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SOME INEQUALITIES FOR THE KHATRI-RAO PRODUCT OF MATRICES

CHONG-GUANG CAO†, XIAN ZHANG†‡, AND ZHONG-PENG YANG§

Abstract. Several inequalities for the Khatri-Rao product of complex positive definite Hermitian matrices are established, and these results generalize some known inequalities for the Hadamard and Khatri-Rao products of matrices.

Key words. Matrix inequalities, Hadamard product, Khatri-Rao product, Tracy-Singh product, Spectral decomposition, Complex positive definite Hermitian matrix.

AMS subject classifications. 15A45, 15A69

1. Introduction. Consider complex matrices $A = (a_{ij})$ and $C = (c_{ij})$ of order $m \times n$ and $B = (b_{ij})$ of order $p \times q$. Let $A$ and $B$ be partitioned as $A = (A_{ij})$ and $B = (B_{ij})$, where $A_{ij}$ is an $m_i \times n_j$ matrix and $B_{kl}$ is a $p_k \times q_l$ matrix ($\sum m_i = m$, $\sum n_j = n$, $\sum p_k = p$, $\sum q_l = q$). Let $A \otimes B$, $A \circ C$, $A \odot B$ and $A \ast B$ be the Kronecker, Hadamard, Tracy-Singh and Khatri-Rao products, respectively. The definitions of the mentioned four matrix products are given by Liu in [1]. Additionally, Liu [1, p. 269] also shows that the Khatri-Rao product can be viewed as a generalized Hadamard product and the Kronecker product is a special case of the Khatri-Rao or Tracy-Singh products. The purpose of this present paper is to establish several inequalities for the Khatri-Rao product of complex positive definite matrices, and thereby generalize some inequalities involving the Hadamard and Khatri-Rao products of matrices in [1, Eq. (13) and Theorem 8], [6, Eq. (3), Lemmas 2.1 and 2.2, Theorems 3.1 and 3.2], and [3, Eqs. (2) and (9)].

Let $S(m)$ be the set of all complex Hermitian matrices of order $m$, and $S^+(m)$ the set of all complex positive definite Hermitian matrices of order $m$. For $M$ and $N$ in $S(m)$, we write $M \geq N$ in the Löwner ordering sense, i.e., $M - N$ is positive semidefinite. For a matrix $A \in S^+(m)$, we denote by $\lambda_1(A)$ and $\lambda_m(A)$ the largest and smallest eigenvalue of $A$, respectively. Let $B^*$ be the conjugate transpose matrix of the complex matrix $B$. We denote the $n \times n$ identity matrix by $I_n$, also we write $I$ when the order of the matrix is clear.

2. Some Lemmas. In this section, we give some preliminaries.

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Lemma 2.1. There exists an \(mp \times \sum m_i p_i\) real matrix \(Z\) such that \(Z^T Z = I\) and
\[
(2.1) \quad A \ast B = Z^T (A \odot B) Z
\]
for any \(A \in S(m)\) and \(B \in S(p)\) partitioned as follows:
\[
A = \begin{bmatrix}
A_{11} & \cdots & A_{1t} \\
\vdots & \ddots & \vdots \\
A_{11} & \cdots & A_{tt}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & \cdots & B_{1t} \\
\vdots & \ddots & \vdots \\
B_{t1} & \cdots & B_{tt}
\end{bmatrix},
\]
where \(A_{ii} \in S(m_i)\) and \(B_{ii} \in S(p_i)\) for \(i = 1, 2, \ldots, t\).

Proof. Let
\[
Z_i = \begin{bmatrix}
O_{i1} & \cdots & O_{i, i-1} & I_{m_i p_i} & O_{i, i+1} & \cdots & O_{it}
\end{bmatrix}^T, \quad i = 1, 2, \ldots, t,
\]
where \(O_{ik}\) is the \(m_i p_k \times m_i p_i\) zero matrix for any \(k \neq i\). Then \(Z_i^T Z_i = I\) and
\[
Z_i^T (A_{ij} \odot B) Z_j = Z_i^T (A_{ij} \odot B_{ii}) Z_j = A_{ij} \odot B_{ij}, \quad i, j = 1, 2, \ldots, t.
\]

Letting \(Z = \begin{bmatrix} Z_1 & \cdots & Z_t \end{bmatrix}\), the lemma follows by a direct computation. \(\square\)

If \(t = 2\) in Lemma 2.1, then Eq. (2.1) becomes Eq. (13) of [1].

Corollary 2.2. There exists a real matrix \(Z\) such that \(Z^T Z = I\) and
\[
(2.2) \quad M_1 \ast \cdots \ast M_k = Z^T (M_1 \odot \cdots \odot M_k) Z
\]
for any \(M_i \in S(m(i))\) \((1 \leq i \leq k, \ k \geq 2)\) partitioned as
\[
(2.3) \quad M_i = \begin{bmatrix}
N_{11}^{(i)} & \cdots & N_{1t}^{(i)} \\
\vdots & \ddots & \vdots \\
N_{t1}^{(i)} & \cdots & N_{tt}^{(i)}
\end{bmatrix},
\]
where \(N_{ij}^{(i)} \in S(m(i)_j)\) for any \(1 \leq i \leq k\) and \(1 \leq j \leq t\).

Proof. We proceed by induction on \(k\). If \(k = 2\), the corollary is true by Lemma 2.1. Suppose the corollary is true when \(k = s\), i.e., there exists a real matrix \(P\) such that \(P^T P = I\) and \(M_1 \ast \cdots \ast M_s = P^T (M_1 \odot \cdots \odot M_s) P\), we will prove that it is true when \(k = s + 1\). In fact,
\[
M_1 \ast \cdots \ast M_{s+1} =
\]
\[
= (M_1 \ast \cdots \ast M_s) * M_{s+1}
\]
\[
= P^T (M_1 \odot \cdots \odot M_s) P * M_{s+1}
\]
\[
= Q^T [P^T (M_1 \odot \cdots \odot M_s) P \odot M_{s+1}] Q \quad (Q^T Q = I)
\]
\[
= Q^T [P^T (M_1 \odot \cdots \odot M_s) P \odot (I_{m(s+1)} M_{s+1} I_{m(s+1)})] Q
\]
\[
= Q^T (P^T \odot I_{m(s+1)}) [(M_1 \odot \cdots \odot M_s) \odot M_{s+1}] (P \odot I_{m(s+1)}) Q.
\]
Letting $Z = (P \odot I_{m(s+1)}) Q$, the corollary follows. □

If the Khatri-Rao and Tracy-Singh products are replaced by the the Hadamard and Kronecker products in Corollary 2.2, respectively, then (2.2) becomes Lemma 2.2 in [6].

**Lemma 2.3.** Let $A$ and $B$ be compatibly partitioned matrices, then $(A \odot B)^* = A^* \odot B^*$.

**Proof.**

\[
(A \odot B)^* = \left( (A_{ij} \odot B_{kl})_{ij} \right)^* = \left( (A_{ij} \odot B_{kl})^* \right)_{ji} = \left( (A^*_i \odot B^*_k)_{ik} \right)_{ji} = (A^*_i \odot B^*_k)_{ji} = A^* \odot B^*. \]

**Definition 2.4.** Let the spectral decomposition of $A \in S^+(m)$ be

\[
A = U_A^* D_A U_A = U_A^* \text{diag}(d_1, \ldots, d_m) U_A,
\]

where $d_i > 0$ for all $i$. For any $c \in \mathbb{R}$, we define the power of matrix $A$ as follows

\[
A^c = U_A^* D_A^c U_A = U_A^* \text{diag}(d_1^c, \ldots, d_m^c) U_A.
\]

**Lemma 2.5.** Let $A \in S^+(m), B \in S^+(p)$ and $c \in \mathbb{R}$, then

i) $A \odot B \in S^+(mp)$, $\lambda_i(A \odot B) = \lambda_1(A) \lambda_i(B)$, and $\lambda_{mp}(A \odot B) = \lambda_m(A) \lambda_p(B)$;

ii) $(A \odot B)^c = A^c \odot B^c$.

**Proof.** Let $A = U_A^* D_A U_A$ and $B = U_B^* D_B U_B$ be the spectral decompositions of $A$ and $B$, respectively. From Lemma 2.3 and [1, Theorem 1(a)], we derive

\[
(2.4)[U_A \odot U_B]^*(U_A \odot U_B) = (U_A^* \odot U_B^*)(U_A \odot U_B) = (U_A^* U_A) \odot (U_B^* U_B) = I_{mp}
\]

\[
(2.5) A \odot B = (U_A^* D_A U_A) \odot (U_B^* D_B U_B) = (U_A^* U_A^*)(D_A \odot D_B)(U_A \odot U_B) = (U_A \odot U_B)(D_A \odot D_B)(U_A \odot U_B).
\]

The lemma follows from (2.4), (2.5), and the definitions of $A \odot B$ and $(A \odot B)^c$. □

If the Tracy-Singh product is placed by the Kronecker product in Lemma 2.5, then ii) of Lemma 2.5 becomes Lemma 2.1 in [6].

**Corollary 2.6.** Let $M_i \in S^+(m(i))$ for $i = 1, 2, \ldots, k$, $n = \prod_{i=1}^{k} m(i)$ and $c \in \mathbb{R}$, then

i) $M_1 \odot \cdots \odot M_k \in S^+(n)$, $\lambda_1(M_1 \odot \cdots \odot M_k) = \lambda_1(M_1) \cdots \lambda_1(M_k)$ and

$\lambda_n(M_1 \odot \cdots \odot M_k) = \lambda_{m(i)}(M_i)$;

ii) $(M_1 \odot \cdots \odot M_k)^c = M_1^c \odot \cdots \odot M_k^c$.

**Proof.** Using Lemma 2.5, the corollary follows by induction. □

If the Tracy-Singh product is replaced by the Kronecker product in Corollary 2.6, then ii) of Corollary 2.6 becomes Eq. (3) in [6].

**Lemma 2.7.** [4], [5] Let $H \in S^+(n)$ and $V$ be a complex matrix of order $n \times m$ such that $V^*V = I_m$, then
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i) \((V^*H^iV)^{1/r} \leq (V^*H^iV)^{1/s}\), where \(r\) and \(s\) are two real numbers such that \(s > r\), and either \(s \notin (-1, 1)\) and \(r \notin (-1, 1)\) or \(s \geq 1 \geq r \geq \frac{1}{2}\) or \(r \leq -1 \leq s \leq -\frac{1}{2}\);

ii) \((V^*H^iV)^{1/s} \leq \Delta(s, r)(V^*H^iV)^{1/r}\), where \(r\) and \(s\) are two real numbers such that \(s > r\) and either \(s \notin (-1, 1)\) or \(r \notin (-1, 1)\), \(\Delta(s, r) = \lambda_n(H)\), \(w = \lambda_n(H)\) and \(\delta = \frac{w}{\theta}\).

iii) \((V^*H^iV)^{1/s} - (V^*H^iV)^{1/r} \leq \Delta(s, r)I\), where \(\Delta(s, r) = \max_{\theta \in [0, 1]} \{[\theta W^s + (1 - \theta)w^s]^{1/s} - [\theta W^r + (1 - \theta)w^r]^{1/r}\}\), and \(r, s, W, w\) and \(\delta\) are as in ii).

3. Main results. In this section, we establish some inequalities for the Khatri-Rao product of matrices.

Theorem 3.1. Let \(M_i \in S^+(m(i))\) (1 \(\leq i \leq k\) be partitioned as in (2.3) and
\(n = \prod_{i=1}^k m(i)\), then

(i) \((M_i^* \cdots \cdots M_k^*)^{1/s} \geq (M_i^* \cdots \cdots M_k^*)^{1/r}\), where \(r\) and \(s\) are as in i) of Lemma 2.7;

(ii) \((M_i^* \cdots \cdots M_k^*)^{1/s} \leq \Delta(s, r)(M_i^* \cdots \cdots M_k^*)^{1/r}\), where \(W = \prod_{i=1}^k \lambda_1(M_i)\) and \(w = \prod_{i=1}^k \lambda_{m(i)}(M_i)\), and \(r, s, \delta\ and \Delta(s, r)\ are as in ii) of Lemma 2.7;

(iii) \((M_i^* \cdots \cdots M_k^*)^{1/s} - (M_i^* \cdots \cdots M_k^*)^{1/r} \leq \Delta(s, r)I\), where \(W = \prod_{i=1}^k \lambda_1(M_i)\) and \(w = \prod_{i=1}^k \lambda_{m(i)}(M_i)\), and \(r, s, \delta\ and \Delta(s, r)\ is as in iii) of Lemma 2.7.

Proof. Let \(H = M_1 \odot \cdots \odot M_k\), then \(H \in S^+(n)\), \(\lambda_1(H) = \prod_{i=1}^k \lambda_1(M_i)\) and \(\lambda_n(H) = \prod_{i=1}^k \lambda_{m(i)}(M_i)\) from i) of Corollary 2.6. Therefore, using ii) of Corollary 2.6, Corollary 2.2, and Lemma 2.7,

\[(M_i^* \cdots \cdots M_k^*)^{1/r} = (Z^T(M_i^* \cdots \cdots M_k^*)Z)^{1/r} = (Z^T(M_1 \odot \cdots \odot M_k)^rZ)^{1/r} \leq (Z^T(M_1 \odot \cdots \odot M_k)^sZ)^{1/s} = (Z^T(M_1^* \cdots \cdots M_k^*)Z)^{1/s} = (M_i^* \cdots \cdots M_k^*)^{1/s},\]

\[(M_i^* \cdots \cdots M_k^*)^{1/s} = (Z^T(M_i^* \cdots \cdots M_k^*)Z)^{1/s} = (Z^T(M_1 \odot \cdots \odot M_k)^sZ)^{1/s} \leq \Delta(s, r)(Z^T(M_1 \odot \cdots \odot M_k)^rZ)^{1/r} = \Delta(s, r)(Z^T(M_1^* \cdots \cdots M_k^*)Z)^{1/r} = \Delta(s, r)(M_i^* \cdots \cdots M_k^*)^{1/r},\]
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\[(M_1 \cdots M_k)^{1/s} - (M_1^T \cdots M_k^T)^{1/r} = \]
\[= (Z^T (M_1 \cdots M_k)^{1/s} - (Z^T (M_1 \cdots M_k)^T)^{1/r} \leq \Delta(s, r)I. \]

If the Khatri-Rao and Tracy-Singh products are replaced by the Hadamard and Kronecker products in Theorem 3.1, respectively, then (i) becomes Theorem 3.1 in [6], and (ii) and (iii) become Theorem 3.2 in [6].

**Theorem 3.2.** Let \(M_i \in S^+(m(i))\) \(1 \leq i \leq k\) be partitioned as in (2.3), then

\[(3.1) \quad (M_1 \cdots M_k)^{-1} \leq M_1^{-1} \cdots M_k^{-1},\]
\[(3.2) \quad M_1^{-1} \cdots M_k^{-1} \leq \frac{(W + w)^2}{4W^2}(M_1 \cdots M_k)^{-1},\]
\[(3.3) \quad M_1 \cdots M_k - (M_1^{-1} \cdots M_k^{-1})^{-1} \leq (\sqrt{W} - \sqrt{w})^2 I,\]
\[(3.4) \quad (M_1 \cdots M_k)^{-1} \leq M_1^2 \cdots M_k^2,\]
\[(3.5) \quad M_1 \cdots M_k \leq \frac{(W + w)^2}{4W^2}(M_1 \cdots M_k)^2,\]
\[(3.6) \quad (M_1 \cdots M_k)^2 - M_1^2 \cdots M_k^2 \leq \frac{1}{4}(W - w)^2 I,\]
\[(3.7) \quad M_1 \cdots M_k \leq (M_1^2 \cdots M_k^2)^{1/2},\]
\[(3.8) \quad (M_1^2 \cdots M_k^2)^{1/2} \leq \frac{W + w}{2\sqrt{W^2}}(M_1 \cdots M_k),\]
\[(3.9) \quad (M_1^2 \cdots M_k^2)^{1/2} - M_1 \cdots M_k \leq \frac{(W - w)^2}{4(W + w)} I,\]

where \(W\) and \(w\) are as in Theorem 3.1.

**Proof.** Noting that \(G \geq H > O\) if and only if \(H^{-1} \geq G^{-1} > O\) [2], we obtain (3.1), (3.2) and (3.3) by choosing \(r = -1\) and \(s = 1\) in Theorem 3.1. Similarly, (3.7), (3.8) and (3.9) can be obtained by choosing \(r = 1\) and \(s = 2\) in Theorem 1. Thereby, using that \(G \geq H > 0\) implies \(G^2 \geq H^2 > 0\), we derive that (3.4) and (3.5) hold.

Liu and Neudecker [3] show that

\[(3.10) \quad V^*A^2V - (V^*AV)^2 \leq \frac{1}{4}(\lambda_1(A) - \lambda_m(A))^2 I\]

for \(A \in S^+(m)\) and \(V^*V = I\). Replacing \(A\) by \(M_1 \cdots M_k\) and \(V\) by \(Z\) in (3.10), we obtain (3.6).

If we replace the Khatri-Rao product by the Hadamard product in (3.1), (3.2), (3.3), (3.4), (3.7), (3.8) and (3.9), then we obtain some inequalities in [6]. If choosing \(t = 2\) and considering the real positive definite matrices in Theorem 3.2, then Theorem 3.2 becomes Theorem 8 in [1]. If choosing \(t = 2\) and replacing the Khatri-Rao product by the Hadamard product in (3.6) and (3.8), respectively, then we obtain Eqs. (2) and (9) of [3].

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