Proof of Atiyah's conjecture for two special types of configurations

Dragomir Z. Djokovic
dragomir@herod.uwaterloo.ca
PROOF OF ATIYAH’S CONJECTURE FOR TWO SPECIAL TYPES OF CONFIGURATIONS∗

DRAGOMIR Z. D–OKOVIĆ†

Abstract. To an ordered \(N\)-tuple \((x_1, \ldots, x_N)\) of distinct points in the three-dimensional Euclidean space Atiyah has associated an ordered \(N\)-tuple of complex homogeneous polynomials \((p_1, \ldots, p_N)\) in two variables \(x, y\) of degree \(N - 1\), each \(p_i\) determined only up to a scalar factor. He has conjectured that these polynomials are linearly independent. In this note it is shown that Atiyah’s conjecture is true for two special configurations of \(N\) points. For one of these configurations, it is shown that a stronger conjecture of Atiyah and Sutcliffe is also valid.

Key words. Atiyah’s conjecture, Hopf map, Configuration of \(N\) points in the three-dimensional Euclidean space, Complex projective line.

AMS subject classifications. 51M04, 51M16, 70G25

1. Two conjectures. Let \((x_1, \ldots, x_N)\) be an ordered \(N\)-tuple of distinct points in the three-dimensional Euclidean space. Each ordered pair \((x_i, x_j)\) with \(i \neq j\) determines a point

\[
\frac{x_j - x_i}{|x_j - x_i|}
\]
on the unit sphere \(S^2\). Identify \(S^2\) with the complex projective line by using a stereographic projection. Hence one obtains a point \((u_{ij}, v_{ij})\) on this projective line and a complex nonzero linear form \(l_{ij} = u_{ij}x + v_{ij}y\) in two variables \(x\) and \(y\). Define homogeneous polynomials \(p_i\) of degree \(N - 1\) by

\[
p_i = \prod_{j \neq i} l_{ij}(x, y), \quad i = 1, \ldots, N.
\] (1.1)

Conjecture 1.1. (Atiyah [2]) The polynomials \(p_1, \ldots, p_N\) are linearly independent.

Atiyah [1], [2] has observed that his conjecture is true if the points \(x_1, \ldots, x_N\) are collinear. He has also verified the conjecture for \(N = 3\). The case \(N = 4\) has been verified by Eastwood and Norbury [4]. For additional information on the conjecture (further conjectures, generalizations, and numerical evidence) see [2], [3].

In order to state the second conjecture, one has to be more explicit. Identify the three-dimensional Euclidean space with \(\mathbb{R} \times \mathbb{C}\) and denote the origin by \(O\). Following Eastwood and Norbury [4], we make use of the Hopf map \(h : \mathbb{C}^2 \setminus \{O\} \to (\mathbb{R} \times \mathbb{C}) \setminus \{O\}\) defined by

\[
h(z, w) = (|z|^2 - |w|^2)/2, z\bar{w}).
\]

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†Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada (djokovic@uwaterloo.ca). The author was supported in part by the NSERC Grant A-5285.
This map is surjective and its fibers are the circles \( \{ (zu, wu) : u \in S^1 \} \), where \( S^1 \) is the unit circle. If \( h(z, w) = (a, v) \), we say that \( (z, w) \) is a lift of \( (a, v) \). For instance, we can take
\[
\lambda^{-1/2}(\lambda, \bar{v}), \quad \lambda = a + \sqrt{a^2 + |v|^2},
\]
as the lift of \( (a, v) \).

Assume that our points are \( x_i = (a_i, z_i) \). For the sake of simplicity assume that if \( i < j \) and \( z_i = z_j \) then \( a_i < a_j \). As the lift of the vector \( x_j - x_i, i < j \), we choose
\[
\frac{1}{\sqrt{\lambda_{ij}}} (\lambda_{ij}, \bar{z}_j - \bar{z}_i),
\]
where
\[
\lambda_{ij} = a_j - a_i + \sqrt{(a_j - a_i)^2 + |z_j - z_i|^2}.
\]
According to the recipe in [2], [3], [4], we always use the lift \((-\bar{w}, \bar{z})\) for the vector \( x_i - x_j \) if \((z, w)\) has been chosen as the lift of \( x_j - x_i \). Hence we introduce the linear forms
\[
l_{ij}(x, y) = \lambda_{ij}x + (\bar{z}_j - \bar{z}_i)y, \quad i < j;
\]
\[
l_{ij}(x, y) = (z_j - z_i)x + \lambda_{ij}y, \quad i > j.
\]

Define \( P \) to be the \( N \times N \) coefficient matrix of the binary forms \( p_i(x, y) \) defined by (1.1) using the above \( l_{ij} \)'s. The second conjecture that we are interested in can now be formulated as follows.

**Conjecture 1.2.** (Atiyah and Sutcliffe [3, Conjecture 2]; see also [4]) If \( r_{ij} = |x_j - x_i| \), then
\[
|\det(P)| \geq \prod_{i < j} (2\lambda_{ij}r_{ij}).
\]

As \( 2\lambda_{ij}r_{ij} = \lambda_{ij}^2 + |z_j - z_i|^2 \), this conjecture can be rewritten as
\[
|\det(P)| \geq \prod_{i < j} \left( \lambda_{ij}^2 + |z_j - z_i|^2 \right).
\]  

(1.2)

Obviously, this conjecture is stronger than Conjecture 1.1.

2. **Two special cases of Atiyah’s conjecture.** We shall prove Atiyah’s conjecture in the following two cases:

(A) \( N = 1 \) of the points \( x_1, \ldots, x_N \) are collinear.

(B) \( N = 2 \) of the points \( x_1, \ldots, x_N \) are on a line \( L \) and the line segment joining the remaining two points has its midpoint on \( L \) and is perpendicular to \( L \).
Let $L$ and $M$ be two perpendicular lines in the three-dimensional Euclidean space intersecting at the origin, $O$. Let $N = m + n$ and assume that the points $x_1, \ldots, x_m$ are on $L$ and $x_{m+1}, \ldots, x_N$ are on $M$ but not on $L$. Set $y_j = x_{m+j}$ for $j = 1, \ldots, n$.

Without any loss of generality, we may assume that $L = \mathbb{R} \times \{0\}$ and $M = \{0\} \times \mathbb{R}$. Write $x_i = (a_i, 0)$ for $i = 1, \ldots, m$ and $y_j = (0, b_j)$ for $j = 1, \ldots, n$. We may also assume that $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$.

The lifts of the nonzero vectors $x_j - x_i$, $i, j \in \{1, \ldots, N\}$ are given in Table 2.1, where we have set

$$
\lambda_{ij} = a_i + \sqrt{a_i^2 + b_j^2}.
$$

<table>
<thead>
<tr>
<th>Vectors $x_j - x_i$</th>
<th>Index restrictions</th>
<th>Lifts</th>
<th>Linear forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_r - x_i$</td>
<td>$1 \leq i &lt; r \leq m$</td>
<td>$(2(a_r - a_i))^{1/2} (1, 0)$</td>
<td>$2(a_r - a_i)x$</td>
</tr>
<tr>
<td>$x_i - x_r$</td>
<td>$1 \leq i &lt; r \leq m$</td>
<td>$(2(a_r - a_i))^{1/2} (0, 1)$</td>
<td>$2(a_r - a_i)y$</td>
</tr>
<tr>
<td>$y_s - y_j$</td>
<td>$1 \leq j &lt; s \leq n$</td>
<td>$(b_s - b_j)^{1/2} (1, 1)$</td>
<td>$(b_s - b_j)(y + x)$</td>
</tr>
<tr>
<td>$y_j - y_s$</td>
<td>$1 \leq j &lt; s \leq n$</td>
<td>$(b_s - b_j)^{1/2} (-1, 1)$</td>
<td>$(b_s - b_j)(y - x)$</td>
</tr>
<tr>
<td>$x_i - y_j$</td>
<td>$1 \leq i \leq m, 1 \leq j \leq n$</td>
<td>$\lambda_{ij}^{-1/2} (\lambda_{ij}, -b_j)$</td>
<td>$\lambda_{ij}x - b_jy$</td>
</tr>
<tr>
<td>$y_j - x_i$</td>
<td>$1 \leq i \leq m, 1 \leq j \leq n$</td>
<td>$\lambda_{ij}^{-1/2} (b_j, \lambda_{ij})$</td>
<td>$b_jx + \lambda_{ij}y$</td>
</tr>
</tbody>
</table>

**Table 2.1** The lifts of the vectors $x_j - x_i$.

The associated polynomials $p_i$ (up to scalar factors) are given by

$$
p_i(x, y) = x^{m-1} y^{i-1} \prod_{j=1}^n (b_jx + \lambda_{ij}y), \quad 1 \leq i \leq m; \quad (2.1)
$$

$$
p_{m+j}(x, y) = (y + x)^{n-j} (y - x)^{j-1} \prod_{i=1}^m (\lambda_{ij}x - b_jy), \quad 1 \leq j \leq n. \quad (2.2)
$$

**Theorem 2.1.** Conjecture 1.1 is valid under the hypothesis (A).

**Proof.** In this case we have $n = 1$. Without any loss of generality we may assume that $b_1 = -1$. After dehomogenizing the polynomials $p_i$ (or $-p_i$) by setting $x = 1$, we obtain the polynomials:

$$
y^{i-1}(1 - \lambda_iy), \quad 1 \leq i \leq m;
$$

$$
\prod_{i=1}^m (y + \lambda_i),
$$
Atiyah’s conjecture

where \( \lambda_i = \lambda_{i1} > 0 \). The coefficient matrix of these polynomials is

\[
\begin{bmatrix}
1 & -\lambda_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & -\lambda_2 & 0 & 0 & 0 \\
0 & 0 & 1 & -\lambda_3 & 0 & 0 \\
\vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & -\lambda_m \\
E_m & E_{m-1} & E_{m-2} & E_{m-3} & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

where \( E_k \) is the \( k \)-th elementary symmetric function of \( \lambda_1, \ldots, \lambda_m \). Its determinant,

\[
1 + \lambda_mE_1 + \lambda_{m-1}E_2 + \cdots + \lambda_1\lambda_2\cdots\lambda_mE_m,
\]

is positive. \( \square \)

**Theorem 2.2.** Conjecture 1.1 is valid under the hypothesis (B).

Proof. In this case \( n = 2 \) and \( b_1 + b_2 = 0 \). Without any loss of generality we may assume that \( b_1 = -1 \). After dehomogenizing the polynomials \( p_i \) (or \( -p_i \)) by setting \( x = 1 \), we obtain the polynomials:

\[
y^{i-1}(1 - \lambda_i^2y^2), \quad 1 \leq i \leq m;
\]

\[
(y + 1) \prod_{i=1}^{m}(y + \lambda_i),
\]

\[
(y - 1) \prod_{i=1}^{m}(y - \lambda_i),
\]

where \( \lambda_i = \lambda_{i1} > 0 \). The coefficient matrix of these polynomials is

\[
\begin{bmatrix}
1 & 0 & -\lambda_1^2 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & -\lambda_2^2 & 0 & 0 & 0 \\
\vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 & -\lambda_m^2 \\
\tilde{E}_{m+1} & \tilde{E}_m & \tilde{E}_{m-1} & \ldots & \ldots & \ldots & \ldots & \ldots \\
(-1)^{m+1}\tilde{E}_{m+1} & (-1)^m\tilde{E}_m & (-1)^{m-1}\tilde{E}_{m-1} & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

where \( \tilde{E}_k \) is the \( k \)-th elementary symmetric function of \( 1, \lambda_1, \ldots, \lambda_m \). Its determinant is \( 2pq \) where

\[
p = 1 + \lambda_m^2\tilde{E}_2 + \lambda_{m-2}^2\lambda_m^2\tilde{E}_4 + \cdots,
\]

\[
q = \tilde{E}_1 + \lambda_m^2\tilde{E}_3 + \lambda_{m-3}^2\lambda_m^2\tilde{E}_5 + \cdots,
\]

and thus it is positive. \( \square \)

3. Atiyah and Sutcliffe conjecture is valid in case (A). In the general setup of the previous section, the Conjecture 1.2 asserts that

\[
|\det(P)| \geq 2^{\binom{G}{2}} \prod_{i,j} (\lambda_{ij}^2 + b_{ij}^2).
\]
where $P$ is the coefficient matrix (of order $N = m + n$) of the polynomials (2.1) and (2.2).

In case (A) this inequality takes the form

$$1 + \lambda_m E_1 + \lambda_{m-1} \lambda_m E_2 + \cdots + \lambda_1 \lambda_2 \cdots \lambda_m E_m \geq \prod_{i=1}^{m} (1 + \lambda_i^2),$$

(3.2)

where, as in the proof of Theorem 2.1, we assume that $b_1 = -1$ and $E_k$ denotes the $k$-th elementary symmetric function of $\lambda_1, \ldots, \lambda_m$. Thus we have

$$\lambda_i = a_i + \sqrt{1 + a_i^2} > 0$$

and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m.$$  (3.3)

Let $E_k^{(2)}$ denote the $k$-th elementary symmetric function of $\lambda_1^2, \ldots, \lambda_m^2$. In view of (3.3), we have

$$\lambda_{m-k+1} \lambda_{m-k+2} \cdots \lambda_m E_k \geq E_k^{(2)}, \quad 0 \leq k \leq m.$$ 

The inequality (3.2) is a consequence of the inequalities just written since

$$\prod_{i=1}^{m} (1 + \lambda_i^2) = \sum_{k=0}^{m} E_k^{(2)}.$$ 

Hence we have the following result.

**Theorem 3.1.** Conjecture 1.2 is valid in case (A).

In case (B) the inequality (3.1) takes the form:

$$\left(1 + \lambda_{m}^2 \tilde{E}_2 + \lambda_{m-2}^2 \lambda_{m}^2 \tilde{E}_4 + \cdots\right) \left(\tilde{E}_4 + \lambda_{m-1}^2 \tilde{E}_3 + \lambda_{m-3}^2 \lambda_{m-1}^2 \tilde{E}_5 + \cdots\right) \geq \prod_{i=1}^{m} (1 + \lambda_i^2)^2,$$

where $\tilde{E}_k$ are as in the proof of Theorem 2.2.

It is easy to verify that this inequality holds for $m = 1$, but we were not able to prove it in general. If we set all $\lambda_i = \lambda > 0$, then the above inequality specializes to

$$\left[(1 + \lambda^2)^m - \sum_{k=0}^{m} \binom{m}{2k+1} (\lambda^{4k+3} - \lambda^{4k+2})\right] \times$$

$$\left[(1 + \lambda^2)^m - \sum_{k=0}^{m} \binom{m}{2k+1} (\lambda^{4k+2} - \lambda^{4k+1})\right] \geq (1 + \lambda^2)^{2m}.$$
Atiyah’s conjecture

Since

\[
\sum_{k \geq 0} \binom{m}{2k + 1} (\lambda^{4k+3} - \lambda^{4k+2}) = \frac{1}{2} (\lambda - 1) \left[ (1 + \lambda^2)^m - (1 - \lambda^2)^m \right],
\]

it is easy to verify that the specialized inequality is valid.

REFERENCES