Basic comparison theorems for weak and weaker matrix splittings

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BASIC COMPARISON THEOREMS FOR WEAK AND WEAKER MATRIX SPLITTINGS

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Abstract. The main goal of this paper is to present comparison theorems proven under natural conditions such as $N_2 \geq N_1$ and $M_1^{-1} \geq M_2^{-1}$ for weak and weaker splittings of $A = M_1 - N_1 = M_2 - N_2$ in the cases when $A^{-1} \geq 0$ and $A^{-1} \leq 0$.

Key words. Systems of linear equation, convergence conditions, comparison theorems, weak splittings, weaker splittings.

AMS subject classifications. 65C20, 65F10, 65F15

1. Introduction. A large class of iterative methods for solving systems of linear equations of the form

$$Ax = b,$$

where $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix and $x, b \in \mathbb{R}^n$, can be formulated by means of the splitting

$$A = M - N \text{ with } M \text{ nonsingular},$$

and the approximate solution $x^{(t+1)}$ is generated as follows

$$Mx^{(t+1)} = Nx^{(t)} + b, \quad t \geq 0,$$

or equivalently,

$$x^{(t+1)} = M^{-1}Nx^{(t)} + M^{-1}b, \quad t \geq 0,$$

where the starting vector $x^{(0)}$ is given.

The above iterative method is convergent to the unique solution $x = A^{-1}b$ for each $x^{(0)}$ if and only if $\rho(M^{-1}N) < 1$, which means that the splitting of $A = M - N$ is convergent. The convergence analysis of the above method is based on the spectral radius of the iteration matrix $\rho(M^{-1}N)$. As is well known, the smaller is $\rho(M^{-1}N)$, the faster is the convergence; see, e.g., [1].

The definitions of splittings, with progressively weaker conditions and consistent from the viewpoint of names, are collected in the following definition.

DEFINITION 1.1. Let $M, N \in \mathbb{R}^{n \times n}$. Then the decomposition $A = M - N$ is called

(a) a regular splitting of $A$ if $M^{-1} \geq 0$ and $N \geq 0$,

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(b) a nonnegative splitting of \( A \) if \( M^{-1} \geq 0, M^{-1}N \geq 0 \) and \( NM^{-1} \geq 0 \),
(c) a weak nonnegative splitting of \( A \) if \( M^{-1} \geq 0 \) and either \( M^{-1}N \geq 0 \) (the first type) or \( NM^{-1} \geq 0 \) (the second type),
(d) a weak splitting of \( A \) if \( M \) is nonsingular, \( M^{-1}N \geq 0 \) and \( NM^{-1} \geq 0 \),
(e) a weaker splitting of \( A \) if \( M \) is nonsingular and either \( M^{-1}N \geq 0 \) (the first type) or \( NM^{-1} \geq 0 \) (the second type),
(f) a convergent splitting of \( A \) if \( \varrho(M^{-1}N) = \varrho(NM^{-1}) < 1 \).

The splittings defined in the successive items extend progressively a class of splittings of \( A = M - N \) for which the matrices \( N \) and \( M^{-1} \) may lose the property of nonnegativity. Distinguishing both types of weak nonnegative and weaker splittings leads to further extensions allowing us to analyze cases when \( M^{-1}N \) may have negative entries if only \( NM^{-1} \) is a nonnegative matrix.

Different splittings were extensively analyzed by many authors, see, e.g., [2] and the references therein.

Conditions ensuring that a splitting of a nonsingular matrix \( A = M - N \) is convergent are unknown in a general case. As was pointed out in [2], the splittings defined in first three items of Definition 1.1 are convergent if and only if \( A^{-1} \geq 0 \), which means that both conditions \( A^{-1} \geq 0 \) and \( \varrho(M^{-1}N) = \varrho(NM^{-1}) < 1 \) are equivalent. We write this formally as the following lemma.

**Lemma 1.2.** Each weak nonnegative (as well as nonnegative and regular) splitting of \( A = M - N \) is convergent if and only if \( A^{-1} \geq 0 \). In other words, if \( A \) is not a monotone matrix, it is impossible to construct a convergent weak nonnegative splitting.

In the case of weak and weaker splittings, the assumption \( A^{-1} \geq 0 \) is not a sufficient condition in order to ensure the convergence of a given splitting of \( A \); it is also possible to construct a convergent weak or weaker splitting when \( A^{-1} \geq 0 \). Moreover, as can be shown by examples the conditions \( A^{-1}N \geq 0 \) or \( NA^{-1} \geq 0 \) may not ensure that a given splitting of \( A \) will be a weak or weaker splitting.

The properties of weaker splittings are summarized in the following theorem.

**Theorem 1.3.** Let \( A = M - N \) be a weaker splitting of \( A \). If \( A^{-1} \geq 0 \), then
1. If \( M^{-1}N \geq 0 \), then \( A^{-1}N \geq M^{-1}N \) and if \( NM^{-1} \geq 0 \), then \( NA^{-1} \geq NM^{-1} \).
2. \[ \varrho(M^{-1}N) = \frac{\varrho(A^{-1}N)}{1 + \varrho(A^{-1}N)} = \frac{\varrho(NA^{-1})}{1 + \varrho(NA^{-1})} . \]

Thus, we can conclude that for a convergent weaker splitting of a monotone matrix \( A \) there are three conditions \( M^{-1}N \geq 0 \) (or \( NM^{-1} \geq 0 \), \( A^{-1}N \geq 0 \) (or \( NA^{-1} \geq 0 \)) and \( \varrho(M^{-1}N) = \varrho(NM^{-1}) < 1 \), and any two conditions imply the third.

The main goal of this paper is to present comparison theorems proven under natural conditions such as \( N_2 \geq N_1 \) and \( M_1^{-1} \geq M_2^{-1} \) for weak and weaker splittings of \( A = M_1 - N_1 = M_2 - N_2 \) in the cases when \( A^{-1} \geq 0 \) and \( A^{-1} \leq 0 \).

### 2. Comparison theorems

When both convergent weaker splittings of a monotone matrix

\begin{equation}
A = M_1 - N_1 = M_2 - N_2
\end{equation}
are of the same type, the inequality
\[(2.2) \quad N_2 \geq N_1\]
implies either
\[A^{-1}N_2 \geq A^{-1}N_1 \geq 0 \quad \text{or} \quad N_2A^{-1} \geq N_1A^{-1} \geq 0.\]

Hence, by the Perron-Frobenius theory of nonnegative matrices (see, e.g., [1]), we have \(\varrho(A^{-1}N_1) \leq \varrho(A^{-1}N_2)\) or \(\varrho(N_1A^{-1}) \leq \varrho(N_2A^{-1})\) and by Theorem 1.3 we can conclude the following result.

**Theorem 2.1.** [2] Let \(A = M_1 - N_1 = M_2 - N_2\) be two convergent weaker splittings of \(A\) of the same type, that is, either \(M_1^{-1}N_1 \geq 0\) and \(M_2^{-1}N_2 \geq 0\) or \(N_1M_1^{-1} \geq 0\) and \(N_2M_2^{-1} \geq 0\), where \(A^{-1} \geq 0\). If \(N_2 \geq N_1\), then
\[\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).\]

This theorem, proven originally by Varga [1] for regular splittings, carries over to the case when both weaker splittings are of the same type. As is pointed out in [3] when both splittings in (2.1) are of different types, the condition (2.2) may not hold.

In the case when \(A^{-1} \leq 0\), then the inequality (2.2) implies either
\[0 \leq A^{-1}N_2 \leq A^{-1}N_1 \quad \text{or} \quad 0 \leq N_2A^{-1} \leq N_1A^{-1}.\]

Hence, one can deduce the following theorem.

**Theorem 2.2.** Let \(A = M_1 - N_1 = M_2 - N_2\) be two convergent weaker splittings of \(A\) of the same type, that is, either \(M_1^{-1}N_1 \geq 0\) and \(M_2^{-1}N_2 \geq 0\) or \(N_1M_1^{-1} \geq 0\) and \(N_2M_2^{-1} \geq 0\), where \(A^{-1} \leq 0\). If \(N_2 \geq N_1\), then
\[\varrho(M_1^{-1}N_1) \geq \varrho(M_2^{-1}N_2).\]

Similarly as in the case of \(A^{-1} \geq 0\), it can be shown that when both splittings in (2.1) are of different types, the condition (2.2) may not arise.

In the case of the weaker condition
\[(2.3) \quad M_1^{-1} \geq M_2^{-1}\]
the contrary behavior is observed. As is demonstrated on examples in [2], when both weak nonnegative splittings of a monotone matrix \(A\) are the same type, with \(M_1^{-1} \geq M_2^{-1}\) (or even \(M_1^{-1} > M_2^{-1}\)) it may occur that \(\varrho(M_1^{-1}N_1) > \varrho(M_2^{-1}N_2)\).

Let us assume that both convergent weaker splittings in (2.1) are of different types such that \(M_1^{-1}N_1 \geq 0\) and \(N_2M_2^{-1} \geq 0\), and let \(v_1 \geq 0\) and \(y_2 \geq 0\) be the eigenvectors such that
\[(2.4) \quad v_1^T M_1^{-1} N_1 = \lambda_1 v_1^T\]
and
\[(2.5) \quad N_2 M_2^{-1} y_2 = \lambda_2 y_2,\]
where \( \lambda_1 = \varrho(M_1^{-1}N_1) \) and \( \lambda_2 = \varrho(M_2^{-1}N_2) = \varrho(N_2M_2^{-1}) \). Multiplying (2.4) on the right by \( A^{-1}y_2 \) and (2.5) on the left by \( v^T_1A^{-1} \), one obtains
\[
v^T_1M_1^{-1}N_1A^{-1}y_2 = \lambda_1v^T_1A^{-1}y_2
\]
and
\[
v^T_1A^{-1}N_2M_2^{-1}y_2 = \lambda_2v^T_1A^{-1}y_2,
\]
and after subtraction we obtain
\[
v^T_1(A^{-1}N_2M_2^{-1} - M_1^{-1}N_1A^{-1})y_2 = (\lambda_2 - \lambda_1)v^T_1A^{-1}y_2.
\]
From (1.1) we have
\[
M^{-1} = (A + N)^{-1} = A^{-1}(I + NA^{-1})^{-1} = (I + A^{-1}N)^{-1}A^{-1},
\]
or
\[
A^{-1} = M^{-1} + M^{-1}NA^{-1} = M^{-1} + A^{-1}NM^{-1}
\]
which implies that
\[
A^{-1}N_2M_2^{-1} - M_1^{-1}N_1A^{-1} = M_1^{-1} - M_2^{-1}.
\]
Hence, one obtains
\[(2.6) \quad v^T_1(M_1^{-1} - M_2^{-1})y_2 = (\lambda_2 - \lambda_1)v^T_1A^{-1}y_2.\]

Let us consider the following cases.

**Case I.** When \( A^{-1} > 0 \), then \( v^T_1A^{-1}y_2 > 0 \).
1. If \( M_1^{-1} > M_2^{-1} \), then \( M_1^{-1} - M_2^{-1} > 0 \) and \( v^T_1(M_1^{-1} - M_2^{-1})y_2 > 0 \), hence \( \lambda_2 - \lambda_1 > 0 \) and \( \lambda_2 > \lambda_1 \).
2. If \( M_1^{-1} \geq M_2^{-1} \), then \( M_1^{-1} - M_2^{-1} \geq 0 \) and
   a) if \( v^T_1(M_1^{-1} - M_2^{-1})y_2 > 0 \), hence \( \lambda_2 - \lambda_1 > 0 \) and \( \lambda_2 > \lambda_1 \).
   b) if \( v^T_1(M_1^{-1} - M_2^{-1})y_2 = 0 \), hence \( \lambda_2 = \lambda_1 \).

**Case II.** When \( A^{-1} \geq 0 \), then \( v^T_1A^{-1}y_2 \geq 0 \).
1. If \( v^T_1(M_1^{-1} - M_2^{-1})y_2 > 0 \), then \( v^T_1A^{-1}y_2 > 0 \), hence \( \lambda_2 - \lambda_1 > 0 \) and \( \lambda_2 > \lambda_1 \).
2. If \( v^T_1(M_1^{-1} - M_2^{-1})y_2 = 0 \), then
   a) for \( v^T_1A^{-1}y_2 > 0 \), \( \lambda_2 - \lambda_1 = 0 \) and \( \lambda_2 = \lambda_1 \).
   b) for \( v^T_1A^{-1}y_2 = 0 \), the relation (2.6) is satisfied for arbitrary values of \( \lambda_1 \) and \( \lambda_2 \).

The following examples of regular splittings illustrate the case II.2.b).

\[
A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = M_1 - N_1 = M_2 - N_2, \quad \text{where}
\]
\[
M_1 = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_1^{-1}N_1 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad v^T_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]
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\[ M_2 = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2^{-1}N_2 = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

Evidently, \( v_1^T(M_1^{-1} - M_2^{-1})y_2 = [1 \ 0] \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \)

and \( v_2^TA^{-1}y_2 = [1 \ 0] \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0. \)

However, a simple modification allows us to avoid this apparent difficulty appearing in the case II.2.b). Assuming a matrix \( B > 0 \), then instead the equations (2.4) and (2.5) the following equations may be taken in consideration

\[
(2.7) \quad \bar{v}_1^T(\varepsilon A^{-1}B + M_1^{-1}N_1) = \bar{\lambda}_1 \bar{v}_1^T
\]

and

\[
(2.8) \quad (\varepsilon BA^{-1} + N_2M_2^{-1})\bar{y}_2 = \bar{\lambda}_2 \bar{y}_2.
\]

Since for \( \varepsilon > 0 \) both matrices \( \varepsilon A^{-1}B + M_1^{-1}N_1 \) and \( \varepsilon BA^{-1} + N_2M_2^{-1} \) are irreducible, their eigenvalues \( \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \) corresponding to spectral radii are strictly increasing functions of \( \varepsilon \geq 0 \) [1], and \( \bar{\lambda}_1 = \lambda_1, \bar{\lambda}_2 = \lambda_2, \bar{v}_1^T = v_1^T \) and \( \bar{y}_2 = y_2 \) with \( \varepsilon = 0 \). Multiplying (2.7) on the right by \( A^{-1}\bar{y}_2 \) and (2.8) on the left by \( \bar{v}_1^T A^{-1} \) and proceeding similarly as with the derivation of (2.6), one obtains finally

\[
(2.9) \quad \bar{v}_1^T(M_1^{-1} - M_2^{-1})\bar{y}_2 = (\bar{\lambda}_2 - \bar{\lambda}_1)\bar{v}_1^T A^{-1}\bar{y}_2.
\]

Since for \( \varepsilon > 0 \) both eigenvectors \( \bar{v}_1 \) and \( \bar{y}_2 \) are positive, it can be concluded that \( \bar{v}_1^T(M_1^{-1} - M_2^{-1})\bar{y}_2 > 0 \) and \( \bar{v}_1^T A^{-1}\bar{y}_2 > 0 \), which implies that \( \lambda_2 - \lambda_1 > 0 \) hence \( \bar{\lambda}_2 > \bar{\lambda}_1 \). Taking the limit for \( \varepsilon \to 0 \), it follows that \( \bar{\lambda}_1 \to \lambda_1 \) and \( \bar{\lambda}_2 \to \lambda_2 \) which allows us to conclude that \( \lambda_2 \geq \lambda_1 \).

In the case when both convergent weaker splittings are of different type but such that \( N_1M_1^{-1} \geq 0 \) and \( M_2^{-1}N_2 \geq 0 \), then instead the equations (2.4) and (2.5) we can consider the equations

\[
N_1M_1^{-1}y_1 = \lambda_1 y_1 \quad \text{and} \quad v_2^TM_2^{-1}N_2 = \lambda_2 v_2^T
\]

providing us the following equation

\[
v_2^T(M_1^{-1} - M_2^{-1})y_1 = (\lambda_2 - \lambda_1)v_2^TA^{-1}y_1,
\]

from which in a similar way we can conclude that \( \lambda_2 \geq \lambda_1 \).

Thus, from the above considerations we obtain the following result.

**Theorem 2.3.** [2] Let \( A = M_1 - N_1 = M_2 - N_2 \) be two convergent weaker splittings of different types, that is, either \( M_1^{-1}N_1 \geq 0 \) and \( N_2M_2^{-1} \geq 0 \) or \( N_1M_1^{-1} \geq 0 \) and \( M_2^{-1}N_2 \geq 0 \), where \( A^{-1} \geq 0 \). If \( M_1^{-1} \geq M_2^{-1} \), then

\[
\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2).
\]
In particular, if \( A^{-1} > 0 \) and \( M_1^{-1} > M_2^{-1} \), then

\[ \rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2). \]

Assuming now that both convergent weaker splittings of different types in (2.1) are derived from a non-monotone matrix \( A \). Referring back to (2.6) the following cases can be analyzed.

**Case III.** When \( A^{-1} < 0 \), then \( v_1^TA^{-1}y_2 < 0 \).

1. If \( M_1^{-1} > M_2^{-1} \), then \( M_1^{-1} - M_2^{-1} > 0 \) and \( v_1^T(M_1^{-1} - M_2^{-1})y_2 > 0 \), hence \( \lambda_2 - \lambda_1 < 0 \) and \( \lambda_2 < \lambda_1 \).
2. If \( M_1^{-1} \geq M_2^{-1} \), then \( M_1^{-1} - M_2^{-1} \geq 0 \) and
   a) if \( v_1^T(M_1^{-1} - M_2^{-1})y_2 > 0 \), hence \( \lambda_2 - \lambda_1 < 0 \) and \( \lambda_2 < \lambda_1 \).
   b) if \( v_1^T(M_1^{-1} - M_2^{-1})y_2 = 0 \), hence \( \lambda_2 - \lambda_1 = 0 \) and \( \lambda_2 = \lambda_1 \).

**Case IV.** When \( A^{-1} \leq 0 \), then \( v_1^TA^{-1}y_2 \leq 0 \).

1. If \( v_1^T(M_1^{-1} - M_2^{-1})y_2 > 0 \), then \( v_1^TA^{-1}y_2 < 0 \), hence \( \lambda_2 - \lambda_1 < 0 \) and \( \lambda_2 < \lambda_1 \).
2. If \( v_1^T(M_1^{-1} - M_2^{-1})y_2 = 0 \), then
   a) for \( v_1^TA^{-1}y_2 < 0 \), \( \lambda_2 - \lambda_1 = 0 \) and \( \lambda_2 = \lambda_1 \).
   b) for \( v_1^TA^{-1}y_2 = 0 \), the relation (2.6) is satisfied for arbitrary values of \( \lambda_1 \) and \( \lambda_2 \).

The following examples of weaker splittings illustrate the case IV.2.b).

\[ A = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} = M_1 - N_1 = M_2 - N_2 \text{ where} \]

\[ M_1 = \begin{bmatrix} -6 & 0 \\ 0 & -7 \end{bmatrix}, \quad N_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad M_1^{-1}N_1 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{2}{5} \end{bmatrix} \text{ and } v_1^T = [0 \quad 1], \]

\[ M_2 = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2^{-1}N_2 = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } y_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

Evidently, \( v_1^T(M_1^{-1} - M_2^{-1})y_2 = [0 \quad 1] \begin{bmatrix} 0 & 0 \\ 0 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \)

and \( v_1^TA^{-1}y_2 = [0 \quad 1] \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \).

Assuming now a matrix \( B < 0 \), and repeating the same procedure as in the case of the case II.2.b), one can obtain again (2.9) from which, taking the limit for \( \varepsilon \to 0 \), we can conclude that \( \lambda_2 \leq \lambda_1 \) for the case IV.2.b). Hence, the following theorem holds.

**Theorem 2.4.** Let \( A = M_1 - N_1 = M_2 - N_2 \) be two convergent weaker splittings of different types, that is, either \( M_1^{-1}N_1 \geq 0 \) and \( N_2M_2^{-1} \geq 0 \) or \( N_1M_1^{-1} \geq 0 \) and \( M_2^{-1}N_2 \geq 0 \), where \( A^{-1} \leq 0 \). If \( M_1^{-1} \geq M_2^{-1} \), then

\[ \rho(M_1^{-1}N_1) \geq \rho(M_2^{-1}N_2). \]

In particular, if \( A^{-1} \leq 0 \) and \( M_1^{-1} > M_2^{-1} \), then

\[ \rho(M_1^{-1}N_1) > \rho(M_2^{-1}N_2). \]
Thus, we see that for the conditions (2.2) and (2.3) passing from the assumption $A^{-1} \geq 0$ to the assumption $A^{-1} \leq 0$ implies the change of the inequality sign in the inequalities for spectral radii.

Finally, it is evident that the following corollary holds.

**Corollary 2.5.** Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak splittings or one of them is weak and the second is weaker, then Theorems 2.1, 2.2, 2.3, and 2.4 hold.

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