On convergence of infinite matrix products

Olga Holtz
holtz@cs.wisc.edu

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1054
ON CONVERGENCE OF INFINITE MATRIX PRODUCTS*

OLGA HOLTZ†

Abstract. A necessary and sufficient condition for the convergence of an infinite right product of matrices of the form

$$A := \begin{bmatrix} I & B \\ 0 & C \end{bmatrix},$$

with (uniformly) contracting submatrices $C$, is proven.

Key words. Infinite matrix products, RCP sets.

AMS subject classifications. 15A60, 15A99

1. Introduction. Consider the set of all matrices in $\mathbb{C}^{d \times d}$ of the form

$$A := \begin{bmatrix} I_s & B \\ 0 & C \end{bmatrix},$$

where $I_s$ denotes the identity matrix of order $s < d$.

Matrices (1) are known to form an LCP set whenever the submatrices $B$ are uniformly bounded and the submatrices $C$ are uniformly contracting, that is, satisfy the condition $\|C\| \leq r$ for some fixed matrix (i.e., submultiplicative) norm $\| \cdot \|$ on $\mathbb{C}^{(d-s) \times (d-s)}$ and some constant $r < 1$; see, e.g., [1]. To recall, a set $\Sigma$ has the LCP (RCP) property if all left (right) infinite products formed from matrices in $\Sigma$ are convergent.

Matrices of the form (1) with uniformly bounded submatrices $B$ and uniformly contracting submatrices $C$ do not necessarily form an RCP set. (They do form such a set if and only if they satisfy a very stringent condition given in Corollary 2.3 below.) However, there exists a simple criterion that can be used to check whether a particular right infinite product formed from such matrices converges.

2. A convergence test.

Theorem 2.1. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of matrices of the form (1) and let

$$\|C_n\| \leq r < 1 \quad \text{for all} \quad n \in \mathbb{N}$$

for some matrix norm $\| \cdot \|$. The sequence $(P_n := A_1 A_2 \cdots A_n)$ converges if and only if so does the sequence $(B_n(I - C_n^{-1}))$. In this event,

$$\lim_{n \to \infty} P_n = \begin{bmatrix} I & \lim_{n \to \infty} B_n(I - C_n)^{-1} \\ 0 & 0 \end{bmatrix}.$$
Proof. To prove the necessity, partition $P_n$ conformably with $A_n$. Then

$$P_n = \begin{bmatrix} I & X_n \\ 0 & C_1 C_2 \cdots C_n \end{bmatrix}, \quad \text{where} \quad X_n := \sum_{i=0}^{n} B_{n-i}(C_{n+1-i}C_{n+2-i} \cdots C_n).$$

If $(P_n)$ converges, then $\lim_{n \to \infty} (X_n - X_{n-1}) = 0$. Also, $\| (I - C_n)^{-1} \| \leq 1/(1 - r)$ for all $n \in \mathbb{N}$. But $X_n = B_n + X_{n-1} C_n$, and thus

$$B_n (I - C_n)^{-1} X_{n-1} = (X_n - X_{n-1})(I - C_n)^{-1} \to 0 \quad \text{as} \quad n \to \infty.$$  

Hence $\lim_{n \to \infty} B_n (I - C_n)^{-1} = \lim_{n \to \infty} X_n$.

To prove the sufficiency, without loss of generality one can assume that $s = d - s$. Indeed, simply replace each $A_n$ by

$$\tilde{A}_n := \begin{bmatrix} I_{\max(s,d-s)} & \tilde{B}_n \\ 0 & \tilde{C}_n \end{bmatrix},$$

where

$$\tilde{B}_n := \begin{cases} B_n & \text{if } s \geq d - s \\ 0_{(d-2s) \times (d-s)} & \text{if } s < d - s \end{cases},$$

and

$$\tilde{C}_n := \begin{cases} C_n & \text{if } s \geq d - s \\ 0_{(2s-d) \times (d-s)} & \text{if } s < d - s \end{cases}.$$  

Then the matrices $\tilde{A}_n$ satisfy all the assumptions of the theorem and the sequence $(\tilde{B}_n (I - C_n)^{-1})$ (the product $P_n$) converges if and only if so does the sequence $(\tilde{B}_n (I - C_n)^{-1})$ (the product $\tilde{P}_n$).

Thus, assume that $s = d - s$. Note that if the sequence $(B_n (I - C_n)^{-1})$ converges, then the sequence $(B_n)$ is bounded, since $\| I - C_n \| \leq 1 + r$ for all $n$. Now, let

$$D_n := X_n - B_n (I - C_n)^{-1}, \quad Y_n := B_{n+1} (I - C_{n+1})^{-1} - B_n (I - C_n)^{-1}$$

for all $n \in \mathbb{N}$. Then

(2) \hspace{1cm} D_{n+1} = (D_n - Y_n) C_{n+1},

hence

$$\| D_{n+1} \| \leq (\| D_n \| + \| Y_n \|) \| C_{n+1} \| \leq (\| D_n \| + \| Y_n \|) r.$$
Repeated use of this inequality gives

$$||D_n|| \leq \sum_{i=1}^{n-1} ||Y_{n-i}|| r^i.$$  

This implies, in particular, that

$$S := \lim_{n \to \infty} \sup_{n \to \infty} ||D_n|| < \infty.$$  

Since $\lim_{n \to \infty} Y_n = 0$, the identity (2) and the upper bound on $||C_n||$ imply that $S \leq rS$, therefore $S = 0$, that is, $\lim_{n \to \infty} D_n = 0$.  

The obtained criterion of convergence can be used to make two more observations in the same spirit.

**Corollary 2.2.** Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of matrices of the form (1) such that the sequence $(C_n)$ converges to a matrix $C$ with spectral radius smaller than 1. Then the sequence $(P_n := A_1 A_2 \cdots A_n)$ converges if and only if so does the sequence $(B_n)$. In this event,

$$\lim_{n \to \infty} P_n = \begin{bmatrix} I & \lim_{n \to \infty} B_n (I - C)^{-1} \\ 0 & 0 \end{bmatrix}.$$  

**Proof.** If $\rho(C) < 1$, then there exists a matrix norm $||\cdot||$ on $\mathbb{C}^{(d-\varepsilon) \times (d-\varepsilon)}$ such that $||C|| < 1$; see, e.g., [2, p. 297, Lemma 5.6.10]. Thus, $||C_n|| \leq r$ for all $n \geq N$ for some $r < 1$ and some $N \in \mathbb{N}$, and the assumption of Theorem 2.1 is then satisfied. The product $P_n$ converges whenever the product $A_N A_{N+1} \cdots$ converges, therefore $(P_n)$ has a limit whenever $(B_n)$ has one. On the other hand, the sequence $((I - C_n)^{-1})_{n=N}^{\infty}$ is bounded, so the necessity argument from the proof of Theorem 2.1 shows that the convergence of $(B_n)$ is also necessary.  

**Corollary 2.3.** A set $\Sigma$ consisting of matrices of the form (1) with uniformly contracting submatrices $C$ is an RCP set if and only if

$$(3) \quad B_1 (I - C_1)^{-1} = B_2 (I - C_2)^{-1} \quad \text{for all} \quad A_1, A_2 \in \Sigma,$$

where

$$A_i = \begin{bmatrix} I & B_i \\ 0 & C_i \end{bmatrix}, \quad i = 1, 2.$$  

**Proof.** Given $A_1, A_2 \in \Sigma$, apply Theorem 2.1 to the product $A_1 A_2 A_1 A_2 \cdots$ to see that the condition (3) is necessary and sufficient for the convergence of such a product. But if it is satisfied for all pairs of matrices from $\Sigma$, then it is sufficient for the convergence of any right product of matrices from $\Sigma$.  

On convergence of infinite matrix products

Acknowledgements. I am grateful to Professor Hans Schneider for his critical reading of the manuscript and to the referee for valuable suggestions.

REFERENCES