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NON-TRIVIAL SOLUTIONS TO CERTAIN MATRIX EQUATIONS

AIIHA LI† AND DUANE RANDALL†

Abstract. The existence of non-trivial solutions $X$ to matrix equations of the form $F(X, A_1, A_2, \ldots, A_s) = G(X, A_1, A_2, \ldots, A_s)$ over the real numbers is investigated. Here $F$ and $G$ denote monomials in the $(n \times n)$-matrix $X = (x_{ij})$ of variables together with $(n \times n)$-matrices $A_1, A_2, \ldots, A_s$ for $s \geq 1$ and $n \geq 2$ such that $F$ and $G$ have different total positive degrees in $X$. An example with $s = 1$ is given by $F(X, A) = X^2 AX$ and $G(X, A) = AXA$ where $\text{deg}(F) = 3$ and $\text{deg}(G) = 1$. The Borsuk-Ulam Theorem guarantees that a non-zero matrix $X$ exists satisfying the matrix equation $F(X, A_1, A_2, \ldots, A_s) = G(X, A_1, A_2, \ldots, A_s)$ in $(n^2 - 1)$ components whenever $F$ and $G$ have different total odd degrees in $X$. The Lefschetz Fixed Point Theorem guarantees the existence of special orthogonal matrices $X$ satisfying matrix equations $F(X, A_1, A_2, \ldots, A_s) = G(X, A_1, A_2, \ldots, A_s)$ whenever $\text{deg}(F) > \text{deg}(G) \geq 1$. $A_1, A_2, \ldots, A_s$ are in $SO(n)$, and $n \geq 2$. Explicit solution matrices $X$ for the equations with $s = 1$ are constructed. Finally, nonsingular matrices $A$ are presented for which $X^2 AX = AXA$ admits no non-trivial solutions.

Key words. Polynomial equation, Matrix equation, Non-trivial solution.

AMS subject classifications. 39B42, 15A24, 55M20, 47J25, 39B72

1. Matrix equations involving special monomials. Given monomials $F(X, A_1, A_2, \ldots, A_s)$ and $G(X, A_1, A_2, \ldots, A_s)$ in the $(n \times n)$-matrix $X = (x_{ij})$ of variables with $n \geq 2$ and with total degrees $\text{deg}(F) > \text{deg}(G) \geq 1$ in $X$, we investigate the existence of non-trivial solutions $X$ to the matrix equation

\[(1.1) \quad F(X, A_1, A_2, \ldots, A_s) = G(X, A_1, A_2, \ldots, A_s).\]

For example, $X^2 AX = AXA$ is such an equation. We note that in this equation, $F(X, A) = X^2 AX$ and $G(X, A) = AXA$ both contain products $AX$ and $XA$. We first record a sufficient condition for non-trivial solutions to the equation (1.1).

Proposition 1.1. Suppose that the monomials $F(X, A_1, A_2, \ldots, A_s)$ and $G(X, A_1, A_2, \ldots, A_s)$ both contain the product $A_i X$ or both contain $XA_i$, for some $i$ with $1 \leq i \leq s$. Whenever $A_i$ is a singular matrix, the matrix equation (1.1) admits non-trivial solutions $X$.

Proof. Let $X$ be any non-zero $(n \times n)$-matrix whose columns belong to the null space of $A_i$ whenever both $F$ and $G$ contain $A_i X$. Similarly, let $X$ be any non-zero matrix whose rows belong to the null space of $A_i^T$ in case both $F$ and $G$ contain $XA_i$. \( \square \)

Our principal result affirms the existence of non-trivial solutions $X$ to matrix equations $F(X, A_1, A_2, \ldots, A_s) = G(X, A_1, A_2, \ldots, A_s)$ whenever $A_1, A_2, \ldots, A_s$ belong to the special orthogonal group $SO(n)$ for any integer $n \geq 2$. We first construct explicit non-trivial solutions for such matrix equations with $s = 1$.

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Proposition 1.2. Every matrix equation $F(X, A) = G(X, A)$ for monomials $F$ and $G$ with different total odd degrees in $X$ admits a non-trivial solution $X$ of the form $A^{p/q}$ whenever $A$ belongs to $SO(n)$ for $n \geq 2$.

Proof. We may assume that $\deg(F) > \deg(G) \geq 1$. We seek a solution $X = A^{p/q}$ to the matrix equation $F(X, A) \cdot (G(X, A))^{-1} = I_n$. The classical Spectral Theorem for $SO(n)$ in [3] affirms that $A = C^{-1}BC$ for matrices $B$ and $C$ in $SO(n)$ where $B$ consists of blocks of non-trivial rotations $R(\theta_i) = \begin{bmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{bmatrix}$ along the diagonal together with an identity submatrix $I_i$. A solution $X$ commuting with powers of $A$ reduces the matrix equation $F(X, A) \cdot (G(X, A))^{-1} = I_n$ to $X^{\deg(F) - \deg(G)} = A^p$ for some integer $p$. Setting $q = \deg(F) - \deg(G)$, we obtain $X = A^{p/q} = C^{-1}B^{p/q}C$ where $B^{p/q}$ consists of blocks of rotations $R(p\theta_i/q)$ along the diagonal together with $I_i$. \( \Box \)

We now establish the existence of non-trivial solutions to many matrix equations via the Lefschetz Fixed Point Theorem. For example, the matrix equation $X^2A_1A_2^2\cdots A_s^2A_1^2 = A_1^2A_2A_1^2XA_2^2$ admits rotation matrices as solutions whenever $A_1$ and $A_2$ belong to $SO(n)$ for any $n \geq 2$.

Theorem 1.3. There is a solution $X$ in $SO(n)$ to any matrix equation $F(X, A_1, A_2, \cdots, A_s) = G(X, A_1, A_2, \cdots, A_s)$, i.e., equation (1.1), with $\deg(F) > \deg(G) \geq 1$ and $n \geq 2$ whenever the $(n \times n)$-matrices $A_i$ belong to $SO(n)$ for $1 \leq i \leq s$.

Proof. Solutions $X$ in $SO(n)$ to the matrix equation (1.1) are precisely the fixed points of the continuous function $H : SO(n) \rightarrow SO(n)$ defined by $H(X) = X \cdot F(X, A_1, A_2, \cdots, A_s) \cdot [G(X, A_1, A_2, \cdots, A_s)]^{-1}$. The existence of fixed points for the map $H$ follows from its non-zero Lefschetz number $L(H)$. We affirm that $L(H) = (\deg(G) - \deg(F))^m$ where $n = 2m$ or $n = 2m + 1$.

Brown in [1, p.49], calculated the Lefschetz number $L(p_k)$ for the $k$th power map $p_k : G \rightarrow G$ defined by $p_k(g) = g^k$ on any compact connected topological group $G$ which is an ANR (absolute neighborhood retract). He proved that $L(p_k) = (1 - k)^\lambda$ where $\lambda$ denotes the number of generators for the primitively generated exterior algebra $H^*(G; \mathbb{Q})$. For $G = SO(n)$, $\lambda = m$ where $n = 2m$ or $n = 2m + 1$; see [4, p.956]. It suffices to show that $H$ is homotopic to $p_k : SO(n) \rightarrow SO(n)$ where $k = \deg(G) - \deg(F) + 1$.

For each $i$ with $1 \leq i \leq s$, let $g_i : [0, 1] \rightarrow SO(n)$ denote any path in $SO(n)$ from $A_i = g_i(0)$ to the identity matrix $I_n = g_i(1)$. Replacing each matrix $A_i$ by the function $g_i$ in $H : SO(n) \rightarrow SO(n)$ produces a homotopy $H_t : SO(n) \rightarrow SO(n)$ for $0 \leq t \leq 1$ with $H_0 = H$ and $H_1 = p_k$. Thus $L(H) = (1 - k)^m = (\deg(G) - \deg(F))^m \neq 0$ so $H$ has a fixed point. \( \Box \)

We now establish the existence of non-trivial solutions $X$ to all matrix equations of the form (1.1) in any $(n^2 - 1)$ components whenever $F$ and $G$ have different odd degrees in $X$ for any $s \geq 1$ and $n \geq 1$. For example, given any $(n \times n)$-matrix $A$, there is a non-zero matrix $X$ such that $X^2AX = AXA$ in at least $(n^2 - 1)$-components. This is a best possible result, since we shall construct matrices $A$ for which $X^2AX = AXA$ admits only the trivial solution. We use the Borsuk-Ulam Theorem following the paper of Lam [2] to prove the following.
Theorem 1.4. Given any monomials $F(X, A_1, A_2, \ldots, A_s)$ and $G(X, A_1, A_2, \ldots, A_s)$ in the $(n \times n)$-matrix $X = (x_{ij})$ together with arbitrary matrices $A_1, A_2, \ldots, A_s$ in $M_n(\mathbb{R})$ for $n \geq 2$ such that $\deg(F)$ and $\deg(G)$ are different odd integers, the matrix equation (1.1) admits a non-trivial solution $X$ in $(n^2 - 1)$ components.

Proof. Set each component of the matrix $F(X, A_1, A_2, \ldots, A_s) - G(X, A_1, A_2, \ldots, A_s)$ equal to zero, except for one fixed component. We obtain $n^2 - 1$ polynomial equations in the $n^2$ variables $x_{ij}$. Now each component of $F(X, A_1, A_2, \ldots, A_s)$ and $G(X, A_1, A_2, \ldots, A_s)$ is a homogeneous polynomial whose degree is given by $\deg(F)$ or $\deg(G)$, respectively. Consequently, every monomial in the $(n^2 - 1)$ polynomial equations has an odd degree, either $\deg(F)$ or $\deg(G)$. Suppose that the system of $n^2 - 1$ polynomial equations in the $n^2$ variables had no non-zero solution. As $X$ ranges over the unit sphere $S^{n^2-1}$ in $\mathbb{R}^{n^2}$, normalization of the non-zero vectors $F(X, A_1, A_2, \ldots, A_s) - G(X, A_1, A_2, \ldots, A_s) \in \mathbb{R}^{n^2-1}$ produces a continuous function $P : S^{n^2-1} \to S^{n^2-2}$. Since $\deg(F)$ and $\deg(G)$ are distinct odd integers, $P$ commutes with the antipodal maps on the spheres. But the classical Borsuk-Ulam Theorem [5, p.266] affirms that no such function $P$ can exist. $\square$

2. The special matrix equation $X^2AX - AXA = 0$. Given any non-zero $(n \times n)$-matrix $A$, consider the matrix equation

\begin{equation}
X^2AX - AXA = 0.
\end{equation}

In this section we discuss solution types of the equation (2.1). We list a few obvious facts about solutions.

Lemma 2.1.
1. If $X \in M_n(\mathbb{R})$ is a solution to (2.1), then $-X$ is a solution too;
2. If $|A| < 0$, then (2.1) has no nonsingular solutions.
3. If $A = B^2$ for some $B \in M_n(\mathbb{R})$, then $X = B$ is a non-trivial solution.
4. If $A^m = I_n$ and $m$ is odd, then $X = A^{-\frac{m-1}{2}}$ is a non-trivial solution.
5. If $A^3 = 0$, then $X = kA$ is a solution to (2.1) for all $k \in \mathbb{R}$.
6. Suppose $P$ is a nonsingular matrix and $B = PAP^{-1}$. Then a matrix $X$ satisfies the equation $X^2AX - AXA = 0$ if and only if $Y = XPXP^{-1}$ satisfies $Y^2BY - BYB = 0$.

By Lemma 2.1(6.), when the matrix $A$ is diagonalizable, the equation (2.1) can be reduced to the diagonal case. We first characterize all solutions for scalar matrices $A$.

Theorem 2.2. Let $A = aI_n \in M_n(\mathbb{R})$, where $n > 1$ and $a \neq 0$. Then the equation (2.1) has non-trivial solutions. Furthermore, the solution set (over the real numbers) consists of matrices in $M_n(\mathbb{R})$ of the form

$$X = Q^{-1} \begin{pmatrix} 
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n 
\end{pmatrix} Q,$$

where $Q$ is a nonsingular matrix with complex entries and $\lambda_i = 0, \sqrt{a}$, or $-\sqrt{a}$ for $i = 1, 2, \ldots, n$. In particular, nonsingular solutions are those with $\lambda_1\lambda_2\cdots\lambda_n$ not
equal to zero. In summary,
1. If \( a^n > 0 \) with \( n > 2 \), then (2.1) has both singular solutions and nonsingular solutions;
2. If \( a^n < 0 \) and \( n > 2 \), then (2.1) has only singular solutions;
3. In case of \( a < 0 \) and \( n = 2 \), there are nonsingular solutions, but no non-trivial singular solutions to (2.1).

Proof. Suppose \( \mathbf{X} \) is a solution to (2.1). Then
\[
\mathbf{X}^2 \mathbf{A} \mathbf{X} - \mathbf{A} \mathbf{X}^2 = a \mathbf{X}^3 - a^2 \mathbf{X} = 0 \iff \mathbf{X}^3 = a \mathbf{X}.
\]

Every matrix \( \mathbf{X} \) satisfying \( \mathbf{X}^3 = a \mathbf{X} \) is diagonalizable over the complex numbers. Suppose \( \mathbf{X} \) is similar to a diagonal matrix \( \mathbf{D} = \text{diag}(\lambda_i) \), then \( \mathbf{X}^3 = a \mathbf{X} \iff \mathbf{D}^3 = a \mathbf{D} \).

This implies \( \lambda_i^3 = a \) or \( \lambda_i = 0 \) for \( i = 1, 2, \ldots, n \). Thus all the solutions to (2.1) are the real matrices similar to these diagonal matrices. Claim 1. is obvious by choosing appropriate (real) \( \lambda_i \)'s. For 2., \( |\mathbf{A}| < 0 \). By Lemma 2.1(2.), equation (2.1) has no nonsingular solutions. The existence of singular solutions over the real numbers is based on the fact that every \( 2 \times 2 \) diagonal matrix of the form
\[
\begin{bmatrix}
\lambda & 0 \\
0 & -\lambda
\end{bmatrix},
\]
where \( \lambda \) is a non-real complex number, can be realized by a complex nonsingular matrix \( \mathbf{Q} \). Assume \( \lambda = \sqrt{-a} \cdot i \), one can check that \( \mathbf{Q} = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \) gives
\[
\mathbf{Q}^{-1} \begin{bmatrix} \sqrt{-a} \cdot i & 0 \\ 0 & -\sqrt{-a} \cdot i \end{bmatrix} \mathbf{Q} = \begin{bmatrix} 0 & \sqrt{-a} \\ \sqrt{-a} & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}).
\]

Since \( n > 2 \), we always can choose at least one diagonal block of \( \mathbf{D} \) to be
\[
\begin{bmatrix}
\sqrt{-a} \cdot i & 0 \\
0 & -\sqrt{-a} \cdot i
\end{bmatrix}
\]
and extend it to a singular solution by choosing at least one zero diagonal element. In case of \( a < 0 \) and \( n = 2 \), nonsingular solutions are similar to
\[
\begin{bmatrix}
0 & \sqrt{-a} \\
-\sqrt{-a} & 0
\end{bmatrix}.
\]
We show by contradiction that in this case (2.1) has no non-trivial singular solutions.

Assume \( \mathbf{0} \neq \mathbf{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \) is a non-trivial solution to (2.1) and \( |\mathbf{X}| = 0 \). Then \( \mathbf{X}^2 = (x_1 + x_4) \mathbf{X} \Rightarrow (x_1 + x_4)^2 \mathbf{X} = a \mathbf{X} \Rightarrow a = (x_1 + x_4)^2 \geq 0 \), a contradiction. \( \square \)

By Lemma 2.1(6.), if \( \mathbf{A} \) is diagonalizable, we only need to consider the solvability of the equation (2.1) for the similar diagonal matrix. Now let us treat diagonal matrices.

Theorem 2.3. Suppose \( \mathbf{A} \) is a non-zero diagonal matrix which has at least one positive entry. Then the equation \( \mathbf{X}^2 \mathbf{A} \mathbf{X} - \mathbf{A} \mathbf{X} \mathbf{X} = \mathbf{0} \) has non-trivial solutions.

Proof. Let \( \mathbf{A} = \text{diag}(\lambda_i) \). Without loss of generality, let \( \lambda_1 > 0 \). Then the diagonal matrix \( \mathbf{X} = \text{diag}(\alpha_i) \) will give non-trivial solutions, where \( \alpha_1 = \sqrt{\lambda_1} \) and for \( i > 1 \), \( \alpha_i = 0 \) or \( \sqrt{\lambda_i} \) if \( \lambda_i > 0 \). When \( \lambda_i \geq 0 \) for all \( i \), we obtain non-trivial solutions \( \mathbf{X} = \text{diag}(\sqrt{\lambda_i}) \). \( \square \)

Corollary 2.4. For \( n > 1 \), the equation (2.1) has non-trivial solutions for all \( n \times n \) positive definite and all positive semidefinite matrices \( \mathbf{A} \).

We end this section with the following proposition.

Proposition 2.5. Suppose \( \mathbf{A} \in \mathbb{M}_n(\mathbb{R}) \) is similar to a block matrix, i.e., there
exists a nonsingular matrix $P$ such that

$$PAP^{-1} = \begin{bmatrix} A_1 & A_2 & \cdots & A_m \end{bmatrix},$$

where each $A_i$ is a square matrix. Suppose $Y_i$ satisfies $Y_i^2A_iY_i - A_iY_iA_i = 0$, for $i = 1, 2, \cdots, m$. Then the matrix $X = P^{-1}BP$ is a solution to $X^2AX - AXA = 0$, where $B$ is a block matrix with blocks $B_i = Y_i$ or $0$. Thus, if at least one of the solutions $Y_i$'s is not zero, we can extend it to non-trivial solutions for the equation $X^2AX = AXA$.

**Theorem 2.6.** Let $A$ be a real $n \times n$ matrix with distinct negative eigenvalues. Then the equation $X^2AX = AXA$ admits only the trivial solution.

**Proof.** Suppose first that $X$ is an invertible solution. Then we have

$$A^{-1}X^2A = XAX^{-1}.$$  

Thus the eigenvalues of $X^2$ are the same as those of $A$. Since the eigenvalues of $A$ are negative and distinct, the eigenvalues of $X$ are all pure imaginary and of distinct modulus. This is impossible.

If $X$ is a singular solution, let $v$ be a null vector of $X$ and observe that $0 = AXAv = XAv$. Thus the null space of $X$ is $A$-invariant. Then there exists an invertible matrix $B$ such that

$$X = B \begin{bmatrix} Y & 0 \\ C & 0 \end{bmatrix} B^{-1} \quad \text{and} \quad A = B \begin{bmatrix} P & 0 \\ D & E \end{bmatrix} B^{-1}.$$  

By Lemma 2.1(6.),

$$\begin{bmatrix} Y & 0 \\ C & 0 \end{bmatrix}^2 \begin{bmatrix} P & 0 \\ D & E \end{bmatrix} \begin{bmatrix} Y & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} P & 0 \\ D & E \end{bmatrix} \begin{bmatrix} Y & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ D & E \end{bmatrix}.$$  

This yields $Y^2PY = PYP$ and by induction $Y = 0$. (See Theorem 3.3 for the $2 \times 2$ case.) This means that

$$\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}^2 = 0 = \begin{bmatrix} 0 & 0 \\ ECP & 0 \end{bmatrix},$$  

which gives $ECP = 0$. Since $E$ and $P$ are invertible, $C = 0$, so $X$ is the trivial solution. \hfill \square

**3. The special case $n = 2$.** In this section, we focus on the equation (2.1) for $2 \times 2$ matrices. Denote

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$
We first consider the existence of non-trivial solutions to (2.1) when $A$ is an orthogonal matrix. When $A$ is orthogonal with $|A| = 1$, the existence of a non-trivial (orthogonal) solution $X = A^{-1}/2$ is given in Proposition 1.2.

**Proposition 3.1.** Let $A$ be an orthogonal matrix in $M_2(\mathbb{R})$ with $|A| = -1$. A non-trivial solution to (2.1) is given by $X = \frac{1}{2} \begin{bmatrix} 1 + a_1 & a_2 \\ a_2 & 1 - a_1 \end{bmatrix}$.

**Proof.** When $|A| = -1$, $A$ is a symmetric matrix with two distinct eigenvalues 1 and $-1$. Thus $A$ is diagonalizable to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. By Lemma 2.1(6.) and Theorem 2.3, (2.1) has a non-trivial solution. A matrix of the form $X = P \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}$ is a non-trivial singular solution to (2.1) when $P$ satisfies $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The solution $X = \frac{1}{2} \begin{bmatrix} 1 + a_1 & a_2 \\ a_2 & 1 - a_1 \end{bmatrix}$ is obtained by finding such a matrix $P$ made of two linearly independent eigenvectors of $A$ via linear algebra (refer to the proof of Theorem 2.2).

Now we discuss more general cases. In the next theorem, we show constructively that the equation (2.1) has non-trivial solutions for a large group of two by two matrices $A$ (over the real numbers).

**Theorem 3.2.** Consider $0 \neq A \in M_2(\mathbb{R})$. The equation (2.1) has non-trivial solutions in the following cases:

1. $A$ has two distinct real eigenvalues, not both negative.
2. $A$ is a scalar matrix.
3. $A$ is a non-scalar matrix with a repeated non-negative eigenvalue.

**Proof.** By Lemma 2.1 and Theorem 2.3, the first is true. The second claim is from Theorem 2.2. For the third, without loss of generality, we may assume

$$A = \begin{bmatrix} a_1 & 0 \\ a_3 & a_1 \end{bmatrix},$$

where $0 \leq a_1$ and $a_3 \neq 0$. If $a_1 = 0$, the matrix $X = \begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix}$ gives a non-trivial solution to (2.1) for any real number $x_3 \neq 0$. If $a_1 \neq 0$, the lower triangular matrix $X = \begin{bmatrix} a_3/(2\sqrt{a_1}) & 0 \\ \sqrt{a_1} & \sqrt{a_1} \end{bmatrix}$ gives a non-trivial solution to (2.1).

We note that by Proposition 2.5, we can extend solutions to (2.1) for the $2 \times 2$ case to solutions for $(n \times n)$-matrices. Finally, we construct non-zero matrices $A$ for which $X^2AX = AXA$ admits only the trivial solution.

**Theorem 3.3.** The equation $X^2AX = AXA$ admits only the trivial solution for any $A \in M_2(\mathbb{R})$ having two distinct negative eigenvalues or having a single negative eigenvalue of geometric multiplicity 1.

**Proof.** For the first case, it is sufficient to assume $A = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$, where $\lambda_1 > \lambda_2 > 0$. Suppose $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ is a solution. Then $|X| = 0$ or $\pm \sqrt{\lambda_1\lambda_2}$ since
A is nonsingular. By comparing the non-diagonal entries of $X^2AX$ and $AXA$, we obtain the following two equations

\begin{align}
&x_2(\lambda_1 x_1^2 + \lambda_1 x_2 x_3 + \lambda_2 x_1 x_4 + \lambda_2 x_2^2 + \lambda_1 \lambda_2) = 0 \\
x_3(\lambda_1 x_1^2 + \lambda_1 x_1 x_4 + \lambda_2 x_2 x_3 + \lambda_2 x_4^2 + \lambda_1 \lambda_2) = 0.
\end{align}

First we assume $0 \neq |X| = \sqrt{\lambda_1 \lambda_2}$. Then $x_2x_3 = x_1x_4 - \sqrt{\lambda_1 \lambda_2}$. Thus (3.1) becomes

\begin{align}
&x_2(\lambda_1 x_1^2 + (\lambda_1 + \lambda_2)x_1 x_4 + \lambda_2 x_2^2 + \lambda_1 \lambda_2 - \lambda_1 \sqrt{\lambda_1 \lambda_2}) = 0 \\
x_3(\lambda_1 x_1^2 + (\lambda_1 + \lambda_2)x_1 x_4 + \lambda_2 x_4^2 + \lambda_1 \lambda_2 - \lambda_2 \sqrt{\lambda_1 \lambda_2}) = 0.
\end{align}

If $x_2x_3 \neq 0$, then equations in (3.2) imply $\sqrt{\lambda_1 \lambda_2} = \lambda_2 \sqrt{\lambda_1 \lambda_2} \implies \lambda_1 = \lambda_2$, a contradiction. If $x_2x_3 = 0$, we compare the (1,1) entries of $X^2AX$ and $AXA$ to obtain $-\lambda_1 x_1^3 = \lambda_2^2 x_1 \implies x_1 = 0 \implies |X| = 0$, a contradiction again. Therefore $|X| \neq \sqrt{\lambda_1 \lambda_2}$.

Now consider the case $|X| = 0$, i.e., $x_1, x_4 = x_2, x_3$. By matrix multiplication, we have

$$X^2AX = -(x_1 + x_4)(\lambda_1 x_1 + \lambda_2 x_4)
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\begin{bmatrix}
\lambda_1 x_1 \\
\lambda_1 x_2 \\
\lambda_2 x_3 \\
\lambda_2 x_4
\end{bmatrix} = AXA.$$

If $x_2 \neq 0$ or $x_3 \neq 0$, then $(x_1 + x_4)(\lambda_1 x_1 + \lambda_2 x_4) = -\lambda_1 \lambda_2$ by comparing the non-diagonal entries. Apply this to the diagonal entries, we obtain $\lambda_1 \lambda_2 x_1 = -\lambda_2^2 x_1$ and $\lambda_1 \lambda_2 x_4 = -\lambda_2^2 x_4 \implies x_1 = x_4 = 0$. Thus $X^2AX = 0 \implies AXA = 0 \implies X = 0$, since $A$ is invertible. This gives only a trivial solution to (2.1). At last, consider the case of $x_2 = 0 = x_3$. Since $x_1x_4 = x_2x_3$, $x_2 = x_3 = 0$. If $x_1 = 0$, compare the (2,2)-entry of $X^2AX$ and $AXA$, we have $\lambda_2 x_1^3 = -\lambda_2^2 x_4 \implies x_4 = 0$. Similarly, $x_3 = 0 \implies x_1 = 0$. Therefore $x_1 = x_2 = x_3 = x_4 = 0$ and $X$ is a trivial solution.

Now assume $A$ has a single negative eigenvalue of geometric multiplicity 1. Let

$$A = \begin{bmatrix}
a_1 & 0 \\
a_3 & a_1
\end{bmatrix}$$

where $a_1 < 0$ and $a_3 \neq 0$. Assume $0 \neq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a solution to (2.1). We claim that $x_2 \neq 0$. If not, the diagonal entries of $X^2AX - AXA$ are $a_1 x_1 (x_1^2 - a_1)$ and $a_1 x_4 (x_4^2 - a_1)$. Since $a_1$ is negative, it forces $x_1 = x_4 = 0$ and then $x_3 = 0$. Now assume $X$ is a singular solution. Then the second row of $X$ is times the first row for some real number $k \neq 0$. By equating the second row minus $k$ times the first row of both $X^2AX$ and $AXA$, we obtain a contradiction. When $X$ is a nonsingular solution, $|X| = a_1$ or $-a_1$. Since $x_2 \neq 0$, $x_3 = \frac{a_1 x_2 + x_3}{x_1 + x_4}$. Then by equating the components of $X^2AX - AXA$, we obtain the following two equations:

$$\begin{cases}
(x_1 + x_4)x_2(a_1 x_1 \pm a_3 x_2 + a_1 x_4) = 0 \\
(x_1 + x_4)(a_1 x_1 x_4 \pm a_3 x_2 x_4 \pm a_1^2 + a_1 x_4) = 0.
\end{cases}$$

This implies $x_1 + x_4 = 0$. Then the (1,1)-component of $X^2AX - AXA$ is $a_1 x_2 a_3$ which can not be zero, a contradiction.

In conclusion, the equation (2.1) has no non-trivial solutions. $\square$

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REFERENCES


