Spring 5-12-2017

Turán Numbers of Vertex-disjoint Cliques in r-Partite Graphs

Anna Schenfisch
University of Wyoming, aschenfi@uwyo.edu

Final Honors Project
Turán numbers of vertex-disjoint cliques in r-partite graphs

Anna Schenfisch

May 12, 2017

1 Abstract

In a broad sense, graph theory has always been present in civilization. Graph theory is the math of connections − at a party, who knows each other? How many handshakes will each person in a meeting have to give before shaking hands with everyone? What is the best way to route traffic through a city's network of roads?

Extremal graph theory is a branch that deals with counting items (called vertices) and connections between two items (called edges) and determining the maximum/minimum number of characteristics needed to satisfy a certain property.

The specific topic of this paper is Turán numbers, a topic of extremal graph theory that attempts to determine the maximum number of edges a graph may have without a specified pattern emerging.

For two graphs, G and H, the Turán number is denoted $ex(G, H)$, and is the maximum number of edges in a subgraph of G that contains no copy of H.

We were able to find and prove a previously unknown Turán number for a certain pattern in a certain graph. To be precise, we found the Turán number of copies of vertex-disjoint cliques in r-partite graphs (part sizes $n_1, ..., n_r$). That is,

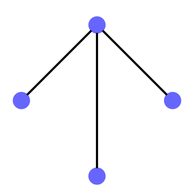
$$ex(K_{n_1,n_2,...,n_r}, kK_r) = \sum_{1\le i<j\le r} n_i n_j - n_1 n_2 + n_2(k-1)$$

This paper will describe the motivation and history of extremal graph theory, discuss definitions and concepts related to the research that was done, go through the proof of our theorem, and finally discuss possible future research as well as general open questions in the field. Note that much of this paper was adapted from a previous paper by the author and other contributors [1].

2 Key concepts

The following glossary may serve as an introduction to those new to the field and as a reminder for those more familiar with it. We encourage the reader to refer back to this section as a reference to both concepts and notation.

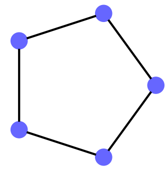
Graph: A graph G is a pair of sets $G = (V, E)$, where V is a fixed set of vertices, and the edge set E is a set of pairs of distinct elements from V. We often write V as $V(G)$ and E as $E(G)$ *(Note: all graphs used in this paper have this definition; that is, they are simple and undirected).*



An example of a graph. The nodes represent vertices and the lines represent edges.

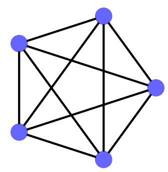
Subgraph: Let G be a graph. A subgraph H of G is a pair of sets $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$, which is itself a graph. If H is a subgraph of G, we write $H \subseteq G$.

Cycle: A graph G is called a cycle if the graph is an alternating sequence of vertices and edges $G = v_1e_1v_2e_2...v_{n-1}e_{n-1}v_ne_nv_{n+1}$, where
$e_i = v_iv_{i+1}$, and if $i < j \leq n$ then $v_i \neq v_j$, and $v_{n+1} = v_1$. A cycle of length n is typically denoted C_n.



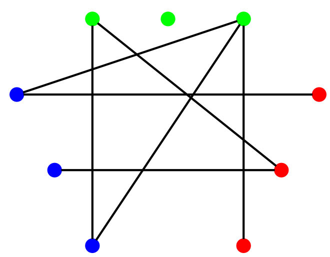
C_5

Complete graph: A graph $G = (V, E)$ is complete if for every pair $x \neq y$ in V we have $xy \in E$. If $|E(G)| = n$, this graph is denoted K_n. A subgraph that is a complete graph is called a **clique of size** n.



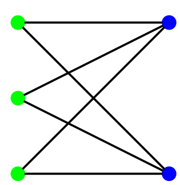
K_5, or clique of size 5

r-Partite graph: A graph G is r-partite if $V(G)$ can be partitioned into r sets, $V_1 \cup V_2 \cup \cdots \cup V_r$ such that if x and y are both in the same V_i, then $xy \notin E(G)$.



A 3-partite (commonly called tripartite) graph. Here, each color indicates to which part a vertex might belong.

Complete r-partite graph: A graph G that is r-partite is called complete r-partite if every pair of vertices among different vertex sets in the partition (commonly called parts) are adjacent (connected by an edge). If $|V_i| = m_i$, then this graph is denoted $K_{m_1,m_2,...,m_r}$.



A complete bipartite graph, $K_{3,2}$

Join: The join of two disjoint graphs G and H occurs when all vertieces in G are connected to all vertices in H. The join of G and H is notated $G + H$.

Weight: For the purposes of this paper, we define the weight of a set of verticies as the number of edges in the subgraph containing exactly these vertecies. For a set of vertices S, the weight of S is denoted as $w(S)$.

Turán number: The Turán number of a pair of graphs S and G with $S \subseteq G$ is the maximum number of edges that a subgraph of G may have and still contain no copy of S. This number may also be called the **extremal number**, and is denoted $ex(G, S)$.

Induced Subgraph: A vertex induced subgraph is a graph consisting of a specified set of vertices along with any edges between these specific vertices. The graph induced by the vertex set S on a graph G is denoted $G[S]$.

3 Motivation and history

The study of graph theory is a fairly recent development, at least in the ways it is defined and researched today. Arguably the first embryonic instance of a graph theory question was posed in 1736, when the Swiss mathematician Leonard Euler published a paper called "The Seven Bridges of Königsberg." In this paper, Euler proved it was not possible to walk through Königsberg crossing each of the city's seven bridges only once. Euler noticed that as the traveler went from one bridge to another, the choice of roads *between* the bridges was irrelevant. With our modern terminology, this is like saying we do not need to pay attention to the shape of an edge between two verticies - all we need to know is that it connects them.

Nearly two centuries later, we finally have our first major extremal problem result made by Mantel in 1907 [2]. He found that if a graph with n verticies (denoted G_n) does not contain a clique of size 3 (denoted K_3) then the number of edges in G_n is no greater than $n^2/2$, or

$$e(G_n) \leq \left[\frac{n^2}{4}\right]$$

This result is much more in line with the type of thinking we see today in graph theory. However, Mantel's result was largely forgotten. It wasn't until years later that graph theory was again picked up, this time by a man named Paul Erdős.

Erdős is considered by many to be one of the fathers of the field. His influence is far reaching - his work influences graph theorists today in how to think about questions, and even what type of questions to ask.

One of Erdős' well known theorems (published in 1938 and colloquially called the C_4-Theorem) is a good example of how graph theory uses techniques and has applications outside of pure graph theory alone. Part of the theorem states:

1. Assume that $n_1 < \cdots < n_k$ are positive integers such that $n_i n_j \neq n_h n_l$ unless $i = h$ and $j = l$ or $i = l$ and $j = h$. What is the maximum number of such integers in $[1, n]$? Denote this maximum by $B(n)$.

In a later a paper, Erdős would find the upper bound

$$B(n) \leq \pi(n) + O\left(\frac{n^{3/4}}{log^{3/2} n}\right)$$

where $\pi(n)$ is the number of primes in $[2, n]$ Note that the last term on the right side represents an error of order $\frac{n^{3/4}}{log^{3/2} n}$; in other words, this bound may deviate from reality by at most some factor of $\frac{n^{3/4}}{log^{3/2} n}$.
At first reading, this question seems to lie purely in the realm of number theory. However, Erdős was able to transform the question into a graph theory one by the following lemma:

Lemma 1. *Every integer $a \in [1, n]$ can be written as*

$$a = bd \quad : \quad b \in B, d \in D$$

where D is the set of integers in $[1, n^{2/3}]$, IP is the set of primes in $(n^{2/3}, n]$ and $B = D \cup IP$.
If we let A be a set that satisfies the condition in 1. Then we can represent every $a \in A$ as $a_i = b_j(i)d_j(i)$. We assume that $b_j > d_j$. If we create a bipartite graph $G(B, D)$ by joining b_j to d_j for each $b_j d_j = a \in A$. In a graph theoretical sense then, each a_j is an edge between its associated vertices. Note that if we were to find a 4-cycle $(b_1 d_1 b_2 d_2)$ in $G(B, D)$, then $a_1 = b_1 d_1$ and $a_2 = b_1 d_2$, $a_3 = b_2 d_2$ and $a_4 = b_2 d_1$ meaning that $a_1 a_3 = a_2 a_4$, which contradicts our assumption. Therefore, the reframed question becomes

2. Given a bipartite graph $G(X, Y)$ with m and n vertices in its color classes, what is the maximum number of edges $G(X, Y)$ may have without containing any C_4?

Erdős proved the following theorem:

Theorem 1. *If $C_4 \in G(X, Y)$, $|X| = |Y| = k$, then*

$$e(G(X, Y)) \leq 3k^{3/2}$$

In yet *another* borrowing from a different field, Erdős turned to geometry, borrowing a lemma from another mathematician, Eszter Klein:

Lemma 2. *Given $p(p+1)+1$ elements, (for some prime p) we can construct $p(p+1)+1$ combinations, taken $(p+1)$ at a time (meaning $(p+1)$-tuples) having no two elements in common.*

Although we have glossed over the details of the proof, the example of Erdős' C_4-Theorem demonstrates how graph theory has connections to other fields (here, we see connections to number theory and finite geometry).

We see the concepts of graph theory being applied to a multitude of other fields, such as arithmetic problems (for example in Sidon numbers [4]), topology (in places such as the Erdős−Stone Theorem [5]), and computer science (Kuratowski's work in networking [6]).

Our theorem deals with Turán numbers, which first appears to be a bit more abstract.

Paul Turán, like Erdős, was a Hungarian mathematician interested in graph theory. He was also highly influential, especially in extremal problems, or problems that seek to find a maximum or minimum number of something (such as edges, colorings, etc.) within a certain constraint. Erdős is even quoted as saying, "Turán initiated the field of extremal graph theory." [7]

Turán's most famous theorem is about the number of edges a graph of n vertices may have without containing a clique of size p. Below, we use the notation $T_{n,p-1}$, which is a Turán graph on n vertices with $p-1$ classes (meaning that it has the maximum possible number of edges without containing a clique of size p).

Theorem 2. *If G_n contains no K_p, then $e(G_n) \leq e(T_{n,p-1})$. In case of equality, $G_n = T_{n,p-1}$.*

While this theorem is for quite specific graphs, it opened the door to more general questions about host graphs not containing specific subgraphs.

Our theorem is a descendant of this idea. Like Turán, we have chosen a specific host graph (an r-partite graph with part sizes $n_1, \ldots, n_r$) and a specific 'forbidden' subgraph (k vertex-disjoint copies of K_r).

Two previous papers should be mentioned which may serve as helpful predecessors to our theorem.

Moon's theorem [8] also has a forbidden graph of a number of vertex disjoint copies of a specific form.

In 2009, Chen, Li, and Tu [9] found the extremal number for k vertex disjoint matchings in bipartite graphs - our theorem extends these results.

4 Theorem

Theorem 3. ***(Main Theorem)*** *For any integers $1 \leq k \leq n_1 \leq \cdots \leq n_r$,*

$$ex(K_{n_1,n_2,\ldots,n_r}, kK_r) = \sum_{1 \leq i < j \leq r} n_i n_j - n_1 n_2 + n_2(k-1).$$

The approach to the proof will fall into two main sections. First, we will show the lower bound, or that indeed, the extremal number is *at least* the number we claim. Then we will show the upper bound, or that the extremal number is *at most* the number we claim. By showing the extremal number is simultaneously at least and at most the value we claim, we will be able to then say that the extremal number is exactly *equal* to the value we claim.

For the lower bound, we consider the graph $((n_1 - (k-1))K_1 \cup K_{k-1,\, n_2}) + K_{n_3,\ldots,n_r}$. In other words, we have $(n_1 - (k-1))$ copies of K_1 (i.e., $(n_1 - (k-1)$ isolated vertices) and the complete bipartite graph between parts of size n_2 and $k-1$. To this, we join the complete r-partite graph between the parts of size n_3 through n_r. Clearly, this is a subgraph of $K_{n_1,...n_r}$. Also note it has the required number of edges. Finally, note that to have kK_r, we would clearly need k vertices from each part involved in the kK_r. However, in our graph, we only have $n_1-(k-1)$ vertices contributing from the part of size n_1, and can therefore not have kK_r. Therefore, since our graph is a subgraph of $K_{n_1,...n_r}$, has the required number of edges, but still does not contain a copy of kK_r, it serves as a proof of the lower bound.

The upper bound will be a bit more of an in-depth process. We consider two cases: $n_2 = n_r$ and $n_2 < n_r$. In the former case, the proof is by induction on $n_1 + k$. In the latter case, the proof is by induction on the total number of vertices in the host graph.

For ease of notation, we define

$$h_k(n_1, n_2, ..., n_r) = \sum_{1\leq i<j\leq r} n_i n_j - n_1 n_2 + n_2(k-1).$$

We begin with the case where $n_2 = n_r$. In order to use induction, we need two base cases, which are established in Lemmas 3 and 4.

Given two disjoint subsets of the vertex set, A, $B \in V(G)$, define AB as the graph formed by the set of edges in G incident to (connected to) a vertex in A and a vertex in B.

Also for ease of notation, given an r-partite graph G with parts $V_1, ..., V_r$, we let $\mathcal{R}(G,r) = \{\{v_1, ..., v_r\} \in V(G) : v_i \in V_i \text{ for all } i \in [r]\}$. That is, $\mathcal{R}(G,r)$ is the set of all r-tuples of vertices with exactly one vertex from each part. We will utilize $\mathcal{R}(G,r)$ throughout to facilitate the counting of edges. For some $S \in \mathcal{R}(G,r)$, define $w(S)$ as the number of edges in the subgraph of G induced by S, that is

$$w(S) = |E(G[S]|.$$

Note that for $S \in \mathcal{R}(G,r)$, an edge $v_i v_j \in V_i V_j$ is counted in $w(S)$ if and only if $\{v_i, v_j\} \subseteq S$. Therefore, summing over all $S \in \mathcal{R}(G,r)$,

$$\sum_{S\in\mathcal{R}(G,r)} w(S) = \sum_{i\leq i<j\leq r} |E(V_i V_j)| \prod_{l\neq i,j} n_l. \tag{1}$$

Lemma 3. *For* $1 \leq n_1 \leq n_2$*,*

$$ex(K_{n_1,n_2,...,n_2}, K_r) = h_1(n_1, n_2, ..., n_2)$$

(Note that the following lemma establishes the base case for the induction on k, since we are looking for a single K_r).

Proof. Suppose $G \subseteq K_{n_1,n_2,...,n_2}$ does not contain a copy of K_r. Then, for all $S \in \mathcal{R}(G,r)$, we would have $w(S) \leq \binom{r}{2} - 1$ (Notice that this is because $\binom{r}{2}$ is the minimum weight of a K_r, so the weight must be at least one fewer to guarantee no K_r). This means that for the sum over all $S \in \mathcal{R}(G,r)$, we would have

$$\sum_{S\in\mathcal{R}(G,r)} w(S) < \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1}. \tag{2}$$

Subtracting (2) from (1) yields

$$\begin{aligned}
0 &\geq \sum_{j=2}^{r} |E(V_1V_j)| n_2^{r-2} + \sum_{i,j\neq 1} |E(V_iV_j)| n_1 n_2^{r-3} - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&= n_1 n_2 r^{r-3} |E(G)| + \sum_{j=2}^{r} |E(V_1V_j)| n_2^{r-3}(n_2 - n_1) - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&\geq n_1 n_2^{r-3} |E(G)| + \left(E|(G)| - \binom{r-1}{2} n_2^2\right) n_2^{r-3}(n_2 - n_1) - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&= n_2^{r-2}|E(G)| - \binom{r-1}{2} n_2^{r-1}(n_2 - n_1) - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1} \\
&= n_2^{r-2}|E(G)| - (r-2) n_1 n_2^{r-1} - \left(\binom{r}{2} - 1\right) n_1 n_2^{r-1}.
\end{aligned}$$

Therefore,

$$|E(G)| \leq n_1 n_2 (r-1) + \binom{r-1}{2} n_2^2 - n_1 n_2 = h_1(n_1, n_2, ..., n_2).$$

We have shown that, although G does not contain a single K_r (as it was defined), the number of edges in G is less than or equal to our theorized extremal number, meaning that the maximum number of edges G may have is our theorized extremal number. Since the maximum number of edges a graph may have without any appearance of a particular subgraph, this shows that $ex(K_{n_1,n_2,...,n_2}, K_r) = h_1(n_1, n_2, ..., n_2)$, as desired. □

Now we will establish the second base case, where we are looking for the same number of K_r's as n_1. As in the Lemma 3, we have all equal part sizes, except for n_1, which may be smaller.

Lemma 4. *For* $1 \leq n_1 \leq n_2$,

$$ex(K_{n_1,n_2,...,n_2}, n_1 K_2 = h_{n_1}(n_1, n_2, ..., n_2).$$

Proof. This lemma will be proved by induction on n_1. The base case of $n_1 = 1$ was shown for all positive integers n_2 by the Lemma 3. Assume the statement is true for $n_1' < n_1$ where $n_1 \geq 2$. Now suppose towards a contradiction that $G \subseteq K_{n_1,n_2,...,n_2}$ contains *more* than $h_{n_1}(n_1, n_2, ..., n_2)$ edges but does not contain a copy of $n_1 K_r$. Also note that simply by the way they are defined, $h_{n_1}(n_1, n_2, ..., n_2)$ is greater than or equal to $h_1(n_1, n_2, ..., n_2)$ (the inequality is not strict here because the lemma was defined for $1 \leq n_1 \leq n_2$, i.e., not with strict inequalities). Symbolically, this is expressed as

$$|E(G)| > h_{n_1}(n_1, n_2, ..., n_2) \leq h_1(n_1, n_2, ..., n_2)$$

Since there are more edges in G than the number of edges we have already established contains a K_r, we know that G contains a copy of K_r.

Now let $S \in \mathcal{R}(G, r)$ such that $G[S] \cong K_r$ (i.e., the subgraph of G induced by S is K_r). Then, if we examine the graph made by removing S from G, we see

$$|E(G \backslash S)| \leq h_{n_1 - 1}(n_1 - 1, n_2 - 1, ..., n_2 - 1),$$

otherwise $G \backslash S$ would contain a copy of $(n_1 - 1)K_r$, and this together with S is a copy of $n_1 K_r$ in G (which we are assuming cannot happen). Therefore,

$$\begin{aligned}
|E(G)| - |E(G \backslash S)| &> h_{n_1}(n_1, n_2, ..., n_2) - h_{n_1 - 1}(n_1 - 1, n_2 - 1, ..., n_2 - 1) \\
&= (r-1)(n_1 + (r-2)n_2) + (r-1)n_2 - \binom{r}{2} - 1.
\end{aligned}$$

Another way of thinking about the above value is the number of edges in $K_{n_1,n_2,...,n_2}$ that have a vertex in S. This implies that all edges in the host graph containing a vertex in S are present in G. Note that this is true for every S such that $G[S] = K_r$ in G.

Let $u_i \in V_i$ and $u_j \in V_J$ with $i \neq j$, if either u_i or u_j is in S, then the edge $u_iu_j \in E(G)$. Otherwise, for $v_i \in S \cap V_i$, let $S' = (X \backslash \{v_i\}) \cup \{u_i\}$. S' induces a copy of K_r in G and therefore $u_iu_j \in E(G)$. Hence $G \cong K_{n_1,n_2,\ldots,n_2}$, and thus contains n_1K_r, a contradiction. □

Now that we have our two necessary base cases, we are ready to prove the main theorem. The proof is split up into two cases: $n_2 = n_r$ and $n_2 < \cdots < n_r$.

Proof. Case 1. Assuming $n_1 = n_r$, we proceed by induction on $n_1 + k$. The base case of $k = 1$ was shown to be true for all positive integers n_1 in Lemma 3. Now assume the statement is true for the parameters n_1', k' such that $n_1' + k' < n_1 + k$ for $n_1 > k \geq 2$. Also assume that $G \subseteq K_{n_1,n_2,\ldots,n_2}$ does not contain a copy of kK_r.

First we will obtain a lower bound on the number of copies of K_r in G. Note that we are not requiring the copies of K_r to be vertex disjoint. Suppose that there are exactly q such copies of K_r in G, then

$$\sum_{S \in \mathcal{R}(G,R)} w(S) \leq q\binom{r}{2} + (n_1n_2^{r-1} - q)\left(\binom{r}{2} - 1\right).$$

Recall that

$$\sum_{S \in \mathcal{R}(G,R)} w(S) = \sum_{j=2}^{r} |E(V_1V_j)|n_2^{r-2} + \sum_{i,j \neq 1} |E(V_iV_j)|n_1n_2^{r-3}.$$

(this is just a rewording of equation (1), made specific to our G where $n_2 = n_r$). Again using the same q as before, this gives

$$q \geq \sum_{j=2}^{r} |E(V_1V_j)|n_2^{r-2} + \sum_{i,j \neq 1} |E(V_iV_j)|n_1n_2^{r-3} - n_1n_2^{r-1}\left(\binom{r}{2} - 1\right). \tag{3}$$

We will use equation (3) to get an upper bound on $|E(G)|$ by counting $\sum\limits_{S \in \mathcal{R}(G,r)} |E(G \backslash S)|$. An edge $v_iv_j \in V_iV_j$ is counted in $|E(G \backslash S)|$ if and only if $v_i \notin S$ and $v_j \notin S$, hence

$$\sum_{S \in \mathcal{R}(G,r)} |E(G \backslash S)| = \sum_{j=2}^{r} |E(V_1V_j)|(n_1 - 1)(n_2 - 1)n_2^{r-2} + \sum_{i,j \neq 1} |E(V_iV_j)|(n_2 - 1)^2 n_1n_2^{r-3}. \tag{4}$$

Using equations (3) and (4), we now have

$$\begin{aligned}\sum_{S \in \mathcal{R}(G,r)} |E(G \backslash S)| + (q + n_1n_2^{r-1}\left(\binom{r}{2} - 1\right)(n_2) - 1 \geq &\sum_{j=2}^{r} |E(V_1V_j)|(n_1 - 1)(n_2 - 1)n_2^{r-2} \\ &+ \sum_{i,j \neq 1} |E(V_iV_j)|(n_2 - 1)^2 n_1n_2^{r-3} \\ &- |E(G)|(n_2 - 1)n_2^{r-2}n_1. \end{aligned} \tag{5}$$

Now for $S \in \mathcal{R}(G,r)$, suppose $G[S]$ is a copy of K_r. Then $|E(G \backslash S)| \leq h_{k-1}(n_2 - 1, n_2 - 1, ..., n_2 - 1)$, else by induction $G \backslash S$ contains a copy of $(k-1)K_r$, and so this together with S yields a copy of kK_r in G. If $G[S]$ is not complete, then since $G \backslash S$ does not contain a copy of kK_r, induction gives $|E(G \backslash S)| \leq h_k(n_1 - 1, n_2 - 1, ..., n_2 - 1)$. Hence

$$\begin{aligned}\sum_{S \in \mathcal{R}(G,r)} |E(G \backslash S)| &\leq q\left(h_{k-1}(n_2 - 1, n_2 - 1, ..., n_2 - 1)\right) \\ &\quad + (n_1n_2^{r-1} - q)\left(h_k(n_2 - 1, n_2 - 1, ..., n_2 - 1)\right) \\ &= q(1 - n_2) + n_1n_2^{r-1}\left(h_k(n_2 - 1, n_2 - 1, ..., n_2 - 1)\right)\end{aligned}$$

and thus, using equation (5) we have

$$|E(G)|(n_2-1)n_2^{r-2}n_1 \leq n_1 n_2^{r-1}\left(h_k(n_2-1,n_2-1,...,n_2-1)+\left(\binom{r}{2}-1\right)(n_2-1)\right).$$

Therefore

$$\begin{aligned}|E(G)| &\leq \frac{n_2}{n_2-1}\left(h_k(n_2-1,n_2-1,...,n_2-1)+\left(\binom{r}{2}-1\right)(n_2-1)\right)\\ &= h_k(n_1,n_2,...,n_2).\end{aligned}$$

Case 2. Assume $n_2 < n_r$. We proceed by induction on the number of total vertices. The base case of $n_1 = n_r$ is true for all positive integers k by case 1. Now assume the statement holds for all parameters $n_1', ..., n_r'$ such that $\sum_{i=1}^r n_i' < \sum_{i=1}^r n_i$. Suppose that $G \subseteq K_{n_1,...,n_r}$ does not contain a copy of kK_r. Let $v_r \in V_r$. The graph $G\backslash\{v_r\}$ does not contain a copy of kK_r, has fewer vertices than G, and $n_2 \leq n_r - 1$. Therefore,

$$\begin{aligned}|E(G)| &= |E(G\backslash\{v_r\})| + d(v_r)\\ &=\leq ex(K_{n_1,...,n_r-1}, kK_r) + d(v_r)\\ &= h_k(n_1, ...n_r-1) + d(v_r)\\ &= \sum_{1\leq i<j\leq r} n_in_j - \sum_{i=1}^{r-1} n_i - n_1n_2 + n)2(k-1_+d(v_r)\\ &\leq \sum_{1\leq i<j\leq r} n_in_j - n_1n_2 + n_2(k-1)\\ &= h_k(n_1,n_2,...,n_r).\end{aligned}$$

□

5 Future research and other open questions

The main theorem relies on the fact that both K_r and $K_{n_1,n_2,...,n_r}$ are r-partite. Certainly the host graph must have more parts than the size of the forbidden clique - in other words, it must be l-partite for $l \geq r$ to have K_r as a subgraph. An interesting generalization would be to calculate $ex(K_{n_1,n_2,...,n_l}, kK_r)$ for $r <$. In [10], De Silva, Heysse, and Young proved that

$$ex(K_{n_1,n_2,...,n_l}, kK_2) = (k-1)\left(\sum_{i=2}^{t} n_i\right),$$

however the Turán number is open for $r \geq 3$. The graph

$$((n_1+n_2k+1)K_1 \cup K_{k1,n_3}) + n_4K_1$$

does not contain kK_3, hence

$$ex(K_{n_1,n_2,n_3,n_4}, kK_3) \geq (n_1+n_2+n_3)n_4 + (k-1)n_3$$

This construction can be easily generalized to r-partite graphs, but it is not clear that this is an extremal construction.

References

[1] J. De Silva, K. Heysse, A. Kapilow, A. Schenfisch, and M. Young: Turán numbers of vertex-disjoint cliques in r-partite graphs, preprint.

[2] W. Mantel: Problem 28, Wiskundige Opgaven, 10 (1907) 60-61.

[3] P. Erdős: On sequences of integers no one of which divides the product of two others and related problems, Mitt. Forsch. Institut Mat. und Mech. Tomsk **2** (1938) 74-82.

[4] P. Erdős: On sequences of integers no one of which divides the product of two others, and some related problems, Mitt. Forschungsinst. Math. u. Mech. Tomsk 2 (1938), 74-82

[5] P. Erdős, A.H. Stone: On the structure of linear graphs. Bulletin of the American Mathematical Society. 52, 12 (1946), 10871091.

[6] Kuratowski, C: Sur l'operation A de l'analysis situs. Fund. Math. 3 (1922), 182-199.

[7] P. Erdős: Paul Turán 1910-1976: His work in graph theory, J. Graph Theory, **1** (1977) 96-101.

[8] J.W. Moon: On independent complete subgraphs in a graph, Canad. J. Math., **20** (1968) 95-102. also in: International Congress of Math. Moscow, (1966), vol 13.

[9] Chen, He, Xueliang Li, and Jianhua Tu. "Complete Solution for the Rainbow Numbers of Matchings." Discrete Mathematics 309.10 (2009): 3370-380. Web.

[10] J. De Silva, K. Heysse, M. Young, Rainbow number for matchings in $r-$partite graphs, preprint.