$. Since $Q$ is orthogonal, the sum of the squares
of the entries in its first $k$ rows equals $k$, and the sum of the squares of the entries in its first $k+1$ columns equals $k+1$. Hence, the sum of the squares of the entries in $V$ equals $1$. It follows that $(x^T x)(y^T y) = 1$. Thus, by replacing $x$ by $(1/\sqrt{x^T x})x$ and $y$ by $\sqrt{x^T y}$, we may assume that $x^T x = 1$ and $y^T y = 1$. Since $Q$ is orthogonal, $y^T$ is orthogonal to each row of $U$, and $x$ is orthogonal to each column of $W$. The corollary now follows.

We now prove Fiedler’s conjecture. We let $\#(A)$ denote the number of nonzero entries in the matrix $A$.

**Theorem 1.3.** Let $Q$ be an orthogonal matrix of order $n \geq 2$ which cannot be written as a direct sum of matrices (no matter how its rows and columns are permuted). Then $Q$ has at least $4n-4$ nonzero entries.

**Proof.** The proof is by induction on $n$. First suppose that $Q$ contains a $k \times l$ zero submatrix for some positive integers $k$ and $l$ with $k+l = n-1$. Without loss of generality we may assume that

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

where $U$ is $k \times k+1$, and $W$ is $l+1 \times l$. By Corollary 1.2, there exist $x$ and $y$ such that $V = xy^T$ and the matrices $U'$ and $W'$ in (1) are orthogonal matrices.

Suppose that $U'$ can be written as a direct sum of two matrices. Then $U'$ contains an $r \times s$ zero submatrix which does not intersect the last row of $U'$ for some positive integers $r$ and $s$ with $r + s = k + 1$. It follows that $Q$ contains an $r \times s + (n - k - 1)$ zero submatrix. Hence, by the observation immediately after Theorem 1.1, $Q$ can be written as a direct sum of matrices. This contradicts our assumptions. Thus, $U'$ cannot be written as a direct sum of matrices. A similar argument shows that $W'$ cannot be written as a direct sum of matrices.

Clearly,

$$\#(Q) = \#(U') + \#(W') - 1 + (\#(y) - 1)(\#(x) - 1).$$
By induction $U'$ has at least $k$ nonzero entries, and $W'$ has at least $4l$ nonzero entries. Thus,

$$\#(Q) \geq 4k + 4l - 1 + (\#(y) - 1)(\#(x) - 1)$$

$$= (4n - 4) - 1 + (\#(y) - 1)(\#(x) - 1).$$

Since $Q$ has no $r \times s$ zero submatrix with $r + s \geq n$, $\#(y) \geq 2$ and $\#(x) \geq 2$. Therefore, $\#(Q) \geq 4n - 4$.

Now suppose that $Q$ does not contain a $k \times l$ zero submatrix for any positive integers $k$ and $l$ with $k + l = n - 1$. If $n = 2$, then each entry of $Q$ is nonzero and hence $\#(Q) \geq 4(n - 1)$. Assume that $n \geq 3$. Then each row and column of $Q$ has at least 3 nonzero entries. Thus, if $n = 3$, then $\#(Q) > 4(n - 1)$.

Assume that $n \geq 4$. If each row and column of $Q$ has at least 4 nonzero entries, then $\#(Q) \geq 4n > 4(n - 1)$. Suppose that some row or column of $Q$ has exactly 3 nonzero entries. We may assume without loss of generality that these occur in columns 1, 2 and 3. Let

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & 0 & \cdots & 0 \\ u & v & w & & & X \end{bmatrix},$$

where $X$ is $n - 1 \times n - 3$.

By Theorem 1.1, the rank of $[u \ v \ w]$ is 2. Without loss of generality we may assume that $u$ and $v$ are linearly independent.

Since each of $u$, $v$ and $w$ is orthogonal to each column of $X$,

$$Q' = [u' \ v' \ X]$$

is an orthogonal matrix of order $n - 1$, where $u'$ and $v'$ are the vectors obtained from $u$ and $v$ by applying the Gram-Schmidt process.

Suppose that $Q'$ can be written as a direct sum of two matrices. Then there exist positive integers $r$ and $s$ with $r + s \geq n - 2$ such that $X$ contains an $r \times s$ zero submatrix. It follows that $Q$ contains an $r + 1 \times s$ zero submatrix, which contradicts our assumptions. Hence $Q'$ cannot be written as a direct sum of matrices.

By the induction hypothesis, $\#(Q') \geq 4n - 8$. Clearly $\#(u') = \#(u)$, and

$$\#(Q) = \#(Q') - \#(v') + 3 + \#(v) + \#(w).$$
Thus it follows that
\[ \#(Q) \geq 4n - 5 + \#(v) + \#(w) - \#(v'). \]

Since rows 2, 3, ..., n of Q are orthogonal to the first row of Q, no row of \([uvw]\) contains exactly one nonzero entry. Thus, each row of \([vw]\) contains at least as many nonzero entries as the corresponding row of \(v'\). Since the second and third columns of Q are orthogonal, some row of \([vw]\) has no zero entries. Thus, for some \(i\), row \(i\) of \([vw]\) has more nonzero entries than row \(i\) of \(v'\). It follows that \(\#(Q) \geq 4n - 4\). \(\square\)

The techniques used in the proof of Theorem 1.3 can be used to classify, as was done in [1], the orthogonal matrices of order \(n\) which cannot be written as a direct sum and which have exactly \(4n - 4\) nonzero entries.

REFERENCES