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SPACES OF RANK-2 MATRICES OVER GF(2)∗

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Abstract. The possible dimensions of spaces of matrices over GF(2) whose nonzero elements all have rank 2 are investigated.

Key words. Matrix, rank, rank-k space.

AMS subject classifications. 15A03, 15A33, 11T35

Let $M_{m,n}(F)$ denote the vector space of all $m \times n$ matrices over the field $F$. In the case that $m = n$ we write $M_n(F)$. A subspace, $K$, is called a rank-$k$ space if each nonzero entry in $K$ has rank equal $k$. We assume throughout that $1 \leq k \leq m \leq n$.

The structure of rank-$k$ spaces has been studied lately by not only matrix theorists but group theorists and algebraic geometers; see [4], [5], [6]. In [3], [7], it was shown that the dimension of a rank-$k$ space is at most $n + m - 2k + 1$, and in [1] that the dimension of a rank-$k$ space is at most $\max(k + 1, n - k + 1)$, when the field is algebraically closed.

In [2] it was shown that, if $|F| \geq n + 1$ and $n \geq 2k - 1$, then the dimension of a rank-$k$ space is at most $n$. Thus, if $k = 2$ and $F$ is not the field of two elements, we know that the dimension of a rank-2 space is at most $n$. In [2] it was also shown that if $n = qk + r$, with $0 \leq r < k$ then if $F$ has an extension of degree $k$ and one of degree $k + r$, then there is a rank-$k$ space of dimension $n$. Thus, for $k = 2$, the only case left to investigate is when $|F| = 2$.

In this paper we shall show that if $m = n = 3$ there is a rank-2 space of dimension $n + 1$ over the field of two elements and that if $n \geq 4$ the dimension of a rank-2 space is at most $n$.

Further, is easily shown that for any field, the dimension of a rank-$m$ space is at most $n$. Thus, henceforth, we assume that $k = 2$, $3 \leq m \leq n$ and that $F = \mathbb{Z}_2$, the field of two elements.

Example 1. Consider the space of matrices

$$
\left\{ \begin{bmatrix}
  a & c & c \\
  d & a + b & c \\
  d & d & b \\
\end{bmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \right\}.
$$

It is easily checked that this is a 4 dimensional rank-2 subspace of $M_3(\mathbb{Z}_2)$.

It follows that for $n = 3$, $n$ is not an upper bound on the dimension of a rank-2 space.

We let $I_k$ denote the identity matrix of order $k \times k$, $O_{k,l}$ the zero matrix of order $k \times l$, and $O_k$ denotes $O_{k,k}$. When the order is obvious from the context, we omit
the subscripts. We shall use the notation ρ(A) to denote the rank of the matrix A. For increasing sequences, \( \alpha \subseteq \{1, 2, \ldots, m\} \), and \( \beta \subseteq \{1, 2, \ldots, n\} \), we will let \( A[\alpha][\beta] \) denote the submatrix of A on rows \( \alpha \) and columns \( \beta \). That is, \( A[i][j][k][l] = \begin{bmatrix} a_{ik} & a_{jl} \\ a_{jk} & a_{jl} \end{bmatrix} \).

**Theorem 2.** If \( n \geq 4 \), \( K \) is a rank 2 space and \( \mathcal{F} = \mathbb{Z}_2 \), then \( \dim K \leq n \).

**Proof.** Without loss of generality, we may assume that \( \begin{bmatrix} I_2 & 0 \\ 0 & O \end{bmatrix} \in K \). We also suppose that \( \dim K > n \).

Suppose that there is some nonzero \( C \in K \) such that \( C = \begin{bmatrix} O_2 & C_2 \\ O & C_4 \end{bmatrix} \). Then \( C_4 = O \) and \( \rho(C_2) = 2 \). Multiplying all elements of \( K \) by appropriate matrices that leave \( \begin{bmatrix} I_2 & 0 \\ O & O \end{bmatrix} \) fixed, we can assume that \( \begin{bmatrix} O & I_2 \\ 0 & O \end{bmatrix} \in K \).

Now, since \( \dim K > n \), there exists \( A \in K \) such that \( a_{1j} = 0 \) for all \( j \) and the rank of \( A \) is 2. Now, let \( B(x, y) = x \begin{bmatrix} I_2 & 0 \\ O & O \end{bmatrix} + y \begin{bmatrix} I_2 & 0 \\ O & O \end{bmatrix} + A \). Then, \( B(x, y)[1, 2, 3][1, 2, 3] = \begin{bmatrix} x \\ a_{21} \\ a_{31} \\ a_{22} + x \\ a_{32} \\ a_{33} \end{bmatrix} \) must have zero determinant for all \( x, y \in \mathcal{F} \). That is

\[
(1) \quad a_{33} x + a_{33} a_{22} x + a_{23} a_{32} x + a_{31} a_{22} y + a_{31} a_{32} y + a_{32} a_{21} y = 0.
\]

Recall that in \( \mathcal{F} = \mathbb{Z}_2 \), \( x^2 = x \) for all \( x \). It follows that for \( x = 0 \) and \( y = 1 \) we have

\[
(2) \quad a_{31} a_{22} + a_{32} a_{21} = 0
\]

and for \( x = 1 \) and \( y = 0 \) we have

\[
 a_{33} + a_{33} a_{22} + a_{32} a_{23} = 0.
\]

Now, we have

\[
y(a_{31} a_{22} + a_{32} a_{21}) + x(a_{33} + a_{33} a_{22} + a_{32} a_{23}) = a_{31} x y
\]

from (1) and each term of the left hand side is zero. Thus \( a_{31} = 0 \).

By considering \( B(x, y)[1, 2, r][1, 2, 3] \) as above, we get that \( a_{r1} = 0 \) for all \( r \).

Similarly, \( B(x, y)[1, 2, r][1, 2, 4] \) must have zero determinant for all \( r \geq 3 \). Thus, \( a_{r4} x + a_{22} a_{44} x + a_{24} a_{42} x + a_{r2} x y = 0 \). If \( x = 1 \) and \( y = 0 \) we get \( a_{r4} + a_{22} a_{44} + a_{24} a_{42} = 0 \) and hence \( a_{r2} x y = 0 \) for all \( x, y \). Thus, \( a_{r2} = 0 \) for all \( r \geq 3 \).

Now, \( B(x, y)[1, 2, r][1, 3, 4] \) must also have zero determinant. That is, \( a_{23} a_{43} x + a_{21} a_{31} y + a_{r2} x y = 0 \). As above we get that \( a_{r3} = 0 \) for all \( r \geq 3 \).

Since \( B(x, y)[1, 2, r][2, 3, s] \) must have zero determinant for \( r, s \geq 3 \), we get \( a_{22} a_{rs} y + a_{r2} x y = 0 \) for all \( x, y \). As above, we get that \( a_{rs} = 0 \) for \( r, s \geq 3 \).
The above contradicts that $A$ has rank 2 since $A$ has only one nonzero row. Thus, there is no matrix $C \in \mathcal{K}$ of the form $C = \begin{bmatrix} 0_2 & C_3 \\ O & C_4 \end{bmatrix}$.

Similarly, there is no matrix in $\mathcal{K}$ of the form $\begin{bmatrix} O_2 & O \\ C_3 & C_4 \end{bmatrix}$.

Since $n \geq 4$ and we have supposed that $\dim \mathcal{K} > n$, there is some rank-2 matrix $A \in \mathcal{K}$ of the form $A = \begin{bmatrix} O_2 & A_2 \\ A_3 & A_4 \end{bmatrix}$. From the above, we know that $A_2$ and $A_3$ are not zero. Since $\rho(A) \geq \rho(A_2) + \rho(A_3)$, we must have $\rho(A_2) = \rho(A_3) = 1$.

Let $R$, $S$ and $Q$ be invertible matrices such that

$$RA_2Q = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\theta} \end{bmatrix}$$

and

$$SA_3R^{-1} = \begin{bmatrix} \alpha & \tilde{\beta} \\ 0 & \tilde{\theta} \end{bmatrix}.$$ 

Let $B = (R \oplus S)A(R^{-1} \oplus Q)$. We have two cases: $\alpha = 0$ (and hence $\beta = 1$) or $\alpha = 1$.

Case 1. $\alpha = 1$. In this case let $E = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \oplus I_{n-2}$ and $F = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \oplus I_m$.

Then $F \left( x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + B \right) E = \begin{bmatrix} x & 0 & 1 & \tilde{\theta} \\ 0 & x & 0 & \tilde{\theta} \\ 1 & 0 & \tilde{\theta} & A_4 \end{bmatrix} = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + FBE$.

Let $C = FBE$. Now, since $\det \left( x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + C \right) [1, 2, r|1, 2, s]$ must be zero, if $r \geq 3$ and $s \geq 3$, we have $c_{rs} = 0$ for all such $(r, s) \neq (3, 3)$ since the coefficient of $x$ must be zero. For $(r, s) = (3, 3)$ we get $x^2 c_{33} + x = 0$, so $c_{33} = 1$. That is

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$ 

With out loss of generality we may assume that $C \in \mathcal{K}$. Since $\dim \mathcal{K} > n \geq 4$, there is some $B \in \mathcal{K}$ which is rank 2 and such that

$$B = \begin{bmatrix} 0 & b_{12} & 0 & b_{14} & \cdots \\ b_{21} & b_{22} & b_{23} & b_{24} & \cdots \\ 0 & b_{32} & 0 & b_{34} & \cdots \\ b_{41} & b_{42} & b_{43} & b_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$
Let $G(x, y) = x \begin{bmatrix} f_2 & 0 & 0 \\ 0 & O & 0 \end{bmatrix} + yC + B$. Since $G(x, y) \in \mathcal{K}$ for all $x$ and $y$, we must have that $\det G(x, y)[rstuvw] = 0$ for any increasing sequences $(r, s, t)$ and $(u, v, w)$, and hence, the coefficient of each term in the polynomial must be $0$. Thus, we obtain

a) $b_{\alpha\beta} = 0$ for all $\alpha, \beta \geq 4$,  

b) $b_{1\beta} = 0$ for all $\beta \geq 4$,  

c) $b_{2\beta} = 0$ for all $\beta \geq 4$,  

d) $b_{3\beta} = 0$ for all $\beta \geq 4$,  

e) $b_{\alpha 1} = 0$ for all $\alpha \geq 4$,  

f) $b_{\alpha 2} = 0$ for all $\alpha \geq 4$, and  

g) $b_{\alpha 3} = 0$ for all $\alpha \geq 4$;

when we take $[rstuvw] = $ 

\begin{itemize}
  \item a) $[23\alpha][23\beta]$, 
  \item b) $[123][12\beta]$, 
  \item c) $[123][13\beta]$, 
  \item d) $[123][23\beta]$, 
  \item e) $[12\alpha][123]$, 
  \item f) $[13\alpha][123]$, and 
  \item g) $[23\alpha][123]$, respectively.
\end{itemize}

Thus

$$B = \begin{bmatrix} 0 & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and hence, $\det G(x, y)[123][123] = x(b_{23}b_{32}) + y(b_{12}b_{23} + b_{22} + b_{21}b_{32} + b_{12}b_{21}) + xy(b_{22})$. Thus $b_{22} = 0$ and $b_{23}b_{32} = 0$. By the symmetry of $x \begin{bmatrix} f_2 & 0 & 0 \\ 0 & O & 0 \\ 0 & 0 & 0 \end{bmatrix} + yC$ we may assume that $b_{32} = 0$. Since $B$ must have rank 2, we have that $b_{21} = 1$ and from the coefficient of $y$, and that $B$ must have rank 2, we get that $b_{32} = b_{12} = 1$. So,

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Now, since $\dim \mathcal{K} \geq 4$ we must have $F \in \mathcal{K}$ of rank 2 such that

$$F = \begin{bmatrix} 0 & 0 & f_{13} & f_{14} & \cdots \\ 0 & f_{22} & f_{23} & f_{24} & \cdots \\ 0 & f_{32} & f_{33} & f_{34} & \cdots \\ f_{41} & f_{42} & f_{43} & f_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ 

For $H(x, y, z) = x \begin{bmatrix} f_2 & 0 & 0 \\ 0 & O & 0 \end{bmatrix} + yC + zB + F$, by considering the minors on $[rstuvw]$ as above, we obtain
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- For all \( \alpha, \beta \geq 4 \), \( f_{\alpha \beta} = 0 \).
- For all \( \beta \geq 4 \), \( f_{1 \beta} = f_{2 \beta} = 0 \).
- For all \( \beta \geq 4 \), \( f_{3 \beta} = 0 \).
- For all \( \alpha \geq 4 \), \( f_{\alpha 1} = f_{\alpha 2} = 0 \).
- For all \( \alpha \geq 4 \), \( f_{\alpha 3} = 0 \).

When we take \([rstuvw] = \)

- \([23\alpha 23\beta] \),
- \([12312\beta] \),
- \([12313\beta] \),
- \([120123] \), and
- \([230123] \), respectively.

Now, \( \det H(x, y, z)[123123] = x(f_{13} + f_{22}f_{33} + f_{32}f_{23}) + y(f_{22} + f_{22}f_{13}) + z(f_{13} + f_{13}f_{32} + f_{33}) + xy(f_{22} + f_{13}) + xz(f_{23}) + yz(f_{32} + f_{23}) \). Thus \( f_{23} = f_{52} = 0 \) and \( f_{22} = f_{13} = f_{33} \). Since the rank of \( F \) must be 2, we have that

\[
F = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

But then,

\[
\begin{bmatrix}
I_2 & 0 \\
0 & 0
\end{bmatrix} + C + F = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

a rank 1 matrix, a contradiction.

Case 2. \( \alpha = 0 \). Here we must have \( \beta = 1 \) so that

\[
B = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

(Note that if \( b_{ij} \neq 0 \) for any \( i, j \) with \( i, j \geq 3 \) then \( x \begin{bmatrix}
I_2 & 0 \\
0 & 0
\end{bmatrix} + B \) has rank 3 or more for some \( x \).

Now, since \( \dim \mathcal{K} \geq 4 \), we have some matrix \( E \in \mathcal{K} \) such that

\[
E = \begin{bmatrix}
0 & e_{12} & 0 & e_{14} & \cdots \\
e_{21} & 0 & e_{23} & e_{24} & \cdots \\
e_{31} & 0 & e_{33} & e_{34} & \cdots \\
e_{41} & e_{42} & e_{43} & e_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

But then, \( \begin{bmatrix}
I_2 & 0 \\
0 & 0
\end{bmatrix} + C + F = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \), a rank 1 matrix, a contradiction.

Case 2. \( \alpha = 0 \). Here we must have \( \beta = 1 \) so that

\[
B = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

(Note that if \( b_{ij} \neq 0 \) for any \( i, j \) with \( i, j \geq 3 \) then \( x \begin{bmatrix}
I_2 & 0 \\
0 & 0
\end{bmatrix} + B \) has rank 3 or more for some \( x \).

Now, since \( \dim \mathcal{K} \geq 4 \), we have some matrix \( E \in \mathcal{K} \) such that

\[
E = \begin{bmatrix}
0 & e_{12} & 0 & e_{14} & \cdots \\
e_{21} & 0 & e_{23} & e_{24} & \cdots \\
e_{31} & 0 & e_{33} & e_{34} & \cdots \\
e_{41} & e_{42} & e_{43} & e_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]
Let \( C(x, y) = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yB + E \). Then, since \( \det C(x, y)[r, s, t|u, v, w] = 0 \) for all strictly increasing \((r, s, t)\) and \((u, v, w)\), each term in the polynomial must be zero, i.e., each coefficient in the polynomial expansion must be zero. Thus, we get

- a) \( e_{33} = e_{21} = 0 \) and \( e_{31} = e_{23} \),
- b) \( e_{a3} = 0 \) for \( \alpha \geq 4 \),
- c) \( e_{a1} = 0 \) for \( \alpha \geq 4 \),
- d) \( e_{2\beta} = 0 \) for \( \beta \geq 4 \),
- e) \( e_{3\beta} = 0 \) for \( \beta \geq 4 \), and
- f) \( e_{a\beta} = 0 \) for \( \alpha, \beta \geq 4 \);

when considering \( \det C(x, y)[r, s, t|u, v, w] = 0 \) for \([r, s, t]|u, v, w] =

- a) \([1, 2, 3][1, 2, 3] \),
- b) \([1, 3, \alpha][1, 2, 3] \),
- c) \([1, 2, \alpha][1, 2, 3] \),
- d) \([1, 2, 3][1, 2, \beta] \),
- e) \([1, 2, 3][2, 3, \beta] \), and,
- f) \([1, 3, \alpha][1, 2, \beta] \), respectively.

Thus,

\[
E = \begin{bmatrix}
0 & e_{12} & 0 & e_{14} & \cdots \\
0 & 0 & e_{23} & 0 & \cdots \\
e_{23} & 0 & 0 & 0 & \cdots \\
0 & e_{42} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Subcase 1. \( e_{33} = 1 \). In this case, since the rank of \( E \) must be 2, and \( e_{31} = e_{23} \), we have that \( e_{4i} = 0 \) and \( e_{j2} = 0 \) for all \( i, j \geq 4 \), and that \( e_{12} = 0 \) by considering that \( \det E[123][134] = 0 \) and \( \det E[13][123] = 0 \), and \( \det E[123][123] = 0 \) respectively. Thus

\[
E = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
O & O & O
\end{bmatrix}
\]

Now, since \( \dim \mathcal{K} > 4 \), there is some nonzero \( F \in \mathcal{K} \) such that

\[
F = \begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & \cdots \\
f_{21} & 0 & 0 & f_{24} & \cdots \\
f_{31} & 0 & 0 & f_{34} & \cdots \\
f_{41} & f_{42} & f_{43} & f_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Let \( G(x, y, z) = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yB + zE + F \). As above we get

- a) \( f_{a\beta} = 0 \) for all \( \alpha, \beta \geq 4 \),
b) $f_{13} = f_{31} = f_{11} = f_{12} = f_{21} = 0$,
c) $f_{3\beta} = f_{2\beta} = f_{1\beta} = 0$ for $\beta \geq 4$, and
d) $f_{\alpha3} = f_{\alpha2} = f_{\alpha1} = 0$ for $\alpha \geq 4$;
when considering that $\det G(x,y,z)[r, s, t][u, v, w]$ must be zero for $[r, s, t][u, v, w] =$
a) $[23\alpha][23\beta]$,
b) $[123][123]$, 
c) $[1\beta][23\alpha]$, and
d) $[1\alpha][123]$, respectively.

But then, $F = O$, a contradiction.

Subcase 2. $e_{23} = 0$. In this case, since the rank of $E$ is 2, we have that $e_{1i} = 1$ for some $i \geq 4$ and $e_{j2} = 1$ for some $j \geq 4$. Here, there are invertible matrices $U$ and $V$ such that $UEV = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} O$, and $U \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} V = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$ and

$UBV = B$. Thus we may assume that $E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} O$.

Now, let $G(x,y,z) = x \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} + yB + zE$. Then,

$G(x,y,z) = \begin{bmatrix} x & 0 & y & z & 0 & \cdots \\ 0 & x & 0 & 0 & 0 & \cdots \\ 0 & y & 0 & 0 & 0 & \cdots \\ 0 & z & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

Since dim $K > n$, there exists $H \in K$, $H \neq 0$ such that $h_{1j} = 0$ for all $j$. Let $K(x, y, z) = G(x, y, z) + H$. We get
a) $h_{\alpha\beta} = 0$ for $\alpha \geq 3$ and $\beta \geq 4$,
b) $h_{2\beta} = 0$ for $\beta \geq 4$,
c) $h_{\alpha1} = 0$, for $\alpha \geq 4$,
d) $h_{\alpha3} = 0$, for $\alpha \geq 4$,
e) $h_{33} = h_{23} = 0$, and
f) $h_{31} = h_{31} = 0$;
when we consider that $\det H(x, y, z)[\gamma][\eta] = 0$ for $[\gamma][\eta] =$
a) $[12\alpha][23\beta]$,
b) $[1\beta][12\alpha]$, 
c) $[1\beta][1\beta]$, and
d) $[1\alpha][12\beta]$, 
e) $[1\beta][1\alpha]$, and
f) \([123, 234]\), respectively.

That is \(H = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & h_{22} & 0 & \cdots \\ 0 & h_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \). But the rank of \(H\) is 1 since \(H \neq 0\), a contradiction since \(H \in \mathcal{K}\) and \(\mathcal{K}\) is a rank 2 space.

We have thus obtained a contradiction in each case, and hence our supposition that \(\dim \mathcal{K} > n\) is false, and the theorem is proved.

**Corollary 3.** Over any field \(F\), for \(m, n \geq 3\), the dimension of any rank-2 subspace of \(\mathcal{M}_{m,n}(F)\) is at most \(n\), except when \(m = n = 3\) and \(F = \mathbb{Z}_2\), in which case, the dimension is at most 4.

**Proof.** By the above theorem and comments, unless \(m = n = 3\) and \(F = \mathbb{Z}_2\), the dimension of any rank-2 space is at most \(n\). If \(\mathcal{K}\) is a rank-2 subspace of \(\mathcal{M}_{3}(\mathbb{Z}_2)\), define \(\mathcal{K}^+ = \{ [A \ 0] | A \in \mathcal{K} \}\). Then \(\mathcal{K}^+\) is a rank-2 subspace of \(\mathcal{M}_{3,4}(\mathbb{Z}_2)\) and hence the dimension is at most 4. Clearly \(\mathcal{K}\) and \(\mathcal{K}^+\) are isomorphic so that the dimension of \(\mathcal{K}\) is at most 4 also.

Another example of a 4 dimensional rank-2 subspace of \(\mathcal{M}_{3}(\mathbb{Z}_2)\) which is not equivalent to the one in Example 1 is given below.

**Example 4.** Consider the space of matrices

\[
\left\{ \begin{bmatrix} a & 0 & c \\ d & a+b & 0 \\ 0 & c+d & b \end{bmatrix} | a, b, c, d \in \mathbb{Z}_2 \right\}.
\]

It is easily checked that this is a 4 dimensional rank-2 subspace of \(\mathcal{M}_{3}(\mathbb{Z}_2)\).

**Conjecture 5.** Over any field, \(F\), the dimension of a rank-\(k\) subspace of \(\mathcal{M}_{m,n}(F)\) is at most \(n\) unless \(m = n = 3\), \(k = 2\) and \(F = \mathbb{Z}_2\).

**References**