Algebra, Calculus, and the ACT

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Algebra, Calculus, and the ACT

Alex S. Krysl

University of Wyoming
Abstract

It is a common saying that “the hardest part of calculus is the algebra”. Unfortunately, I found that many students lack the necessary, prerequisite algebra skills and knowledge in order to utilize completely the novel calculus concepts learned. For calculus to be effective, algebraic manipulation presents itself as an essential precondition.

As an example, students apply exponent rules throughout the whole differentiation and integration process—like the power rule. For students who lacked a solid background or basis in algebraic concepts like exponent rules, factoring, rewriting equations, and graphing functions, I observed their learning taking place in the calculus classroom as laborious and arduous.

There is another catch here: in high school, many of the students taking this first-year calculus course are juniors preparing to take the ACT. However, the ACT omits calculus from its tests. Teachers are required to prepare their students for the mathematics portion of the ACT, all the while progressing and teaching calculus.

So, the question becomes: How do teachers prepare students to take the ACT while continuing to propel them forward in their knowledge and application of calculus? Through my student teaching experience, I found that through applying a method called “Just-In-Time Review”, combined with specific ACT preparation, students improved their algebraic knowledge while enhancing their learning of calculus and preparing for the ACT.

This work is a collection of all of the pieces of my EdTPA and the majority of my research and data surrounding calculus and my lesson plans in my classroom. I will propose some methods or ideas that will help teachers be successful in regards to both the ACT and their calculus—mathematics—course.
About the School Where You Are Teaching

1. In what type of school do you teach? (Type an “X” next to the appropriate description; if “other” applies, provide a brief description.)
   
   Middle school: _____
   High school: ___X___
   Other (please describe): _____

2. Where is the school where you are teaching located? (Type an “X” next to the appropriate description.)

   City: ______
   Suburb: ___X___
   Town: ______
   Rural: ______

3. List any special features of your school or classroom setting (e.g., charter, co-teaching, themed magnet, remedial course, honors course) that will affect your teaching in this learning segment.

   [ For the high school specifically, about 60% of the students qualify for free or reduced lunch. In addition, 42% of the students are of minority ethnicity. Of the three traditional high schools in the district (LCSD #1), Cheyenne South H.S. is easily the most affected by poverty and low SES situations.

   This is an honors course. Many of these students are sophomores or juniors that are on an accelerated track in mathematics. We move quickly through much of the course material, in order to prepare students for AP Calculus BC—which is the next class in the progression of the mathematics curriculum. ]

4. Describe any district, school, or cooperating teacher requirements or expectations that might affect your planning or delivery of instruction, such as required curricula, pacing plan, use of specific instructional strategies, or standardized tests.

   [ The mathematics department here at South High School has some organizational and curricula stipulations that will affect this class. My school uses concept quizzes based upon specific concepts in the subject area. The concept quizzes are the specific implementation of standards referenced grading practices in the mathematics department. These concepts are different for each subject. I will have to construct and modify concepts and concept quizzes that span and assess the calculus content in order to adhere to departmental standards and expectations.

   ________________

   1 If you need guidance when making a selection, reference the NCES locale category definitions (https://nces.ed.gov/surveys/ruraled/definitions.asp) or consult with your placement school administrator.
Additionally, I will use a specific note taking process and form which all of the South mathematics teachers utilize. It is called a Unit at a Glance. At the end of each unit, students are given a note sheet with essential questions and the titles of the topics covered by the previous unit. They use this sheet to take end of the unit notes by revising the notes and concepts of the previous unit. Particularly for mathematics at South High School, the Unit at a Glance’s are part of our school’s goals to implement Avid note taking strategies in the classroom through the Cornell Way. For the Unit at a Glance’s specifically, they serve to help students revise their notes, receive written feedback from an instructor, and address written feedback—as designated in the Cornell Way. I will be required to provide time to do this as a review. In addition, I will need to put my concepts and essential questions in the format to which these students are accustomed.

Furthermore, the school district where I am student teaching has standardized tests called GVC Common Assessments. (GVC stands for Guaranteed and Viable Curriculum.) These tests are given quarterly. What I teach in this course will be somewhat dependent upon what the GVC assessment.

About the Class Featured in this Learning Segment

1. What is the name of this course?
   [ Honors Calculus A ]

2. What is the length of the course? (Type an “X” next to the appropriate description; if “other” applies, provide a brief description.)
   One semester: __X__
   One year: _______
   Other (please describe):

3. What is the class schedule (e.g., 50 minutes every day, 90 minutes every other day)?
   [ 80 minutes twice a week, and 44 minutes once a week ]

4. Is there any ability grouping or tracking in mathematics? If so, please describe how it affects your class.
   [ Yes, because it is an accelerated Honors program which starts as early as 7th grade. Most of the students have been in the Honors pathway all the way up until this course. This means that the course is quicker than the majority of the math courses offered at the high school level. This course is usually taken by juniors—who often move onto AP Calculus BC for their senior year. ]

5. Identify any textbook or instructional program you primarily use for mathematics instruction. If a textbook, please provide the title, publisher, and date of publication.
   [ We will utilize a textbook: Calculus of a Single Variable, Brooks Cole, 2010 ]

6. List other resources (e.g., electronic whiteboard, graphing calculators, online resources) you use for mathematics instruction in this class.
   [ I will be using the SmartBoard for notes and writing out solution to problems. Also, I will utilize a document cam to show alternative documents and work on the SmartBoard. Everybody in the course has a graphing calculator (i.e. TI 83, 84, 84 plus, etc.), which we will use to graph functions and evaluate limits. Additionally, we will use the calculators to evaluate functions at certain values and create tables by which to determine limits numerically. ]
About the Students in the Class Featured in this Learning Segment

1. Grade-level composition (e.g., all seventh grade; 2 sophomores and 30 juniors):
   [ The class is composed of 5 sophomores, 7 juniors, and 1 senior ]

2. Number of
   - students in the class: __13__
   - males: __6__ females: ___7__

3. Complete the charts below to summarize required or needed supports, accommodations, or modifications for your students that will affect your instruction in this learning segment. As needed, consult with your cooperating teacher to complete the charts. Some rows have been completed in italics as examples. Use as many rows as you need.

   Consider the variety of learners in your class who may require different strategies/supports or accommodations/modifications to instruction or assessment (e.g., students with Individualized Education Programs [IEPs] or 504 plans, students with specific language needs, students needing greater challenge or support, students who struggle with reading, students who are underperforming or those with gaps in academic knowledge).

   For Assessment Task 3, you will choose work samples from 3 focus students. At least one of these students must have a specified learning need. Note: California candidates must include one focus student who is an English language learner.²

<table>
<thead>
<tr>
<th>Students with IEPs/504 Plans</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEPs/504 Plans: Classifications/Needs</td>
</tr>
<tr>
<td>ADHD</td>
</tr>
<tr>
<td>Dyslexia</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Students with Specific Language Needs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language Needs</td>
</tr>
</tbody>
</table>

² California candidates—If you do not have any English language learners, select a student who is challenged by academic English.
<table>
<thead>
<tr>
<th>Other Learning Needs</th>
<th>Number of Students</th>
<th>Supports, Accommodations, Modifications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Struggling Algebra Skills (and Credit Recovery)</td>
<td>1</td>
<td>This student is retaking the class for credit recovery. The student has forgotten most of the material and really struggled the previous school year in the same class. The student struggles with their algebra skills and operations. The student remembers some concepts, but struggles with others, and needs reinforcement and instruction in many areas even preceding this course.</td>
</tr>
</tbody>
</table>
1. **Central Focus**
   
a. Describe the central focus and purpose of the content you will teach in the learning segment.

   [The purpose of this content is to give an introduction to limits, tangent lines, and calculus. The central focus will be on limits and tangent lines as they lead to an understanding of derivatives in the future. I am introducing and teaching limits because they permeate all of calculus. Finally, the central focus will be on evaluating limits analytically—using algebra—because calculus begins and depends upon the algebraic manipulation of limits in order to solve and simplify for the slope of a function.]

   b. Given the central focus, describe how the standards and learning objectives within your learning segment address

   - conceptual understanding,
   - procedural fluency, **AND**
   - mathematical reasoning and/or problem-solving skills.

   [My standards for this content were about limit properties and evaluating limits analytically. In other words, I wanted students to understand the basic properties and methods of solving limits. These standards and learning objectives come directly from the AP Calculus curriculum. The learning objective include: (with the “Students will be able to” sentence starter)

     - “estimate limits of functions” using numerical and graphical information
     - “express limits symbolically using correct notation”
     - “determine limits of functions” using basic theorems of limits and algebraic rules (i.e. sums, products, differences, quotients, and composite functions).
     - “determine limits of functions” using “algebraic manipulation”

   In order to find the limit definition of the derivative in the future, students must first understand limits and be able to understand them at a mathematical and conceptual level in order to be ready for derivatives—and other calculus concepts dealing with limits in the future. In addition, when evaluating limits analytically through algebraic manipulation. The same methods of algebraic manipulation that appear in the introduction of limits, are the same ones that appear when finding the limit definition of the derivatives. The procedures are the same; so, this unit acts as a preparation in procedural fluency when finding the derivative of a function using limits. Additionally, the alternate definition of the derivative requires students to manipulate algebraically limits in order to solve for the derivative of a function at a single point using limits. These algebraic manipulations become commonplace in the limit-derivative world. Since the limit and alternate definitions of the derivatives always cause a limit to be of indeterminate form—in other words 0/0—the practice of mathematical procedures regarding the calculation of indeterminate limits becomes imperative. While the students are being introduced to limits, they also are undergoing preparation for the resulting, logical step toward full-blown calculus.

   The connections to mathematical reasoning are vast. There are three ways to evaluate a limit. One is done using a table (numerical); another is done using a graph (graphical); and lastly, it is]
done using an equation (analytical). Students should be able to reason through a limit in multiple ways that are mathematically viable. ]

c. Explain how your plans build on each other to help students make connections between concepts, computations/procedures, AND mathematical reasoning or problem-solving strategies to build understanding of mathematics.

[ My first lesson plan consists of an introduction to the idea of local linearity through a discovery activity. From there, I will define local linearity by having the students explore secant lines and tangent lines and use them to approximate functions at a point. The students start to investigate limits by calculating the slopes of secant lines as the 2 points used become closer and closer. These slopes approach the slope of the tangent line, which previews the connection between limits and derivatives. As a result, the students review how to calculate the slope of a secant line, which continues to appear throughout the use of limit and derivatives. Next, this leads us into the concept of a limit.

Ultimately, the conceptual underpinnings of limits lead students to a deeper understanding and appreciation of derivative—and eventually integrals. For example, the ideas of getting infinitely close relate directly to how the limit of the secant line equals the slope of the tangent line. The distance between the two points that determine the secant line go to zero and the slope of the secant line approaches the value of the slope of the tangent line. This conceptual understanding is key for discussions about the average and instantaneous rates of change that occur later in the semester. The goal is for the students to understand that limits provide the basis, reasoning, and models for derivatives.

A conceptual foundation to calculus as a whole is the idea of slope as it relates to being infinitely instantaneous. This is why I started with a discovery lesson based upon the equations of lines. Since local linearity is first explored through the equations and slopes of lines as they relate to the scale of the graph, students are prepared for the ultimate goal of “Limit Land”, which is “Derivative Land”—if you will. Furthermore, the subsequent lesson and homework introduces the slope of the tangent line as being approached by the slope of the secant line as the distance between the two points becomes smaller and smaller. As a result, the limit definition of the derivative is not pulled from out in left field; instead, the students will have experience with limit notation and vocabulary in the context of slope. The transition to limits, derivatives, and calculus is smoothed through the introduction of slope within local linearity. ]

2. Knowledge of Students to Inform Teaching

For each of the prompts below (2a–c), describe what you know about your students with respect to the central focus of the learning segment.

Consider the variety of learners in your class who may require different strategies/support (e.g., students with IEPs or 504 plans, English language learners, struggling readers, underperforming students or those with gaps in academic knowledge, and/or gifted students).

a. Prior academic learning and prerequisite skills related to the central focus—Cite evidence of what students know, what they can do, and what they are still learning to do.
Prior to calculus, students should know how to calculate the slope of a line. Students should also know how to graph an equation of a line (and other various function families). Students should have experience with exponents and rationalizing the denominator, which is important for evaluating limits analytically. Students should know how to read a graph and a table in order to evaluate limits using those tools/methods.

Students are still learning how to multiply complex binomials and manipulate algebraic equations/expressions. This will be important as students learn to rationalize the numerator in order to solve for a limit analytically. In terms of algebraic manipulation, students will also continue to learn how to rewrite fractions, especially complex fractions. As calculus progresses, students will be constantly required to simplify complex fractions in order to solve limits and manipulate equations (to make the calculus possible or simpler).

I do have a couple students who algebra skills and abilities are quite low (especially low for going into a calculus, college level course) ]

b. Personal, cultural, and community assets related to the central focus—What do you know about your students’ everyday experiences, cultural and language backgrounds and practices, and interests?

Many of the students desire to go into engineering, finance, or some sort of STEM related job or major. These students are motivated and excited to learn calculus as it relates to their future career or degree.

Additionally, there is some excitement and mystery surrounding calculus in the minds of the students. They think of calculus as some higher understanding as they have heard of derivatives and integrals, but have little or no idea of what they actually mean or connect to in life.

Finally, I have a couple of students who are incredibly quick and eager to learn the mathematics of calculus. They are light years ahead in terms of the prerequisites and algebra skills needed to learn and succeed in calculus. ]

c. Mathematical dispositions—What do you know about the extent to which your students perceive mathematics as “sensible, useful, and worthwhile”?3

1. perceive mathematics as “sensible, useful, and worthwhile”
2. persist in applying mathematics to solve problems
3. believe in their own ability to learn mathematics

I have multiple students who often question why we are learning certain mathematical concepts or principles. Some of the students ask this question genuinely; however, others ask this question out of a disdain or skepticism of the relevance of the mathematics. The majority of this calculus class does perceive mathematics as not only sensible and useful but also worthwhile and enjoyable. Many of these students enjoy mathematics and have excelled in it over their junior high and high school careers.

On the other hand, I have a handful of students who struggle with algebra and who have some doubts about being able to learn mathematics. Often, these students are not as likely to persist in when attempting to problem solve something new. They are quick to ask for help from me or my mentor teacher instead of thinking through it themselves or collaborating with a partner. ]

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3 From The Common Core State Standards for Mathematics
3. Supporting Students’ Mathematics Learning

Respond to prompts below (3a–c). To support your justifications, refer to the instructional materials and lesson plans you have included as part of Planning Task 1. In addition, use principles from research and/or theory to support your justifications.

a. Justify how your understanding of your students’ prior academic learning; personal, cultural, and community assets; and mathematical dispositions (from prompts 2a–c above) guided your choice or adaptation of learning tasks and materials. Be explicit about the connections between the learning tasks and students’ prior academic learning, their assets, their mathematical dispositions, and research/theory.

[The algebra in this section of the unit is quite difficult if a student has minimal or no background with the concepts or skills. Therefore, I chose to take an entire extra block day in order to introduce “Evaluating Limits Analytically”. Rationalizing a numerator or denominator is something that most of the students will not remember and will need additionally practice and time—while some unfortunately will have to learn it as completely new material. Some of my students will struggle with algebra will also be overwhelmed and discouraged if I were to bombard them with so much algebra and new notation within a couple of days. Moreover, taking some extra time during this section will allow me to go at a quicker pace when doing the limit and alternate definitions of the derivative (which require the majority of the same algebraic concepts and skills).

Also, I chose to take some time to review secant line and equations of tangent lines in order to frontload the central ideas of slopes, which is the main reason for finding a derivative in the first place. I did this before heading into limits because it will help create an overall focus for the entire quarter and semester of Calculus. Additionally, some of my students who need additional time to process things algebraically, conceptually, and visually (along with reading). Research shows that “just in time” remediation or teaching of concepts that should have been learned previously—in this case algebra—creates the need and desire to learn within the students. Instead of doing a complete algebra review before the beginning of calculus, I will continue to update and reteach algebraic concepts and skills that relate to the problems and concepts contained in calculus and limits.]

b. Describe and justify why your instructional strategies and planned supports are appropriate for the whole class, individuals, and/or groups of students with specific learning needs.

Consider the variety of learners in your class who may require different strategies/support (e.g., students with IEPs or 504 plans, English language learners, struggling readers, underperforming students or those with gaps in academic knowledge, and/or gifted students).

[ We will spend time learning limits through multiple different means. First, we will use graphs in order to help those who learn best visually. In addition, we will be using our calculators in order to provide some kinesthetic learning opportunities in regards to limits. Also, the students will use their calculators as a tool in order to avoid some of the problems that come with pencil and paper calculations. This will help my student with dyslexia as she will not have to spend as much time writing and reading off her notes.
Furthermore, the calculators will help fill in some of the gaps in understanding as it pertains to algebra by allowing some of my lower level (in terms of algebra) students to use them as a resource for understanding and solving the concepts.

With the couple of students that are gifted, I will ask higher level questions that tend toward establishing algebraic rules for the phenomenon that we will observe with limits and calculus.

c. Describe common mathematical preconceptions, errors, or misunderstandings within your central focus and how you will address them.

[ A common mathematical preconception is that the limit notation is functional notation. It is not indicating a function, but rather an operation, which requires an operation—much like the sine function. Since the students are freshly out of trigonometry, I will relate limit notation to trigonometric notation and other similar operation notation that the students have observed before calculus.

Additionally, students will struggle with distributing across binomials and other expression while evaluating limits analytically. I will address these problems by explaining and showing the mistakes commonly made during the unit. In addition, I will be meticulous in showing my work when I am doing example in front of the class at the Smart Board. Students tend to copy what the teacher models for them. Therefore, I will be sure to show my work in a comprehensive fashion that includes mathematically correct notation.]

4. Supporting Mathematics Development Through Language

As you respond to prompts 4a–d, consider the range of students’ language assets and needs—what do students already know, what are they struggling with, and/or what is new to them?

a. Language Function. Using information about your students’ language assets and needs, identify one language function essential for students to develop conceptual understanding, procedural fluency, and mathematical reasoning or problem-solving skills within your central focus. Listed below are some sample language functions. You may choose one of these or another language function more appropriate for your learning segment.

<table>
<thead>
<tr>
<th>Compare/Contrast</th>
<th>Justify</th>
<th>Describe</th>
<th>Explain</th>
<th>Prove</th>
</tr>
</thead>
</table>

Please see additional examples and non-examples of language functions in the glossary.

[ Students must be able to describe the process of taking a limit in order to be successful within this central focus. If the students cannot describe how one takes a limit, both conceptually and mathematically, they will be unable to grow and succeed in the problem solving and mathematical reasoning skills regarding limits. The process of finding a limit graphically requires an understanding that can be expressed and described in both words and mathematical notation. The concept of a limit is monumental within calculus and cannot be dismissed as optional learning or knowledge if a student is to continue within the subject/content area. ]
b. Identify a key learning task from your plans that provides students with opportunities to practice using the language function identified above. Identify the lesson in which the learning task occurs. (Give lesson day/number.)

[ For homework after learning about local linearity—which is highly connected to limits—students are required to give a personal explanation of what local linearity means to them. This will be done on Worksheet 1.1 Local Linearity. Many of them will use both words and mathematical notation in order to answer the problem. This short paragraph will clue me about their understanding concerning local linearity (and limits).

Moreover, the students will be required to solve for the secant lines that approximate functions at a certain value. As the points defining the secant lines become closer and closer, the slope of the secant lines will start to approach the slope of the tangent line at that specific point. This serves as an introduction to limit notation (and really a derivative).

This will be Lessons #2 and #3. ]

c. **Additional Language Demands.** Given the language function and learning task identified above, describe the following associated language demands (written or oral) students need to understand and/or use:

- **Vocabulary and/or symbols**
- **Mathematical precision** (e.g., using clear definitions, labeling axes, specifying units of measure, stating meaning of symbols), appropriate to your students’ mathematical and language development
- **Plus** at least one of the following:
  - Discourse
  - Syntax

[ The word approach is used in limits and mathematics to designate that a value, function, series, or variable is getting really close to (but not quite exactly equal to) a certain value. Students will be required to use this language and notation on the worksheet as they calculate the slopes of secant lines, which will eventually approach the slope of the tangent line at a certain x-value. The students will be required to use this language and the arrow used in limit notation in order to refer to the concept and calculations of local linearity.

The specific syntax I will require will be this: \( As \ x \to c, m \to d \) which would be read accordingly: “As \( x \) approaches \( c \), \( m \) (the slope) approaches \( d \).” Each problem on this worksheet requires a statement containing this syntax. This worksheet—the 1.1 Local Linearity Worksheet—will ultimately be an introduction and preparation for limit notation (and eventually derivative notation), which comes in the subsequent lesson. While at first painful, the repetition and usage of this syntax will prime the students for the novel mathematical notation that appears in the world of limits. ]

d. **Language Supports.** Refer to your lesson plans and instructional materials as needed in your response to the prompt.

- Identify and describe the planned instructional supports (during and/or prior to the learning task) to help students understand, develop, and use the identified language

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4 For an elaboration of “precision,” refer to the “Standards for Mathematical Practice” from The Common Core State Standards for Mathematics (June 2010), which can be found at [http://www.corestandards.org/assets/CCSSI_MathStandards.pdf](http://www.corestandards.org/assets/CCSSI_MathStandards.pdf).
demands (function, vocabulary and/or symbols, mathematical precision, discourse, or syntax).

[ I plan to model the very first part of the first problem on the worksheet for my students in order to introduce the concept and the new notation that they will be utilizing in the future. The modeling will come after an introduction to local linearity using the Smart Board for notes and discussion. As a result, the students will have no doubt about which syntax and mathematical notation/vocabulary to use in this situation. ]

5. Monitoring Student Learning

In response to the prompts below, refer to the assessments you will submit as part of the materials for Planning Task 1.

a. Describe how your planned formal and informal assessments will provide direct evidence of students’ conceptual understanding, procedural fluency, AND mathematical reasoning and/or problem-solving skills throughout the learning segment.

[ Concerning formal assessment, after the student learn the first official concept, which is Limit Properties, the students will be taking concept quizzes which test their knowledge of the current concepts in the unit. Thus, I can evaluate my students’ conceptual understanding and procedural fluency and my communication and teaching of the concepts. This also serves as a formative assessment that allows me to differentiate or remediate my instruction accordingly. For example, the second concept concerns evaluating limits analytically. I must assess my students ability to manipulate limits algebraically so that they are prepared with the prerequisite skills for determine the derivative of a function using limits. This concept tests the conceptual knowledge of the students concerning evaluating limits; however, it also develops procedural fluency as they solve and become familiar with the different types of manipulations and methods necessary to solve an indeterminate limit algebraically.

Another example that touches on the mathematical reasoning or problem-solving skills necessary is the first concept of evaluating a limit. A limit can be evaluated three different ways. Often, one of the three ways—graphically, numerically, and algebraically—ends up being easier than the other one. On the summative and formative assessments, when the students is given the freedom to choose how to evaluate a limit, he or she must use mathematical reasoning and problem solving to determine the limit of the function. When there are little to no guideline, the students must think critically about the problem, and then use their problem-solving skills in order to determine an answer or a solution to the problem.

I will informally assess my students through questions and discussion over the homework for each concept and learning objective. Additionally, informal assessment will occur as we collaborate as a class and in groups when we work on examples or homework during class time. Informal assessment especially will be helpful regarding graphical limits and continuity. Graphical limits and continuity are those topics or concepts that can seem quite complicated; however, after some discussion, experience, struggle, and clarification, they become less formidable in the scope of things. The homework I have planned and the class time discussing continuity and graphical limits allow for meaningful conversation and dialogue to occur concerning these profound, conceptual ideas. ]
b. Explain how the design or adaptation of your planned assessments allows students with specific needs to demonstrate their learning.

Consider the variety of learners in your class who may require different strategies/support (e.g., students with IEPs or 504 plans, English language learners, struggling readers, underperforming students or those with gaps in academic knowledge, and/or gifted students).

[ The formal assessment, (which are also formative) are open-ended in nature and allow the students to show their work or explain themselves in the way that allows them to demonstrate their understanding, without sacrificing mathematical notation or language.

Also, with those who struggle with algebra, it allows them multiple opportunities and occasions to practice using their algebra skills and knowledge in a setting that prepares them for the summative assessment at the end of a unit. ]
TASK 2: INSTRUCTION COMMENTARY

1. Which lesson or lessons are shown in the video clip(s)? Identify the lesson(s) by lesson plan number.
   [ This is lesson plan #1. ]

2. Promoting a Positive Learning Environment
   Refer to scenes in the video clip(s) where you provided a positive learning environment.

   a. How did you demonstrate mutual respect for, rapport with, and responsiveness to students with varied needs and backgrounds, and challenge students to engage in learning?

   [ In the second video clip, I have students come up to the SmartBoard in order to draw and explain their thinking and reasoning considering the equations of the lines on the board. This is an exploratory activity where the students are working in pairs or trios in order to write equations for the three lines graphed (see attachment below). I challenged the students to engage in learning by explaining their answers in front of the entire class. The students were in groups so they could ask questions and help each other out when confused. ]

   I demonstrate mutual respect for students by asking questions and requiring responses from all of the students. I did not single students out by asking individual questions that only pertained to that person. Instead, I asked pertinent, general questions to the whole class like: “How will the scale affect the equation of that line?” or “How will the scale affect the slope of the function?”

   Another part of promoting a positive learning environment was utilizing my mentor teacher in the classroom discussions and activities during the first weeks of my student teaching experience—including this lesson. The students were most comfortable with her because of their previous experience and knowledge of her. It was helpful to keep my mentor teacher involved at first because of her experience, insight, and connection in relation to both the calculus and the students.

3. Engaging Students in Learning
   Refer to examples from the video clip(s) in your responses to the prompts.

   a. Explain how your instruction engaged students in developing
      - conceptual understanding,
      - procedural fluency, AND
      - mathematical reasoning and/or problem-solving skills.
My instruction engaged students in developing conceptual understanding by challenging their perception of scale and their perception of linearity. After confronting the issue of scale, I asked questions about how the scale would affect the equations of the lines. Having the students verbalize and recognize the effect of changing the scale of graph deepens the conceptual understanding preceding local linearity.

Additionally, my students developed procedural fluency in writing linear equations. They had to write the equations of the three lines to begin with in the first clip; then, they come back and rewrite equations for those same three lines with the new scale factor.

In terms of mathematical reasoning and problem-solving skills, this exploration activity lends itself toward developing careful reasoning concerning graphs, scales, and functions. At the end of the video, I challenge the students to match the actual equations of the graphs with the linear equations, while giving justification for each. Furthermore, I required the students to give justification for their equations for the lines. I asked specific questions such as: “Why did you write -1 or 2/3rds in that equation?” These types of clarification or justification questions and moments provide students with the opportunity to reflect and expand their cognitive processing.

b. Describe how your instruction linked students’ prior academic learning and personal, cultural, and/or community assets with new learning.

[ Students had previous knowledge of graphing and writing linear equations. I connected their experience with this to a brand new idea of local linearity by connecting the ideas that non-linear graphs can appear linear when we zoom in closely on the graph.]

Additionally, most of the students had familiarity with a scale of simply one unit. However, throughout this lesson, we attempted to get new experience with a graph of a different scale. From here on out, the students paid close attention to the scale of every graph I put in front of them. Doing this activity helped them recognize the tedious nature of reading graphs and taking limits of graphs. ]

4. Deepening Student Learning during Instruction

Refer to examples from the video clip(s) in your explanations.

a. Explain how you elicited and built on student responses to promote thinking and develop conceptual understanding, procedural fluency, AND mathematical reasoning and/or problem-solving skills.

[ When discussing how the scale influences the slope, I took time to write down student responses to why the slope would not change in spite of the scale. I wanted to take time to address the students’ ideas for how and why the equations would change for the function. Additionally, we wanted the students to be shocked when we revealed the actual equations of the graphs, which are not linear in the traditional sense. These graphs only appear linear when looked at closely. All of the questioning and exploratory process led up to the students realizing that non-linear functions can appear linear when we really zoom in on the graphs.]

In addition, from this video, in the next class period, we used our graphing calculators and the SmartBoard to recreate this scenario by zooming in on the functions. I built upon their responses from the end of the video by having them justify each equation for each graph. Their
precursory bewilderment instead turned into mathematical reasoning and understanding due to their curiosity and determination to comprehend the phenomenon of local linearity. ]

b. Explain how you used representations to support students' understanding and use of mathematical concepts and procedures.

[ On the SmartBoard, I had a large-scale representation of the graph attached below. I used this for the expressed purpose of having the students write out equations and draw representation of their thinking. I was able to use the SmartBoard in order to talk about the ideas of slope, x-intercepts, and scale.

Moreover, each of the students had their own graph in front of them on which they could display their thinking and compare results with their partners. Therefore, everybody was on the same page while discussing the graphs and concepts. Interacting with the graphs allowed the students to come to a better—even more personal—understanding of local linearity.

Specifically in relation to slope, the students and I were able to use the SmartBoard pens in order to draw the separate vertical and horizontal pieces of the slopes of the graphs. This was helpful when the scale was revealed for the graph, as students were able to distinguish that the slope stayed the same despite the change in the scale. ]

5. Analyzing Teaching

Refer to examples from the video clip(s) in your responses to the prompts.

a. What changes would you make to your instruction—for the whole class and/or for students who need greater support or challenge—to better support student learning of the central focus (e.g., missed opportunities)?

Consider the variety of learners in your class who may require different strategies/support (such as students with IEPs or 504 plans, English language learners, struggling readers, underperforming students or those with gaps in academic knowledge, and/or gifted students).

[ In the first video clip, I introduced the task of writing equations to match the graphs; however, right after, I interrupted the students in order to restate the directions. I think this moment was unneeded and even distracting.

Additionally, I think that I would have done the whole lesson over one day if I could have done so. I think the students were in the right frame of mind as I revealed the actual equations of the graphs. The students could have taken another 5-10 minutes after that to connect the actual equations to the graphs.

Subsequently, we could have had some of the students come up and explain why the new equations matched up with each corresponding graph. I had the students come up and explain their first written equations. However, I did not have the students come up and explain the updated equations after revealing the scale. I glossed over the connections between the scale and the equations too quickly. The students would have understood the scale and its effect in a better way if I had continued to have them come up to the board in order to explain their answers.
Furthermore, in order to challenge the whole class, I wish I had switched up partners in the middle of the activity in order to provide for different perspectives and methods for thinking about the ideas of scale, local linearity, and equations of lines. I think I missed a great opportunity for students to collaborate and experience some freedom in mathematics.

I would change the involvement and interjection of my mentor teacher if I had done this lesson later in the year. However, this lesson was one of my first lessons as it was within my first 2 weeks of student teaching. Therefore, the novelty of my person to the classroom environment cannot be ignored when evaluating the lesson. Her interjections helped fueled conversation; in addition, I believe they eased the learning process for many of the students. I was very appreciative of her insight and assistance. Furthermore, the students are quite comfortable with my mentor teacher and expect to hear her chime in on most anything—as they value her two cents on most anything. Thus, I would not change her involvement at this point, because she contributed to the classroom environment and discussion.

b. Why do you think these changes would improve student learning? Support your explanation with evidence of student learning AND principles from theory and/or research.

[I believe that doing all of this on the same day, within the same activity, would have benefitted student learning because they were present and engaged in the learning situation at the time. When students are present and engaged in the content and the learning, they are going to understanding the concepts at a higher level. Also, the long break two days (block day schedule). When the conceptual understanding of the impact of scale and zooming comes into play, it makes for a more appropriate transition into local linearity and limits for calculus.

Furthermore, allowing the students time and space in order to think and collaborate is important. They do not need my voice interrupting the much-needed space to process mathematically. In addition, it takes away from the time that they could be conversing and comparing with their neighbor.

Students learn at a deeper level when they interact with their peers. Multiple viewpoints and perspectives expand the knowledge of the students. Additionally, the switching of partners forces students to encounter different ways of thinking about a concept.]


This were the equations I wrote on the whiteboard:

\[ f(x) = x^3 + 0.002 \]
\[ g(x) = \frac{2}{3}x - 0.001 \]
\[ h(x) = \sin(2x) \]
1. **Analyzing Student Learning**

   a. Identify the specific learning objectives measured by the assessment you chose for analysis.

   [ 1. Students will be able to find the equation of a tangent line at a point of a polynomial.

   2. Students will be able to determine the slope of a function at a point.

   3. Students will be able to recognize when a function is discontinuous and therefore where a derivative does not exist.
   ]

   b. Provide a graphic (table or chart) or narrative that summarizes student learning for your whole class. Be sure to summarize student learning for all evaluation criteria submitted in Assessment Task 3, Part D.

   [ 

<table>
<thead>
<tr>
<th>Student ID</th>
<th>Pretest Score</th>
<th>Post-Test Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student A</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Student B</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Student C</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>Student D</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Student E</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Student F</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Student G</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Student H</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>Student I</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>Student J</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Student K</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Student L</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td><strong>Average Score</strong></td>
<td><strong>1.5</strong></td>
<td><strong>5.666666667</strong></td>
</tr>
</tbody>
</table>

   This chart shows the progression of learning from pre-test to post-test. The scores are out of 10 possible points. I gave the pre-test at the beginning of the semester in January. Then, I
administered the post-test toward the beginning of March after the students had just learned the power rule.

Without question, in looking at this chart, there is much room to grow for myself as a teacher, and for my students in terms of their understanding. There is a rise in the scores; however, many of the students experienced some difficulty with the third problem—the problem on the second page of the assessment. The third problem on the assessment is difficult as it is a piecewise function. We spent some time on piecewise functions; however, not related as much to the power rule or finding derivatives. I believe that this affected some of the scores of my students as I did not prepare them adequately with experience with both the power rule and piecewise functions before the post-test assessment.

<table>
<thead>
<tr>
<th>Honors Calc 2B</th>
<th>Question #1 Average</th>
<th>Question #2 Average</th>
<th>Question #3 Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-Test</td>
<td>1.00</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td>Post-Test</td>
<td>1.92</td>
<td>1.33</td>
<td>2.42</td>
</tr>
</tbody>
</table>

This test shows the differences in averages from pre-test to post-test in terms of question. As is obvious, the improvement for questions 1 and 2 is substantial—as both question #1 and question #2 were out of two points. The majority of the students were able to understand and solve both of these questions appropriately and adequately. However, as we can observe, despite the apparent growth, the third question is still lacking as the question has a total value of 6 points (3 points for the slope, and 3 points for each equation of the tangent line. Since the third question contained the piecewise function, it demonstrates where the students’ misconceptions emanated. The students had a rough time knowing which part of the function to use and decided upon which method of finding the derivative to use (i.e. power rule, alternate definition of the derivative, and limit definition of the derivative).

My students definitely learned different methods for finding the derivative of a function at a specific point. This can be seen from the spike in the points from pre-test to post-test on questions 1 and 2. Both of these questions dealt with the slope (derivative) of a function at specific points. My students understand that they were to find the slope of the functions by whatever means they could muster. Some did this the easy way (power rule); and some did it the hard way (limit definition of the derivative). Either way, the students understood what they were doing or they understood the concept they were demonstrating knowledge upon.

However, this was not the case for the application topic of finding the equation of the tangent line. Question 3 tested this concept. The students did okay finding the slopes of the function; but, the students, on the whole, did poorly when it came to calculating the equations of the tangent lines. This indicates that I did not do a great job of preparing my students for applying the derivative (slope) in specific situations. My students had determined the equations of tangent lines before; however, they had minimal experience with doing so in the context of the power rule.
This last table shows the results of each individual student for each question on both the pre-test and the post-test.

Students G, J, K, and L all struggled on the third question in both the pre-test and post-test. These students had difficulty with the piecewise function. They were confused about which part of the function to use for certain parts of the slope. This reflects poorly upon me, as I did not spend as much time talking about piecewise functions and the process of taking a derivative with a difficult — complex — piecewise like problem 3 contains.

Students A, C, E, and H were able to calculate the slope at certain points along the piecewise function; however, they had troubles using the slopes to determine the equations of the tangent lines at those points. This is why these students received half or less than half of the possible points for question 3. They were able to do the first half of the required task, but failed to apply the derivative when calculated at a certain point.

All of the students were able to recognize the slope of a linear function. Moreover, the majority of the students were able to find the slope of a function using the power rule or the alternate definition of the derivative. Evidenced by the improvement of results on questions 1 and 2 from pre-test to post-test, many students were able to find the slope of a function at a specific x-value. The students who struggled on question 2 had difficulties with the asymptote of the rational function at x = 0.

Furthermore, the majority of the students improved on their overall understanding of the third question with the piecewise function. There was some definite improvement as nobody even scored a point on that problem on the pre-test. However, many students had a good idea of what they were doing and at the very least demonstrated some understanding surrounding derivatives, slopes, and tangent lines.

]  

3. Use evidence found in the student work samples and the whole class summary to analyze the patterns of learning for the whole class and differences for groups or individual learners relative to
   - conceptual understanding,
   - procedural fluency, **AND**
   - mathematical reasoning and/or problem-solving skills.
Consider what students understand and do well, and where they continue to struggle (e.g., preconceptions, common errors, common struggles, confusions, and/or need for greater challenge).

I have many gaps in understanding among students at this point in time—right after the post-test. For example, many of my students are new to the power rule. Multiple students used the alternate definition of the derivative in order to find the slope at a point. One student even used the limit definition of the derivative in order to find the derivative function, instead of utilizing the power rule—which is insane when you consider the amount of time it takes to do the power rule as compared to the time it takes to do the limit definition of the derivative. Furthermore, some of the students correctly used the power rule and evaluated it at a point to find the derivative.

All of these are valid ways of finding the slope at a point. For example, Student I used the power rule to find the slope in this case. The student first finds the derivative functions and then evaluates it appropriately at \( x = -1 \). This is by far the easiest method of finding the slope in this case. Student I used all of their work and demonstrated his knowledge and application of exponent rules. The students carefully showed all of their work on this problem.

However, if you look at Student L’s work, the student has a tough time with their exponent rules as the student incorrectly writes \( \frac{1}{x} \) as \( x^{(1/2)} \) power. Student L is my student that has dyslexia and is generally slower in understanding concepts and reading problems. In addition, this student does not do their homework because they believe it is worthless since it counts for so little of their overall grade—as per district and building policy. Writing the exponent incorrectly causes their derivative to be incorrect despite using the power rule in the right manner. As a result, this indicated to me that my students needed a review—or maybe even a lesson on exponent rules before moving on to the power rule. The procedural fluency surrounding the power rule and exponent properties obviously needs some assistance and review. Additionally, the student had a misconception about the objective. Student L did not find the slope of the function at point A either, which can be done without even finding the derivative.

Finally, Student J attempts both the power rule and the alternate definition of the derivative in order solve for the slope at \( x = -1 \). Firstly, for the power rule, the student does the power rule correctly at first, but then incorrectly rewrote the equation by attempting to take the reciprocal of the negative exponent. As a result, the final answer of \( \frac{1}{2} \) is wrong. Additionally, this same student tried the alternate definition of the derivative for the slope as well. However, the student stopped halfway through due to his omission of limit notation. The absence of limit notation and derivative notation caused the student to forget what he/she was calculating. The student got the right answer when using the alternate definition; but, the student instead turned to the power rule. I believe that if the student would have had written in the derivative notation, the student would have been more confident and more cognizant with his answer.

d. If a video or audio work sample occurs in a group context (e.g., discussion), provide the name of the clip and clearly describe how the scorer can identify the focus student(s) (e.g., position, physical description) whose work is portrayed.

[ N/A ]
2. Feedback to Guide Further Learning

Refer to specific evidence of submitted feedback to support your explanations.

a. Identify the format in which you submitted your evidence of feedback for the 3 focus students. (Delete choices that do not apply.)
   - Written directly on work samples or in separate documents that were provided to the focus students
   - In audio files
   - In video clip(s) from Instruction Task 2 (provide a time-stamp reference) or in separate video clips

If a video or audio clip of feedback occurs in a group context (e.g., discussion), clearly describe how the scorer can identify the focus student (e.g., position, physical description) who is being given feedback.

[ N/A ]

b. Explain how feedback provided to the three focus students addresses their individual strengths and needs relative to the learning objectives measured.

[ All of the feedback for these students can be seen on the samples I provided at the end.]

For Student J, I encouraged them by pointing out that they got the right answer when using the alternate definition of the derivative. However, I also reminded the student to use correct notation. Lastly, on the student’s work concerning the power rule, I circled the two steps where he messed up and pointed out that the work did not match up. I left the discovery of his error to him, as I want my students to realize, identify, and correct their own mistakes.

Regarding Student L, I circled the two steps where they went wrong. Then, I asked how they got form that first step to the subsequent step. This is to address the ineptitude surrounding exponent rules, which is really the problem in this case. Then, I encouraged the student by pointing out that they did the power rule correctly, which is the objective that I was testing. The student is doing the power rule in a procedurally correct manner, which I thought was important to point out so that they understood that their error was found in the algebra.

Since Student I utilized the power rule correctly, I encouraged them with a comment about how they performed the power rule appropriately.

[ For Student J, I will use this feedback in order to encourage this student to use correct notation in the future. This student does like to skip steps and omit notation. I will use this assessment and feedback as an important, subtle reminder that correct notation often leads to correct answer—as well as easier error recognition.

For Student L, I will use this feedback to remind them about exponent rules and the importance of doing homework. This student refuses to do homework and yet always want to ace my assessment. This student gets frustrated when they do not do well on the assessments. I am going to attempt to use my feedback and this assessment as a reminder that homework and]
studying does help us progress in mathematics. In addition, this feedback will motivate me to review exponent rules with not just this student but also the entire class.

For Student I, my feedback will hopefully be a positive encouragement to continue working hard. In addition, I will make a point to remind this student to utilize the power rule in every possible opportunity, as it is easier to use to find the derivative.

3. Evidence of Language Understanding and Use

When responding to the prompt below, use concrete examples from the clip(s) and/or student work samples as evidence. Evidence from the clip(s) may focus on one or more students.

You may provide evidence of students’ language use from ONE, TWO, OR ALL THREE of the following sources:

1. Use the video clip(s) from Instruction Task 2 and provide time-stamp references for evidence of language use.

2. Submit an additional video file named “Language Use” of no more than 5 minutes in length and cite language use (this can be footage of one or more students’ language use). Submit the clip in Assessment Task 3, Part B.

3. Use the student work samples analyzed in Assessment Task 3 and cite language use.

a. Explain and provide concrete examples for the extent to which your students were able to use or struggled to use the
   - selected language function,
   - vocabulary and/or symbols, AND
   - mathematical precision, discourse, or syntax
   to develop content understandings.

[ Student L and Student J do not use the correct notation concerning the derivative. As a result, this indicates to me that there is still a gap in understanding in terms of vocabulary and mathematical precision and syntax. The recognition that the derivative represents the slope of the function is lacking at the very least. However, with both of the students, they did recognize that in order to find the slope of the function, they needed to use some sort of rule of differentiation—like the power rule or the alternate definition of the derivative—in order to find the slope of non-linear function at a specific point. Therefore, there are at least some connections between differentiation and slope. However, the mathematical precision and language surrounding such for these two students is not where I want it to be.

On the other hand, for students like Student I, all of the mathematical precision and syntax is intact and completely correct. It is obvious that the student understands the task and comprehends the connections between derivative notation and language and determining the slope of an original function. This is evidence in the work by the correct f-prime notation in finding the derivative and then his correct work in evaluating the derivative at a specific value. ]
4. Using Assessment to Inform Instruction
   a. Based on your analysis of student learning presented in prompts 1b–c, describe next steps for instruction to impact student learning:
      - For the whole class
      - For the 3 focus students and other individuals/groups with specific needs

Consider the variety of learners in your class who may require different strategies/support (e.g., students with IEPs or 504 plans, English language learners, struggling readers, underperforming students or those with gaps in academic knowledge, and/or gifted students needing greater support or challenge).

[ For the whole class, I responded by reviewing exponent rules in preparation for the extensive future use of the power rule. Overall, many of my students had gaps in their understanding surrounding exponents. As a result, we took a good chuck of class after this in order to reteach and relearn the basic properties and rules surrounding exponents. Furthermore, I reminded the students to use the power rule to their convenience as it is easier and quicker to use to find the derivative of a function—even when only finding the slope of a function at one point. We had a conversation where I told them that they should use the power rule every time they could unless I stipulated otherwise. Additionally, I will be presenting and introducing more problems in the course concerning the application of the derivative. It is important to be able to calculate the derivative; however, in the real world, and for the AP test, it is imperative to use the derivative to either find the slope at a point, or find the equation of a tangent line at a point, or calculate the average rate of change over an interval, etc. This assessment showed me that I need to provide more opportunities for my students to apply their knowledge surrounding derivatives.

For the Student J, I need to continue to challenge him with his mathematical notation and language. This means I need to pay close attention to his homework and continue to provide feedback if notation continues to be an issue. This student does not like to show all of their work. I must encourage the student to continue to show their work in order to demonstrate their understanding and develop proficiency.

For Student L, I need to encourage her to do her homework if she wants her grade to improve and to reflect her understanding. I need to challenge her to come in and see me if she has issues or questions about the content and concepts. This comes through a personal conversation.

b. Explain how these next steps follow from your analysis of student learning. Support your explanation with principles from research and/or theory.

[ These next steps follow from my analysis of student learning since many of the students struggled with exponents within the power rule. If the algebra is the problem in completing the calculus, then I need to reteach or remind the students of the algebraic skills that we need in order to understand and complete the objectives.

For Student J, I will continue to encourage them to show their work because showing the proper steps and notation helps students remember the concepts and procedures better.

For Student L, she still has a long way to go in terms of comprehension and proficiency over the current concepts in calculus. She has been able to skate by in the past by not doing much homework and then performing satisfactorily on the quizzes or assessments. However, this
philosophy has been catching up with her in calculus. She takes longer than the average student to process through ideas and concepts. This extra time is taken during homework and classwork. As a result, she might change her habits in order to experience success within this class.

My assessment was testing the application of the derivative. As a result, my response to garner more exposure for my students is appropriate as many of them will need to apply the derivative in future college classes, the AP test, and STEM fields. Thus, future homework, objectives, lessons, and assessments will focus upon applications or uses of the derivative.
Hello! My name is Alex Krysl. I am a Secondary Education—Mathematics Major here at the University of Wyoming. This past semester I student taught at South High School in Cheyenne, Wyoming under Jayne Wingate. I had an amazing experience teaching 4 different classes with everything from Algebra to Geometry to AP Calculus BC. Nonetheless, I am only going to share some of my observations, experiences, and findings from the two Honor Trigonometry and Calculus courses that I taught for virtually my whole student teaching involvement. The majority of my students in these classes were juniors. In our school district, like most, juniors take the ACT as part of their testing requirement for the state and district. However, as some of you may know, the ACT does not cover calculus level concepts or content. So you may be wondering, how does taking a calculus class prepare these students for the most important testing situation of their lives? This test gives them free money and tremendous opportunities at the collegiate level. Thankfully, both Calculus and the ACT intersect within the world of algebra. I am going to share how I connected these ideas in some of the unit plans I had to develop for my EdTPA.

Now to some of you, math may be revolting. The thought of Calculus may fill you with dread. But actually, I am here to tell you that calculus is not that hard to do. Let me give you an example. In Calculus, we have a special thing called a derivative which is really an equation that automatically gives the slope of a curve or function at any point. There is something called the power rule. The power rule is a general form for any polynomial (a variable to some power) or any variable to some exponential power. Here is the rule: if $f(x) = x^n$ then the derivative is $f'(x) = nx^{n-1}$. You see the pattern. The first exponent is dropped down in front to multiply the first term, and then the exponent is decreased by 1. This is so easy a third grader could learn the
rule and follow it. However, the problem is...the application or usage of this rule often requires algebraic knowledge and tools beyond the abilities of most third graders. The hardest part of calculus is the algebra. Let me give you a couple examples:

\[ f(x) = \frac{2x^4 + 3x^2 - 5}{4x^2} \quad f(x) = \frac{x^5 - 3x + 4}{2\sqrt{x}} \]

My students struggled with both of the functions because they require algebraic manipulation using the rules of exponents before the simple power rule can be used. However, I believe they would have struggled even more if I had not done something called Just-in-Time review. Just-in-Time review is a researched educational philosophy that says that if you review the skills and knowledge necessary for a future unit or topic right before starting that topic, the students will be more successful on the whole. I was able to institute this idea before introducing the power rule to my students. I did this by reviewing the rules of exponents alongside a set of ACT questions regarding exponents. For example, we reviewed that:

- \( x^a x^b = x^{a+b} \)
- \( \frac{x^a}{x^b} = x^{a-b} \)
- \( (x^a)^b = x^{ab} \)
- \( \frac{a}{x^b} = \sqrt[b]{x^a} \)

The ACT always has some questions related to exponents and exponential rules. Here are some examples of those types of questions:
The second question there really relates to the power rule questions that I used to quiz my students. However, I was able to do an exponents review, while preparing students for the ACT in calculus class.

I wish I would have done something like this a little earlier in my student teaching stint. Another difficult topic (because of the algebra, not the calculus) is the limit definition of the derivative. It often requires factoring. Nevertheless, many of my students did not remember how to factor, or were very poor at it. I think it would have behooved the class to review factoring right before this concept. The review could have been done with ACT-like factoring questions. I believe their understanding of the topic and their scores on my quizzes would have been much higher if we would have done a factoring ACT review.

In the future, I would suggest to future mathematics teachers, who are teaching higher-level math courses, to evaluate the places where they can continue to teach algebra and other
ACT math topics in relation to their curriculum. Algebra is pervasive in calculus, and calculus teacher can continue to do their jobs while simultaneously preparing them for the most important test of their high school careers.
Unit Plan: Limits in Calculus

**Big Idea:**

The concept of limits is the foundation of calculus, as both derivatives and integrals are based upon limits. Limits are used to determine the behavior of functions. Limits can be calculated and explored through graphs, tables, and functions in a wide range of subjects, fields, and topics. For example, when traveling in a car, if we want to calculate our speed at a certain instant, we can use a limit to measure or calculate the instantaneous speed of the car at that specific moment.

**Concepts:**

I have listed the concepts specific to each lesson plan before the timeline of each respective lesson plan.

**Standards:**

I have listed the different, varying Common Core Standards and AP Calculus Standards under each respective lesson plan.

**Classroom Situation and Set-Up**

This unit plan will be for the second semester of an Honors Trigonometry and Calculus class. There will be two classes of approximately 24 students. The room will be set up with 28 individual desks and chairs in 7 pods of 4 desks each. This will help me create mini-groups for my activities, investigations, and simulations. I will have a smart board and two adjacent white boards at the “front” of the room. Each student will have a math binder for notes, assignments, assessments, and sections. I will assume that the students already have a background in calculating the slope of a line given two points. Furthermore, I will expect them to have an understanding of different graph functions and function families from Algebra 2 and College Algebra, since graphing is often utilized to evaluate and learn limits. The students should have
experience and knowledge about algebraic manipulation (i.e. conjugation) from College Algebra. The lesson plans will be assumed for eight separate class periods. The duration of such class periods will be determined for each lesson, and will be a part of each lesson plan description.

At Cheyenne South High School, the schedule is majority block, with the block days being A and B days with periods one-hour and 25-minutes long each. However, C days, usually Fridays, are only 45-minute long class periods. I will designate for each lesson the type of day that it will be.

Accommodations:

I have a student named Allison in one of the classes who is an ESL student. The specific accommodation(s) will be designated in each lesson plan for that lesson—if needed.

Research of Learning and Understanding of Limits

The word “Calculus” can strike fear into the hearts of many students. Also, there is a bit of mystique surrounding calculus, which contributes to the trepidation and hesitancies when it comes to students. From personal experience, the revelation of the integral sign was an incredible moment as the meaning of such was exposed to 17, 18-year old college students for the first time. The understanding and learning of Calculus is the bridge to college math and science. As one homeschooling parent commented, “Calculus is definitely the 10-ton gorilla of math courses for a high school student.” (Pride, 2006, p. 32).

Calculus is the field of mathematics of change and infinity. The modern teaching of calculus follows classes related to functions and algebra. Limits, specifically, connect to concepts of continuity of functions. David Tall (1990) evaluates a questionnaire given math students asking them about the continuity of five different functions, which can be seen below:
He states that, “Mathematically f1, f2 and f3 are continuous, whilst f4 and f5 are not (pg. 6). However, demonstrating that the concept of continuity is not clear, the questionnaire incorrectly stated that f2 was continuous. It is clear that \( f(x) = 1/x \) is not uniformly continuous, as it has an asymptote at \( x = 0 \) where the left and right limits at 0 are not equal. So, if a questionnaire from a scholarly article can record this question/answer wrong, it is not at all surprising that high school and college students have misconceptions concerning continuity.

Tall (1990) points out that the “majority are ‘right answers for wrong reasons’” and acknowledges that the “function f2 often causes dispute even amongst seasoned mathematicians” (pg. 6). This is because all of the functions are continuous at some points. What the questionnaire really wants to ask is: Which of following function are uniformly continuous? Or Which of the following functions are continuous at \( x = 0 \)? Students can get tripped up on the question because technically, in the above example, all of the functions except for f5 are continuous at most points. Student must learn to delineate between uniformly continuous and continuous at a point—or over an interval. Additionally, he records that many of the students are relying upon
the graph instead of the equation to determine continuity, when in fact, the concept image of continuity of a function can be misleading. This is why students should not only deal with continuity on the graphical level; algebraic reasoning is needed to understand continuity.

However, continuity is only half of the story in terms of calculus’s relationship with functions. Calculus is also deeply integrated with rates of change, especially among functions. Researchers have found students of multiple ages have difficulty conceptualizing the idea of rate of change (Herbert & Pierce, 2008; Teuscher & Reys, 2012). Teuscher and Reys (2012) make the observation that, “A commonality among the discussions by these researchers is that typically, students are introduced to rate of change with a formula for slope of a line.” In other words, most students end up viewing slope as simply a formula which spits out a meaningful output value which teachers designate as “the slope”. This implies that rate of change in mathematics is rarely first encountered by discovery and through deep, conceptual understanding.

Furthermore, as Herbert and Pierce (2008) indicate that students tend to practice inputting numbers and calculating the slope of a line with little to no focus on the interpretation of what the result means within a given context or without regard for the units of measure. The only exception to this is most likely with linear functions as it is a focus of the Common Core standards and many curriculums nationwide. Students are then usually introduced to high powers of polynomials and exponential equations and functions. However, what is missing is an emphasis on slope. The emphasis on slope for these high level functions fails to come until calculus. This is quite problematic as students enter the domain of calculus. Derivatives, integrals, limits, and most other calculus concepts are rooted in an awareness and a knowledge of rates of change.
Teuscher and Reys (2012) even note that after students are introduced to rates of change in calculus or pre-calculus, “the students may see this as a new concept and not related to the slope of a line.” Teachers of calculus need to tread lightly in light of these facts. Definite, purposeful lesson and ideas must be put forth to connect calculus and rates of change. It is an uphill battle for most teachers, however, because the majority of algebra classes—where functions are taught—do not discuss slope on the whole. As a result, many students are capable of using the slope formula to find the rate of change; however, they are unable to interpret the meaning of rate of change in a contextual situation, with a given graph, or more importantly with nonlinear situations (Teuscher & Reys, 2012).

Tall (1990) recommends using “examples and non-examples of a mathematical concept or process” to help learners of calculus—and mathematics in general—to understand and learn the general properties and principles embodied by said examples. He suggests a magnify program to start, where a class examines examples and non-examples of “local straightness”, which is nowadays better known as local linearity. The suggestion is “to magnify a tiny portion of a graph to investigates examples and non-examples of ‘local straightness’” (p. 10). Infinitesimals, and therefore calculus and limits, are based upon this idea of zooming infinitely close at certain points on functions to evaluate their rate of change, continuity, behavior, etc.

Local Linearity or straightness is best used as a cognitive root for derivatives and limits. Blume and Heid (2008) suggest using local linearity to show that the tangent line is almost indistinguishable from the function graph—to the point where it becomes a good approximation of the original function over that small interval, or at that point. The usual way the concept of a derivative and limit is introduced is through a sequence of secant lines to a function converging to a unique tangent line—which can be defined as the limit of the secant lines. Concerning
technology and local linearity, Blume and Heid (2008) suggest a computer algebra system (CAS) or a graphing calculator in order for the students to experience and interact with multiple functions and examples of this phenomenon of local linearity. This technology can help represent some of the concepts and ideas foundational to calculus.

Concerning limits specifically, Bernard Cornu (1999) says that “one of the greatest difficulties in teaching and learning the limit concept lies not only in its richness and complexity, but also in the extent to which the cognitive aspects cannot be generated purely from the mathematical definition” (p. 153). In other words, remembering and using the mathematical formulae for the definition of a limit is only one portion needed for adequate understanding. The acquisition of the fundamental conception of a limit is an entirely different animal. The limit concept goes beyond the mathematical symbols representing a limit; in fact, it delves into the analytical branch of mathematics. The problem with many calculus textbooks is that they focus on the operations and the algebra of limits as they relate to differentiation and integration rather than the connections to mathematical analysis (Cornu, 1999).

In respect to the limit concept, students have many different understandings of the words ‘tends to’ and ‘limit’ (Schwarzenberger & Tall, 1978). ‘Tends to’ is a word that in modern times has been replaced in the mathematical world by ‘approaches’. Students interpret ‘tends to’ to mean to approach with reaching something, just reaching something, or eventually staying away from something. With the mathematical concept of a limit, a limit never actually reaches the point of interest—but it gets infinitely close. Additionally, the word ‘limit’ was interpreted as: an impassible limit which is reachable, an impassible limit which is impossible to teach, a point which one approaches, without actually reaching it (Cornu, 1999). Obviously, it is not difficult to understand why student have misconceptions regarding language regarding limits.
However, there are some among the scholars that would advise against using limits altogether. R. Michael Range (2011) suggests a teaching method for calculus that abstains from using any limits—even in the calculation and creation of tangent lines, derivative rules, and factorization of functions. He also refrains from using infinitesimals in his construction of derivatives. This method is highly dependent upon analysis, and is quite complicated and filled with rich, complicated mathematical language and ideas. While he points out that using algebra can occasionally be easier than messing with limits, the analytical basis on his calculus ends up being more complicated in the long run (Range, 2011). Additionally, this type of calculus is not as effective or integral as limit-based calculus in preparing students for college-based mathematics (even if you don’t think that is a valid reason, limits abound across the engineering, technological, scientific, and mathematical realms).

A study done by Beste Güçler (2012) investigated the characteristics of one instructor’s discourse on limits and compared his discourse with those of the students. The study showed the difficulties in word use in relation to limits. Güçler (2012) comments that the study highlights “the conceptual challenges surrounding the interplay between the dynamic and static aspects of limit” (p. 451-452). A static conception of a limit is based upon the $\varepsilon$ and $\delta$ formal definition of a limit. This is the definition championed by analysis. However, the informal definition of a limit, which this unit plan utilizes, is known as the dynamic conception of a limit. Much of the scholastic world disagrees on which methods impart a better understanding of a limit to students (Denbel, 2014). Denbel (2014) does agree that the dynamic conception is easier and more natural for students to develop in their understanding of limits; however, Denbel also opines that “the main difficulty is for students to pass from a dynamic conception to a formal understanding of limits” (p. 28). Back to Güçler’s study, the instructor, Dr. Brenner, spent most of his class time
on dynamic limits. As a result, the majority of his students focused on the limit being a process and not a specific number (i.e. as $x$ approaches a certain number, the function is getting closer and closer to this number, as opposed to, the limit of the function at this point is $a$). Güçler (2012) designates two concepts of the limit: a “limit is a number” and “limit is a process” (p. 451). The latter definition is often related to the informal, dynamic conception of a limit, while the former is more connected to the formal, static definition of a limit. However, it is entirely possible to comprehend both concepts of a limit using the informal, dynamic definition. Students in this unit will struggle understanding both concepts as they seem counterintuitive. Teachers should utilize careful language and should emphasize the limit as both a process and a number. The process will happen naturally if the course is introduced using local linearity and the informal definition of limits. But a limit as a number can be focused upon by using sequences and polynomials.
References


Lesson Plan #1: Introduction: Seeing is Believing

Description:

On an overhead, I will have the following equations graphed with a window of -0.004 to 0.004 for the horizontal axis (x) and -0.003 to 0.003 for the vertical axis (y), with a scale of 0.001. The graph with originally show the grid lines but no the scale. The students will have to determine the equations of the graphs. The students will begin to explore whether non-linear functions can appear linear. Also, the students will complete a pre-test. This will be 1-hour and 25-minute lessons on a B-day.

- \( f(x) = x^3 + 0.002 \)
- \( g(x) = \frac{2}{3}x - 0.001 \)
- \( h(x) = \sin(2x) \)

Concepts:

- Local Linearity is connected to limits in the idea of getting infinitely close to a point.
- It is possible for non-linear functions to appear linear.

Standards:

CCSS.MATH.CONTENT.HSA.CED.A.1

Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

APC.1
The student will define and apply the properties of elementary functions, including algebraic, trigonometric, exponential, and composite functions and their inverses, and graph these functions, using a graphing calculator. Properties of functions will include domains, ranges, combinations, odd, even, periodicity, symmetry, asymptotes, zeros, upper and lower bounds, and intervals where the function is increasing or decreasing.
## Time Sequence for Lesson Plan #1:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>5ish minutes</td>
<td>Introduction of Activity</td>
<td>• I will have graphs displayed on the smart board before the start of class.</td>
<td>What can you say about the functions? How do the functions differ?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Split the students into pairs within their desk groups of 4.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Distribute handout with graphs.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Equation Construction</td>
<td>• The students will work in pairs to determine the equations of the graphs.</td>
<td>How does the scale affect your understanding of the function? How do you know when a function is linear?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• After determining equations for each function, the class offer up suggestions for equations to the functions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Revelation</td>
<td>• I will reveal to the students that the scale is 0.001.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Write it up on the White Board for Allison.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• I will ask if the functions are still linear since I expect the students to come up with linear equations at first.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Write the equations on the board.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Ask the students to work in pairs to determine which function is which, and to give a justification for each.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Discussion</td>
<td>• Use Zoom-Trig on the graphing calculator to see the regular functions.</td>
<td>What types of graphs would this not work for? Can we zoom in on any function to make it look linear? Does zooming in on a function make it linear at that point?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• As a class, discuss the appearance of linearity among the functions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Introduction of term Local Linearity</td>
<td></td>
</tr>
</tbody>
</table>
### 30 minutes or remaining time:
- Pre-Test (Appendix A)
- Local Linearity Worksheet (Appendix A)

<table>
<thead>
<tr>
<th>Pre-Test and Homework</th>
<th>Why might we want the functions to look linear?</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Announce the start of a new unit: Calculus!</td>
<td></td>
</tr>
<tr>
<td>• Hand out Pre-test and local linearity sheet.</td>
<td></td>
</tr>
<tr>
<td>• If students finish early, they can work on their homework below.</td>
<td></td>
</tr>
<tr>
<td>• Worksheet due as homework in two class periods (beginning of Lesson Plan #3).</td>
<td></td>
</tr>
<tr>
<td>• Only assign Problem 1 of the worksheet as a paragraph writing problem.</td>
<td></td>
</tr>
</tbody>
</table>

### Accommodations:

For Allison, I will provide directions for the activity in paper form, ahead of time (which can be seen in Appendix A). After everyone gets going on the activity, I will check with her and her partner to ensure that the directions are understood and to answer any questions. Also, her partner(s) will be able to doubly explain and help her with the activity. Furthermore, Allison will have extra time to complete the Post-Test. She will also be allowed to ask any clarifying questions relating to the problems on the test.
Appendix A

Honors Calculus    Seeing is Believing

Determine the equation for each of the following graphs:

Additional Written Directions for Allison:

1. Write out a math equation for each line.

2. Discuss with your partner your equations.
Worksheet 1.1 Local Linearity (10 points)

1. Describe what the concept of “Local Linearity” means to you? (see rubric below)

For each of the following, determine a linear function that models the behavior of the given non-linear function in the vicinity of $x = a$. Show all work and write your final answer in correct mathematical language. Problem 2 is worth 2 points and Problems 3 and 4 are worth 3 points.

2. $y = x^2$ when: i. $x = 1$ ii. $x = -3$
3. \( y = 2x^3 + 3 \) when: i. \( x = 0 \)  ii. \( x = 2 \)  iii. \( x = -1 \)

4. \( y = \sin(x) \) when: i. \( x = 0 \)  ii. \( x = \frac{\pi}{3} \)  iii. \( x = \frac{\pi}{2} \)

Rubric for Problem 1:

<table>
<thead>
<tr>
<th>Score</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Student writes out a paragraph clearly and correctly explaining local linearity in mathematical and general terms.</td>
</tr>
<tr>
<td>2</td>
<td>Student writes out a somewhat unclear explanation or only writes a sentence of</td>
</tr>
<tr>
<td>1</td>
<td>Student incorrectly explain local linearity and shows little effort or understanding of the concept.</td>
</tr>
<tr>
<td>0</td>
<td>Student does not write out any explanation.</td>
</tr>
<tr>
<td></td>
<td>Score</td>
</tr>
</tbody>
</table>
Answer the following questions using the graph(s) provided:

1. What is the slope of this function at point A and point B?
2. What is the slope of the function at $x = 0$? At $x = 3$? And $x = 2$?

$\quad (where \ f(x) = (x - 2)^2 - 1)$
3. What is the slope of the function at $x = 0$? At $x = -1$? At $x = 1$?

$$f(x) = \frac{1}{x}$$
4. What is the slope of the function at $x = -3$? At $x = 1$? At $x = 2$? At $x = 3$?

where $f(x) = \begin{cases} 
  x^2, & \text{when } x \leq 1 \\
  3, & \text{when } 1 < x \leq 2 \\
  x, & \text{when } 2 < x
\end{cases}$
<table>
<thead>
<tr>
<th>Problem #</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
</tbody>
</table>
Lesson Plan #2—Local Linearity

Description:
In this subsequent lesson, the students will be connecting some of the ideas of linearity to functions and slope and calculus. We will be exploring secant lines and tangent lines as it relates to one and two points on a function. We will observe the behavior of the function and the equation(s) of the secant line(s) as we pick points closer and closer together on the function. Then, the students will use their calculators to do the calculations for the secant lines of the function(s). Furthermore, students will come up with their own definitions of local linearity and will be required to construct linear equations that work for non-linear functions at certain points. This lesson will be over 1 hour and 25-minute class periods (B-day).

Concepts:
- The slopes of secant lines can be used to approximate the slope of the tangent line at a point on a function.
- Secant lines can be used to approximate the values of functions at certain points or over certain intervals.
- A curve can appear to be linear at a specific point on a curve (i.e. if we either zoom in really close at that point or we draw the line that is tangent to the curve at that point).

Standards:
APC.1 The student will define and apply the properties of elementary functions, including algebraic, trigonometric, exponential, and composite functions and their inverses, and graph these functions, using a graphing calculator. Properties of functions will include domains, ranges, combinations, odd, even, periodicity, symmetry, asymptotes, zeros, upper and lower bounds, and intervals where the function is increasing or decreasing.
### Time Sequence for Lesson #2:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
</table>
| 20-25 minutes | Introduction to Local Linearity | • Tell the students to get out their notes.  
• I will be at the SmartBoard to begin the lesson.  
• Draw a random f(x) and one points on the f(x): (a, f(a)).  
• Discuss how to write an equation of a line that “models” the behavior of f(x) at x = a.  
• Eventually, someone suggest using another point, which I will draw (x, f(x)).  
• Discuss the formula for slope as change in y over change in x and the equation with the two points given.  
• Discuss what happens as (x, f(x)) comes closer to point (x, f(x)). | How do we calculate the slope of a secant line?  
How can we approximate a tangent?  
What do the secant and tangent lines tell us about the function? |
| 20ish Minutes | Example as a Class | • On the SmartBoard, I will introduce a concrete example: “Write a line that models f(x) = x² when x = 2.”  
• The students will calculate different slopes for three different points (or really change in x’s). In this case, Δx = 0.1, 0.01, 0.001  
• Ask them to finish the statement: “As x approaches 2, m (or the slope) approaches ___”  
• Have them calculate the equation of the tangent line at x = 2. | How close should the other point be to (a, f(a)) or the point we are interested in?  
What happens to the slope as we continue to make that second point closer to the original points?  
How does this example relate to our activity from last class period? |
| 15ish Minutes | Finish-up Worksheet | • Tell students to retrieve worksheet from last lesson.  
• Allow them to work on the worksheet for problem 2 in pairs.  
• I will be roaming: asking and answering questions. | How would you personally describe Local Linearity to someone? |
<table>
<thead>
<tr>
<th>Remaining time</th>
<th>Calculator Training and Worksheet Finish-Up</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Graphing Calculators</td>
<td>• Show students how to use their graphing calculators to do the calculations for the secant lines at different Δx’s.</td>
</tr>
<tr>
<td></td>
<td>1. Enter f(x) into y= (y1=)</td>
</tr>
<tr>
<td></td>
<td>2. Go back to the working screen</td>
</tr>
<tr>
<td></td>
<td>3. Choose “vars”, and arrow over to “Y-vars”</td>
</tr>
<tr>
<td></td>
<td>4. Select function</td>
</tr>
<tr>
<td></td>
<td>5. Select y1</td>
</tr>
<tr>
<td></td>
<td>6. Enter y</td>
</tr>
<tr>
<td></td>
<td>• Make sure students understand the procedure for calculations.</td>
</tr>
<tr>
<td></td>
<td>• Allow extra time for working on the worksheet.</td>
</tr>
</tbody>
</table>

**Accommodations:**

I will print out the notes at the end of the class period for Allison to have as additional help in recording and understanding the course material. During the calculator training, I will go over to her to make sure that she was able to get it—although the visual graphing calculator on the SmartBoard should help as well.
Lesson Plan #3: Introduction to Limits and Properties of Limits

Description:

For this lesson, I will provide an overview and quick introduction to Pre-Calculus and Calculus (and therefore limits). I will introduce limit notation and we will begin to look at specific examples thru the usage of graphs, tables, and algebraic equations. Also, we will take a look at the three different situations where the limit of f(x) does not exist (DNE). We will look at specific examples for those three different situations. Then, we will review different properties of functions to introduce specific properties of limits. Trigonometric Expressions and their limits will be explored. This will be a lecture with whole-class examples and discussions, with a homework assignment given at the end for the next class period. This lesson will be over 1 hour and 25-minute class periods (B-day).

Concepts:

- Limits help us calculate the slope of a tangent line at a specific point or function, by taking the limit of the slope of the secant line for a point \( a \) on a function.
- Calculus and limits help us focus on mathematics at an infinite level, both infinitely small (or close) and large (like sums).
- Limits can be found graphically (from graphs), numerically (from tables), and analytically (from equations).
- Limits sometimes do not exist at a point. For a limit to exist, \( f(x) \) has to approach the same number on both sides of a function or equation.
- A limit is “the value” of a function in the close vicinity of a certain point.
- Limits can be used to determine the behavior of a function for both defined and undefined values.
Standards:

CCSS.MATH.CONTENT.HSA.REI.A.2

Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

APC.1
The student will define and apply the properties of elementary functions, including algebraic, trigonometric, exponential, and composite functions and their inverses, and graph these functions, using a graphing calculator. Properties of functions will include domains, ranges, combinations, odd, even, periodicity, symmetry, asymptotes, zeros, upper and lower bounds, and intervals where the function is increasing or decreasing.

APC.2
The student will define and apply the properties of limits of functions. Limits will be evaluated graphically and algebraically. This will include:
   a) limits of a constant;
   b) limits of a sum, product, and quotient;
   c) one-sided limits; and
   d) limits at infinity, infinite limits, and non-existent limits.

APC.3
The student will use limits to define continuity and determine where a function is continuous or discontinuous. This will include
   a) continuity in terms of limits;
   b) continuity at a point and over a closed interval;
   c) application of the Intermediate Value Theorem and the Extreme Value Theorem; and
   d) geometric understanding and interpretation of continuity and discontinuity.

Time Sequence for Lesson Plan #3:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>3ish Minutes</td>
<td>Review of Local Linearity</td>
<td>• Discuss and review Local Linearity as a class.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Open up for questions.</td>
<td></td>
</tr>
<tr>
<td>5-10 minutes</td>
<td>Pre-Calculus and Calculus Differences</td>
<td>• Officially introduce Calculus.</td>
<td>What is Calculus?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• T-chart of aspects/differences of Pre-Calculus and Calculus.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Calculus explained as a mathematics of change.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Explanation of Limits as the jump to Calculus.</td>
<td></td>
</tr>
</tbody>
</table>
### 20 Minutes
- **SmartBoard**

<table>
<thead>
<tr>
<th>Limit Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Redraw random ( f(x) ) with two points and secant line from previous lesson(s).</td>
</tr>
<tr>
<td>Discussion about what happens when ( (x, f(x)) ) is brought closer to ( (a, f(a)) ). (i.e. secant line becomes tangent line)</td>
</tr>
<tr>
<td>Limit notation introduced for the slope of a tangent line</td>
</tr>
<tr>
<td>Explanation of the 3 different ways to find or evaluate a limit (i.e. graphically with a graph, numerically with a table, and analytically with an equation).</td>
</tr>
<tr>
<td>Graphical Example of ( f(x) = \frac{x^3 - 1}{x - 1} ) ( \text{And evaluating the limit at } x = 1. )</td>
</tr>
<tr>
<td>Numerical Example by creating a table in order to calculate: ( \lim_{x \to 0} \frac{x}{\sqrt{(x + 1)} - 1} )</td>
</tr>
</tbody>
</table>

### Will \( x \) in a limit ever actually be \( a \) in a limit? What happens when we take the limit of the slope at a certain point? How can we evaluate or find limits?

### 15-20 Minutes
- **SmartBoard**

<table>
<thead>
<tr>
<th>Limits that don’t exist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example on the SmartBoard of ( f(x) = \frac{1}{x^2} )</td>
</tr>
<tr>
<td>Take the ( \lim_{x \to 0} f(x) ).</td>
</tr>
<tr>
<td>Discuss if the limit exists or not, and discuss why.</td>
</tr>
<tr>
<td>Introduce the three different situations when limits do not exist (DNE)</td>
</tr>
<tr>
<td>1. When the function approaches different numbers from the left and right.</td>
</tr>
<tr>
<td>2. When the function oscillates between two numbers as it approaches a value</td>
</tr>
<tr>
<td>3. When a function increases/decreases without a bound as it approaches a value (i.e. asymptote).</td>
</tr>
</tbody>
</table>

### What happens with the limit doesn’t approach a finite value? What might indicate that a limit does not exist (DNE)?

### 15 Minutes
- **SmartBoard**

<table>
<thead>
<tr>
<th>Examples of DNE Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw a piecewise function graph with a hole where it is not continuous.</td>
</tr>
</tbody>
</table>

### What do you notice about the functions where the left and right limits are
• Discuss why limit at the point with the hole DNE.
• Also, graph the example of \( \lim_{x \to 0} \frac{|x|}{x} \) and discuss why the limit DNE
• Provide two more examples for them to try on their own, namely:
  1. \( \lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} \)
  2. \( \lim_{x \to 2} f(x) \), where 
     \[
     f(x) = \begin{cases} 
     1, & x \neq 2 \\
     0, & x = 2 
     \end{cases}
     \]

### 20 Minutes
- SmartBoard
- See Properties of Limits Notes (Appendix B)
- Transition to talking about Functions: continuous and non-continuous
- Use the Smartboard to write notes for the different properties of limits and the basic equalities of limits.
- Introduce basic limits for trigonometric expressions
- See Appendix C for notes and examples that will be written

### 2 minutes
- Textbook
- White Board
- Lesson 3 HW (Appendix B)
- Assign homework on the white board from textbook:
  Pg. 67: 5-25 odd, 27-34, 37-44

### Accommodations:
Concerning Allison, I will print out the notes from the SmartBoard for her at the end of the class period.
1.3 Properties of limits

Remember: A limit is 'the value' of a function in the vicinity of a certain point.

Functions can be:
1. "well behaved" or continuous
2. non-continuous

The limit of continuous functions can be found by direct substitution.

\[
\lim_{x \to c} f(x) = f(c)
\]

Basic limits: If b \neq c are real numbers, then

1. \( \lim_{x \to c} b = b \)
2. \( \lim_{x \to c} x = c \)
3. \( \lim_{x \to c} x^a = c^a \)

Ex:

\[
\lim_{x \to 3} 2 = 2 \quad \lim_{x \to 3} x = 3 \quad \lim_{x \to 3} x^2 = 3^2 = 9
\]

Properties of limits

1. \( \lim_{x \to c} h \cdot f(x) = h \cdot \lim_{x \to c} f(x) \) - The limit of a constant times a function is the constant times the limit.

2. \( \lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \) - The limit of a sum is the sum of the limits.

3. \( \lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) \) - The limit of a product is the product of the limits.
4. \( \lim_{x \to c} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \quad \text{The limit of a quotient is the quotient of the limits} \)
\( \times \lim_{x \to c} g(x) \neq 0 \)

5. \( \lim_{x \to c} \left[ f(x) \right]^n = \left[ \lim_{x \to c} f(x) \right]^n \quad \text{The limit of the power is the power of the limit} \)

Examples

1. \( \lim_{x \to 2} (x^2 + 3x + 4) = 2^2 + 3(2) + 4 = 14 \)
   Using properties = \( \lim_{x \to 2} x^2 + \lim_{x \to 2} 3x + \lim_{x \to 2} 4 \)
   \( (\lim_{x \to 2} x)^2 + 3 \lim_{x \to 2} x + \lim_{x \to 2} 4 \)
   \( 2^2 + 3(2) + 4 = 14 \)

2. \( \lim_{x \to 1} 5(x + 2) = 5 \lim_{x \to 1} (x + 2) \)
   = 5 \( (1 + 2) \)
   = 15

3. \( \lim_{x \to -3} (x^2 + 3x + 1) = (-3)^2 + 3(-3) + 1 \)
   \( 4 - 6 + 1 = -1 \)
1. Limits of Trigonometric Expressions

\[
\lim_{x \to c} \sin(x) = \sin(c)
\]

* Use direct substitution

1. \[
\lim_{x \to \pi/2} \sin(x) = \sin(\pi/2) = 1
\]

2. \[
\lim_{x \to 0} 3 \cos(x) = 3 \lim_{x \to 0} \cos(x) = 3 \cos(0) = 3
\]

3. \[
\lim_{x \to \pi/2} (\tan^2(x)) = (\lim_{x \to \pi/2} \tan(x))^2 = (\tan(\pi/2))^2 = \infty = 3
\]
In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.

1. \( h(x) = -x^2 + 4x \)
   (a) \( \lim_{x \to +2} h(x) \)
   (b) \( \lim_{x \to -1} h(x) \)

2. \( g(x) = \frac{12(\sqrt{x} - 3)}{x - 9} \)
   (a) \( \lim_{x \to +4} g(x) \)
   (b) \( \lim_{x \to -4} g(x) \)

3. \( f(x) = x \cos x \)
   (a) \( \lim_{x \to +0} f(x) \)
   (b) \( \lim_{x \to -1} f(x) \)

4. \( f(t) = |t - 4| \)
   (a) \( \lim_{t \to +4} f(t) \)
   (b) \( \lim_{t \to -4} f(t) \)

In Exercises 5–22, find the limit.

5. \( \lim_{x \to +2} x^3 \)

6. \( \lim_{x \to +4} x^4 \)

7. \( \lim_{x \to 0} (2x - 1) \)

8. \( \lim_{x \to -3} (3x + 2) \)

9. \( \lim_{x \to +1} (x^2 + 3x) \)

10. \( \lim_{x \to +1} (-x^3 + 1) \)

11. \( \lim_{x \to +1} (2x^2 + 4x + 1) \)

12. \( \lim_{x \to +1} (3x^2 - 2x^2 + 4) \)

13. \( \lim_{x \to +1} \sqrt{x + 1} \)

14. \( \lim_{x \to +1} \sqrt[3]{x + 4} \)

15. \( \lim_{x \to +1} (x + 3)^2 \)

16. \( \lim_{x \to +1} (2x - 1)^3 \)

17. \( \lim_{x \to +1} \frac{1}{x - 2} \)

18. \( \lim_{x \to +1} \frac{2}{x + 2} \)

19. \( \lim_{x \to +1} \frac{x}{x^2 + 4} \)

20. \( \lim_{x \to +1} \frac{2x - 3}{x + 5} \)

21. \( \lim_{x \to +1} \frac{3x}{\sqrt{x^2 + 2}} \)

22. \( \lim_{x \to +1} \frac{\sqrt{x} + 2}{x - 4} \)

In Exercises 23–26, find the limits.

23. \( f(x) = 5 - x, \ g(x) = x^3 \)
   (a) \( \lim_{x \to +3} f(x) \)
   (b) \( \lim_{x \to +3} g(x) \)
   (c) \( \lim_{x \to +3} g(f(x)) \)

24. \( f(x) = x + 7, \ g(x) = x^2 \)
   (a) \( \lim_{x \to +3} f(x) \)
   (b) \( \lim_{x \to +3} g(x) \)
   (c) \( \lim_{x \to +3} g(f(x)) \)

25. \( f(x) = 4 - x^2, \ g(x) = \sqrt{x} + 1 \)
   (a) \( \lim_{x \to +3} f(x) \)
   (b) \( \lim_{x \to +3} g(x) \)
   (c) \( \lim_{x \to +3} g(f(x)) \)

26. \( f(x) = 2x^2 - 3x + 1, \ g(x) = \sqrt{x} + 6 \)
   (a) \( \lim_{x \to +3} f(x) \)
   (b) \( \lim_{x \to +3} g(x) \)
   (c) \( \lim_{x \to +3} g(f(x)) \)

In Exercises 27–36, find the limit of the trigonometric function.

27. \( \lim_{x \to +\pi/2} \sin x \)

28. \( \lim_{x \to +\pi/2} \tan x \)

29. \( \lim_{x \to +\pi/3} \cos \frac{\pi x}{3} \)

30. \( \lim_{x \to +\pi/2} \sin \frac{\pi x}{2} \)

31. \( \lim_{x \to +\pi/6} \sec 2x \)

32. \( \lim_{x \to +\pi/2} \cos 3x \)

33. \( \lim_{x \to +\pi/3} \sin x \)

34. \( \lim_{x \to +\pi/3} \cos x \)

35. \( \lim_{x \to +\pi/6} \tan \left(\frac{\pi x}{4}\right) \)

36. \( \lim_{x \to +\pi/6} \sec \left(\frac{\pi x}{6}\right) \)

In Exercises 37–40, use the information to evaluate the limits.

37. \( \lim_{x \to +\pi} f(x) = 3 \)
   \( \lim_{x \to +\pi} g(x) = 2 \)
   (a) \( \lim_{x \to +\pi} [5g(x)] \)
   (b) \( \lim_{x \to +\pi} [g(x) + g(x)] \)
   (c) \( \lim_{x \to +\pi} [f(x)g(x)] \)
   (d) \( \lim_{x \to +\pi} \frac{f(x)}{g(x)} \)

38. \( \lim_{x \to +\pi} f(x) = \frac{3}{2} \)
   \( \lim_{x \to +\pi} g(x) = \frac{1}{2} \)
   (a) \( \lim_{x \to +\pi} [4f(x)] \)
   (b) \( \lim_{x \to +\pi} [f(x) + g(x)] \)
   (c) \( \lim_{x \to +\pi} [f(x)g(x)] \)
   (d) \( \lim_{x \to +\pi} \frac{f(x)}{g(x)} \)

39. \( \lim_{x \to +\pi} f(x) = 4 \)
   (a) \( \lim_{x \to +\pi} [f(x)] \)
   (b) \( \lim_{x \to +\pi} \sqrt[3]{f(x)} \)
   (c) \( \lim_{x \to +\pi} [3f(x)] \)
   (d) \( \lim_{x \to +\pi} \frac{f(x)^3}{2} \)

40. \( \lim_{x \to +\pi} f(x) = 27 \)
   (a) \( \lim_{x \to +\pi} \sqrt{f(x)} \)
   (b) \( \lim_{x \to +\pi} \frac{f(x)}{18} \)
   (c) \( \lim_{x \to +\pi} [f(x)]^2 \)
   (d) \( \lim_{x \to +\pi} \frac{f(x)^3}{23} \)

In Exercises 41–44, use the graph to determine the limit visually (if it exists). Write a simpler function that agrees with the given function at all but one point.

41. \( g(x) = \frac{x^2 - 1}{x} \)

42. \( h(x) = \frac{-x^2 + 3x}{x} \)

(a) \( \lim_{x \to +1} g(x) \)
(b) \( \lim_{x \to +1} h(x) \)

43. \( g(x) = \frac{x^3 - 2}{x - 1} \)

44. \( f(x) = \frac{x}{x^2 - x} \)

(a) \( \lim_{x \to +1} f(x) \)
(b) \( \lim_{x \to +2} f(x) \)

(c) \( \lim_{x \to +0} f(x) \)
(d) \( \lim_{x \to +3} f(x) \)
Lesson Plan #4: Determining Limits Analytically

Description:
We will be exploring how to find a limit using algebra and by manipulating equations and expressions. We will introduce the idea of $\frac{0}{0}$ as a warning when solving limits analytically.

We will explore the results and situations for solving the limit when this happens. We will provide some guidelines for finding a limit analytically using many different algebraic techniques or processes learned in previous math classes. This lesson will be over 1 hour and 25-minute class periods (B-day).

Concepts:
- Limits can be determined analytically (by manipulating equations algebraically).
- When direct substitution of a limit gives $\frac{0}{0}$, it is a warning when solving limits analytically. It is in **Indeterminate Form**. This means that the limit can be manipulated algebraically to reveal the answer for a limit.

Standards:

**CCSS.MATH.CONTENT.HS.A.SSE.B.3**
Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

**CCSS.MATH.CONTENT.HS.A.REI.A.2**
Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

**CCSS.MATH.CONTENT.HS.A.REI.B.3**
Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
APC.1
The student will define and apply the properties of elementary functions, including algebraic, trigonometric, exponential, and composite functions and their inverses, and graph these functions, using a graphing calculator. Properties of functions will include domains, ranges, combinations, odd, even, periodicity, symmetry, asymptotes, zeros, upper and lower bounds, and intervals where the function is increasing or decreasing.

APC.2
The student will define and apply the properties of limits of functions. Limits will be evaluated graphically and algebraically. This will include:
   a) limits of a constant;
   b) limits of a sum, product, and quotient;
   c) one-sided limits; and
   d) limits at infinity, infinite limits, and non-existent limits.

APC.3
The student will use limits to define continuity and determine where a function is continuous or discontinuous. This will include
   a) continuity in terms of limits;
   b) continuity at a point and over a closed interval;
   c) application of the Intermediate Value Theorem and the Extreme Value Theorem; and
   d) geometric understanding and interpretation of continuity and discontinuity.

Time Sequence for Lesson Plan #4:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
</table>
| 2ish minutes | Homework Turn-In | • Have students turn in homework from previous class period.  
               • Have students get out their notes. | Why do we want to solve limits analytically?  
Why not just use a graph?  
How are f(x) and g(x) similar and different?  
What is significant about the difference(s)?  
What is the analytic situation (with direct substitution) where the limit does not exist? |
| 30 minutes  
• SmartBoard  
• Graphing Calculator (maybe) | Determining the Limit Analytically | • Start at the Smartboard with the graphs of two functions:  
1. \( f(x) = \frac{x^2-1}{x-1} \)  
2. \( g(x) = x + 1 \)  
• Have the students evaluate both limits (at \( x=1 \)) using direct substitution  
• Discussion of \( \frac{0}{0} \) as a warning algebraically. |  |
• Eventually, someone mentions 0 in the denominator as a problem
• Introduce *Factor and Cancel* and *Simplifying* as ways to change zero from being in the denominator.
• Work through following examples as a class:
  1. \( \lim_{{x \to -3}} \frac{x^2 + x - 6}{x + 3} \)
  2. \( \lim_{{x \to 1}} \frac{x^4 - 1}{x - 1} \)

20 Minutes
- **SmartBoard**
- **Notes or Guideline for Finding a Limit (Appendix C)**

<table>
<thead>
<tr>
<th>Guideline for Finding a Limit</th>
<th>• Set out Guideline for Finding a Limit Analytically • Follow Notes for Finding a Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is the first thing that you should do if you are given a limit to solve? If the limit initially equals ( \frac{0}{0} ), what should you do algebraically?</td>
<td></td>
</tr>
</tbody>
</table>

30 Minutes
- **SmartBoard**
- **White Board**

<table>
<thead>
<tr>
<th>Examples of Situations in Guideline</th>
<th>• Do these examples together as a class: ( \lim_{{x \to 0}} \frac{\sqrt{(x + 1)} - 1}{x} ) This one requires a rationalization of the numerator. ( \lim_{{x \to 0}} \frac{\sqrt{(x - 1)} - 2}{x - 5} ) This one also requires a rationalization of the numerator. ( \lim_{{x \to 0}} \frac{1}{x + 4} - \frac{1}{4} ) This one requires a simplification of complex fractions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Which method should we use in this example? What are the things in these expressions that we can manipulate algebraically?</td>
<td></td>
</tr>
</tbody>
</table>

2 minutes
- **Textbook**
- **White Board**

<table>
<thead>
<tr>
<th>Assignment of Homework</th>
<th>• Assign homework on the white board from textbook: Pg. 68: 45-61 odd</th>
</tr>
</thead>
</table>
Accommodations:

I will print out the notes from the SmartBoard for Allison at the end of the class period.
Appendix C

Guidelines for finding a limit

→ Use direct substitution
If result is: # = limit
$\% \Rightarrow$ limit does not exist
$\% \Rightarrow$ Indeterminate
Use algebra to simplify using
→ Factor
→ Rationalize numerator
→ Simplify complex fractions

In Exercises 45–48, find the limit of the function (if it exists). Write a simpler function that agrees with the given function at all but one point. Use a graphing utility to confirm your result.

45. \( \lim_{{x \to -1}} \frac{x^2 - 1}{x + 1} \)
46. \( \lim_{{x \to -1}} \frac{2x^2 - x - 3}{x + 1} \)
47. \( \lim_{{x \to 2}} \frac{x^3 - 8}{x - 2} \)
48. \( \lim_{{x \to -1}} \frac{x^1 + 1}{x + 1} \)

In Exercises 49–64, find the limit (if it exists).

49. \( \lim_{{x \to 0}} \frac{x}{x^2 - x} \)
50. \( \lim_{{x \to 0}} \frac{3x}{x^2 + 2x} \)
51. \( \lim_{{x \to 4}} \frac{x - 4}{x^2 - 16} \)
52. \( \lim_{{x \to 3}} \frac{3 - x}{x^2 - 9} \)
53. \( \lim_{{x \to -3}} \frac{x^2 + x - 6}{x^2 - 9} \)
54. \( \lim_{{x \to -1}} \frac{x^2 - 5x + 4}{x^2 - 2x - 8} \)
55. \( \lim_{{x \to 4}} \frac{\sqrt{x} + 5 - 3}{x - 4} \)
56. \( \lim_{{x \to 3}} \frac{\sqrt{x + 1} - 2}{x - 3} \)
57. \( \lim_{{x \to 0}} \frac{\sqrt{x + 5} - \sqrt{3}}{x} \)
58. \( \lim_{{x \to 0}} \frac{\sqrt{x + 5} - \sqrt{3}}{x} \)
59. \( \lim_{{x \to 0}} \frac{[1/(3 + x)] - (1/3)}{x} \)
60. \( \lim_{{x \to 0}} \frac{1/(x + 4) - (1/4)}{x} \)
61. \( \lim_{{\Delta x \to 0}} \frac{2(x + \Delta x) - 2x}{\Delta x} \)
62. \( \lim_{{\Delta x \to 0}} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \)
63. \( \lim_{{\Delta x \to 0}} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} \)
64. \( \lim_{{\Delta x \to 0}} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \)
Lesson Plan #5: Special Trig Limits

Description:
This lesson focuses on trigonometric limits that are not as readily obvious. Also, the Sandwich Theorem will be introduced. Then, we will harken back to the unit circle in order to solve the most important trig limit, which will lead us into the many of the other trig limits. We will work thru these examples as a whole class. This will be for 45-minute class periods (C-day).

Concepts:

- If two function “sandwich” \( f(x) \leq h(x) \leq g(x) \) for all \( x \), and the limit of \( f(x) \) and \( g(x) \) approach the same value as \( x \) approaches \( c \) \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L \), then \( \lim_{x \to c} h(x) = L \).

- When direct substitution of a limit gives \( \frac{0}{0} \), it is a warning when solving limits analytically. It is in Indeterminate Form. This means that the limit can be manipulated algebraically to reveal the answer for a limit.

- Limits can be found graphically (from graphs), numerically (from tables), and analytically (from equations).

Standards:

CCSS.MATH.CONTENT.HSA.SSE.B.3
Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

CCSS.MATH.CONTENT.HSA.REI.A.2
Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
**CCSS.MATH.CONTENT.HS.A.REI.B.3**

Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

**APC.1**
The student will define and apply the properties of elementary functions, including algebraic, trigonometric, exponential, and composite functions and their inverses, and graph these functions, using a graphing calculator. Properties of functions will include domains, ranges, combinations, odd, even, periodicity, symmetry, asymptotes, zeros, upper and lower bounds, and intervals where the function is increasing or decreasing.

**APC.2**
The student will define and apply the properties of limits of functions. Limits will be evaluated graphically and algebraically. This will include:

- **a)** limits of a constant;
- **b)** limits of a sum, product, and quotient;
- **c)** one-sided limits; and
- **d)** limits at infinity, infinite limits, and non-existent limits.

**APC.3**
The student will use limits to define continuity and determine where a function is continuous or discontinuous. This will include

- **a)** continuity in terms of limits;
- **b)** continuity at a point and over a closed interval;
- **c)** application of the Intermediate Value Theorem and the Extreme Value Theorem; and
- **d)** geometric understanding and interpretation of continuity and discontinuity.

**Time Sequence for Lesson Plan #5:**

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
</table>
| 2ish minutes       | Homework Turn-In          | • Have students turn in homework from previous class period.  
                       |                                                          | • Have the students get out their notes.             | What is the limit of this function?  
                       |                                                          |                                                      | How could you solve for this limit in different ways?  
                       |                                                          |                                                      | How can we solve this limit analytically?                |
| 5 minutes          | Initial Problem           | • Start at the Smartboard and pull up graphing calculator.  
                       | SmartBoard Graphing Calculators                       | • Ask students to graph:  
                       |                                                          | $f(x) = \frac{\sin(x)}{x}$ on their graphing calculators. |                                                    |
Discuss its continuity and how to solve for the limit:  
\[ \lim_{x \to 0} f(x) \]
Eventually, the class realizes that we are unsure how to solve it analytically.

| 10 minutes | Sandwich Theorem | Introduce Sandwich Theorem on SmartBoard  
Tell students that we will use the theorem in order to solve for \[ \lim_{x \to 0} \frac{\sin(x)}{x} \]
What is the intuition in this theorem? How would you describe this theorem to someone else?

| 10-15 Minutes | Using Unit Circle and Sandwich Theorem | Draw out the unit circle with similar triangles to set up Sandwich Theorem  
Follow Notes for Sandwich Theorem and Sine over x
What do you notice about the areas of the triangles and the corresponding sector? How can we set up an inequality and use the Sandwich Theorem with these areas? What are the areas equal to algebraically (with trigonometric expressions)?

| 10-15 Minutes | Examples of Other Special Trig Limits | Do these examples together as a class: \[ \lim_{x \to 0} \frac{1 - \cos(x)}{x} \]  
\[ \lim_{x \to 0} \frac{\tan(x)}{x} \]  
\[ \lim_{x \to 0} \frac{\sin(4x)}{x} \]  
\[ \lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)} \]
What trig identities can you use to simplify the limit? What limits do you already know? How can you manipulate these limits to make them simpler?

| 2 minutes | Assignment of Homework | Assign homework on the white board from textbook: |
| • Lesson 5 HW  (Appendix D) | • Pg. 68: 65-75 odd, 97-99 |

**Accommodations:**

I will print out the notes from the SmartBoard for Allison at the end of the class period.
Look at the areas of the two triangles and the corresponding sector.

$$\triangle AOP < \text{Sector AOP} < \Delta AOT$$

$$\frac{1}{2} \sin \theta \leq \pi \theta^2 \leq \frac{1}{2} \theta \times \left( \frac{\sin \theta}{\sin \theta} \right)$$

Multiply by 2

$$\sin \theta \leq \pi \theta^2$$

Divide by \( \sin \theta \)

$$1 \leq \frac{\theta}{\sin \theta}$$

Take reciprocal

(Signs switch)

$$\frac{1}{\pi} \leq \frac{1}{\theta}$$

$$\Rightarrow \frac{1}{\pi} \leq \frac{1}{\theta}$$

Switch order

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \leq \lim_{\theta \to 0} \frac{\sin \theta}{\theta}$$

Take limit

$$\frac{1}{\pi} \leq \lim_{\theta \to 0} \frac{\sin \theta}{\theta}$$

The limit sine is between two functions whose limit is one.

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$
In Exercises 65–76, determine the limit of the trigonometric function (if it exists).

65. \( \lim_{x \to 0} \frac{\sin x}{5x} \)

66. \( \lim_{x \to 0} \frac{3(1 - \cos x)}{x} \)

67. \( \lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^2} \)

68. \( \lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta} \)

69. \( \lim_{x \to 0} \frac{\sin^2 x}{x} \)

70. \( \lim_{x \to 0} \frac{\tan^2 x}{x} \)

71. \( \lim_{h \to 0} \frac{(1 - \cos h)^2}{h} \)

72. \( \lim_{\phi \to \pi} \phi \sec \phi \)

73. \( \lim_{x \to \pi/2} \frac{\cos x}{\cot x} \)

74. \( \lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x} \)

75. \( \lim_{t \to 0} \frac{\sin 3t}{2t} \)

76. \( \lim_{x \to 0} \frac{\sin 2x}{\sin 3x} \) \( \text{Hint: Find } \lim_{x \to 0} \left( \frac{2 \sin 2x}{2x} \right) \left( \frac{3x}{3 \sin 3x} \right) \).

**WRITING ABOUT CONCEPTS**

97. In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

98. Give an example of two functions that agree at all but one point.

99. What is meant by an indeterminate form?

100. In your own words, explain the Squeeze Theorem.
Rubric for Problems 97-99 on the Lesson 5 HW:

<table>
<thead>
<tr>
<th>Problem #</th>
<th>1</th>
<th>½</th>
<th>0</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>97</td>
<td>Student writes out a correct explanation, mentioning a hole or elevated point (point discontinuity) being at the one specific point for one of the functions.</td>
<td>Student incorrectly explains what is happening at that one point.</td>
<td>Student does not write out any explanation.</td>
<td></td>
</tr>
<tr>
<td>98</td>
<td>Student writes out a correct example of functions that agree at all but one point.</td>
<td>Student writes out an incorrect example of functions that agree at all but one point.</td>
<td>Student does not write out examples of functions that agree at all but one point.</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>Student writes out a correct explanation of indeterminate form, connecting it to solving limits analytically and needing to do more work algebraically.</td>
<td>Student only writes out a partial explanation of indeterminate form.</td>
<td>Student does not explain indeterminate form.</td>
<td></td>
</tr>
</tbody>
</table>
Lesson Plan #6: Limits and Continuity

Description:
In this lesson, we will be looking at limits and how they relate to continuity of functions. We will review the definition of continuity in light of functions. Then, we will review the different types of discontinuities. This will lead us to talking about the requirements for f(x) to be continuous at a certain point x = a. After the subsequent assignment of homework, we will move on to discussing one-sided limits. Then, we will look at specific example and applications of one-sided limits. This lesson will be over 1 hour and 25-minute class periods (B-day).

Concepts:
- Limits help us categorize and determine the types of discontinuities in functions.
- Some discontinuities are “removable” by algebraic manipulations (i.e. holes and elevated points), some discontinuities are “non-removable” (i.e. jumps and asymptotes).
- for f(x) to be continuous at a point x = a.
  1. f(x) must be defined at x = a (No Hole)
  2. The limit \( \lim_{x \to c} f(x) \) must exist (No jump or asymptote)
  3. The \( \lim_{x \to c} f(x) = f(a) \), which means that the limit equals the value (No elevated points)
- One-sided limits help us determine the nature of a function at a certain point as we approach that value of x from either the left or right direction. Some functions only have a one-sided limit, from the left or the right.
Standards:

CCSS.MATH.CONTENT.HS.A.SSE.B.3
Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

CCSS.MATH.CONTENT.HS.A.REI.A.2
Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

CCSS.MATH.CONTENT.HS.A.REI.B.3
Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

APC.1
The student will define and apply the properties of elementary functions, including algebraic, trigonometric, exponential, and composite functions and their inverses, and graph these functions, using a graphing calculator. Properties of functions will include domains, ranges, combinations, odd, even, periodicity, symmetry, asymptotes, zeros, upper and lower bounds, and intervals where the function is increasing or decreasing.

APC.2
The student will define and apply the properties of limits of functions. Limits will be evaluated graphically and algebraically. This will include:
   a) limits of a constant;
   b) limits of a sum, product, and quotient;
   c) one-sided limits; and
   d) limits at infinity, infinite limits, and non-existent limits.

APC.3
The student will use limits to define continuity and determine where a function is continuous or discontinuous. This will include
   a) continuity in terms of limits;
   b) continuity at a point and over a closed interval;
   c) application of the Intermediate Value Theorem and the Extreme Value Theorem; and
   d) geometric understanding and interpretation of continuity and discontinuity.
Time Sequence for Lesson Plan #6:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
</table>
| 2 minutes           | Homework Turn-In | • Have students turn in homework from previous class period.  
• Have students get out notes. | When is a function continuous?  
How do you know?  
When is a function not continuous?  
Which of these discontinuities are removable and which are non-removable?  
Which discontinuities do you recognize and do you think you can give a concrete example of a function with each one?  
What are the function values and limit values in these situations? |
| 15 minutes          | Determining the Limit Analytically | • Discuss when a function is continuous, and specific examples of continuous functions.  
• Review, draw, and list the different types of discontinuities (and if they are removable or not with algebra).  
1. Hole (which is removable algebraically)  
2. Jump  
3. Point, also known as an elevated point (which is removable with algebra).  
4. Asymptote  
• Also, discuss the limits for each of these discontinuities. | |
| 15 Minutes          | Requirements for f(x) to be continuous at a point x = a. | • List the requirements for f(x) to be continuous at a point x = a.  
4. f(x) must be defined at \( x = a \) (No Hole)  
5. The limit \( \lim_{{x \to c}} f(x) \) must exist (No jump or asymptote)  
6. The limit \( \lim_{{x \to c}} f(x) = f(a) \), which means that the limit equals the value (No point discontinuity), | What do we need to have for f(x) to be continuous at a certain point?  
How do we make sure that there is not a hole in the function?  
How can we use limits to show that no jump or asymptotes or point discontinuities exist? |
### Assignment of Homework
- Assign homework on the white board from textbook:
  - Pg. 78: 1-21 odd, 27, 35-51 odd

### One-Sided Limits
- Remind students about left and right limits and how they relate.
- Introduce idea that some functions will only have a left or right limit.
- Work thru examples as a class:
  \[
  \lim_{x \to -1^-} \frac{\sqrt{x} + 1}{2 - x} \\
  \lim_{x \to 2^+} \frac{x^2 - 4}{x^2 - 4} \\
  \lim_{x \to 2} f(x)
  \]
  for
  \[
  f(x) \begin{cases} 
    x^2 - 3x & x \leq 2 \\
    4 - x & x > 2
  \end{cases}
  \]
  and
  \[
  \lim_{x \to 2^+} \frac{|x - 2|}{x - 2}
  \]

### Accommodations:
I will print out the notes from the SmartBoard for Allison at the end of the class period.
1.4 Exercises

In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) \( \lim_{x \to c} f(x) \)  
(b) \( \lim_{x \to c_-} f(x) \)  
(c) \( \lim_{x \to c_+} f(x) \)

1. \[
\begin{array}{c}
\text{Graph 1}
\end{array}
\]

2. \[
\begin{array}{c}
\text{Graph 2}
\end{array}
\]

3. \[
\begin{array}{c}
\text{Graph 3}
\end{array}
\]

4. \[
\begin{array}{c}
\text{Graph 4}
\end{array}
\]

5. \[
\begin{array}{c}
\text{Graph 5}
\end{array}
\]

6. \[
\begin{array}{c}
\text{Graph 6}
\end{array}
\]
In Exercises 7–26, find the limit (if it exists). If it does not exist, explain why.

7. \( \lim_{x \to 5} \frac{1}{x^2 + 8} \)
8. \( \lim_{x \to 3} \frac{-3}{x + 5} \)
9. \( \lim_{x \to 5^+} \frac{x - 5}{x^2 - 25} \)
10. \( \lim_{x \to 2^-} \frac{2 - x}{x^2 - 4} \)
11. \( \lim_{x \to 3} \frac{x}{\sqrt{x^2 - 9}} \)
12. \( \lim_{x \to 9} \frac{\sqrt{x - 3}}{x - 9} \)
13. \( \lim_{x \to 0^+} \frac{|x|}{x} \)
14. \( \lim_{x \to 10^-} \frac{|x - 10|}{x - 10} \)
15. \( \lim_{\Delta x \to 0} \frac{1}{x + \Delta x} \frac{x}{\Delta x} \)
16. \( \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + x + \Delta x - (x^2 + x)}{\Delta x} \)

17. \( \lim_{x \to 2} f(x), \text{ where } f(x) = \begin{cases} \frac{x + 2}{2}, & x \leq 3 \\ \frac{12 - 2x}{3}, & x > 3 \end{cases} \)
18. \( \lim_{x \to 2} f(x), \text{ where } f(x) = \begin{cases} \frac{x^2 - 4x + 6}{x - 2}, & x < 2 \\ \frac{-x^2 + 4x - 2}{x - 2}, & x \geq 2 \end{cases} \)
19. \( \lim_{x \to 2} f(x), \text{ where } f(x) = \begin{cases} \frac{x^3 + 1}{x + 1}, & x < 1 \\ \frac{x - 1}{x}, & x \geq 1 \end{cases} \)
20. \( \lim_{x \to 1^-} f(x), \text{ where } f(x) = \begin{cases} x, & x \leq 1 \\ 1 - x, & x > 1 \end{cases} \)
21. \( \lim_{x \to \pi} \cot x \)
22. \( \lim_{x \to \pi/2} \sec x \)
23. \( \lim_{x \to 7} (5|x - 7|) \)
24. \( \lim_{x \to 2} (2x - [x]) \)
25. \( \lim_{x \to 5} (2 - |1 - x|) \)
26. \( \lim_{x \to 1} \left(1 - \left\lfloor -\frac{x}{2} \right\rfloor \right) \)

In Exercises 27–30, discuss the continuity of each function.

27. \( f(x) = \frac{1}{x^2 - 4} \)
28. \( f(x) = \frac{x^2 - 1}{x + 1} \)
29. \( f(x) = \frac{\sqrt{x}}{x + 1} \)
30. \( f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases} \)

In Exercises 31–34, discuss the continuity of the function on the closed interval.

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>31. ( g(x) = \sqrt{49 - x^2} )</td>
<td>([-7, 7])</td>
</tr>
<tr>
<td>32. ( f(t) = 3 - \sqrt{9 - t^2} )</td>
<td>([-3, 3])</td>
</tr>
<tr>
<td>33. ( f(x) = \begin{cases} 3 - x, &amp; x \leq 0 \ 3 + \frac{4}{x}, &amp; x &gt; 0 \end{cases} )</td>
<td>([-1, 4])</td>
</tr>
<tr>
<td>34. ( g(x) = \frac{1}{x^2 - 4} )</td>
<td>([-1, 2])</td>
</tr>
</tbody>
</table>

In Exercises 35–60, find the x-values (if any) at which \( f \) is not continuous. Which of the discontinuities are removable?

35. \( f(x) = \frac{6}{x} \)
36. \( f(x) = \frac{3}{x - 2} \)
37. \( f(x) = x^2 - 9 \)
38. \( f(x) = x^2 - 2x + 1 \)
39. \( f(x) = \frac{1}{x^2 + 1} \)
40. \( f(x) = \frac{1}{x^2 + 1} \)
41. \( f(x) = 3x - \cos x \)
42. \( f(x) = \cos \frac{\pi x}{2} \)
43. \( f(x) = \frac{x}{x^2 - x} \)
44. \( f(x) = \frac{x}{x^2 - 1} \)
45. \( f(x) = \frac{x}{x^2 + 1} \)
46. \( f(x) = \frac{x - 6}{x^2 - 36} \)
47. \( f(x) = \frac{x + 2}{x^2 - 3x - 10} \)
48. \( f(x) = \frac{x - 1}{x^3 + x - 2} \)
49. \( f(x) = \frac{|x + 7|}{x + 7} \)
50. \( f(x) = \frac{|x - 8|}{x - 8} \)
51. \( f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases} \)
52. \( f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases} \)
Lesson Plan #7: Intermediate Value Theorem and Review

❖ This lesson will be my second lesson plan that I want graded.

Description:

This lesson will introduce and cover the Intermediate Value Theorem for the whole class period. This theorem is connected to both limits and continuity. The second half of the lesson will be for review. The students will review their unit review and will be able to ask question and work together to complete their unit reviews, in preparation for their end-of-the-unit quiz/test. Finally, there will also be a post-test as a follow-up to the pre-test in this unit. This lesson will be over 1 hour and 25-minute class periods (B-day).

Concepts:

- Intermediate Value Theorem (IVT): If f(x) is continuous on \([a, b]\), there exists a number \(c\) where \(a \leq c \leq b\) such that \(f(c)\) is between \(f(a)\) and \(f(b)\).

Standards:

APC.1
The student will define and apply the properties of elementary functions, including algebraic, trigonometric, exponential, and composite functions and their inverses, and graph these functions, using a graphing calculator. Properties of functions will include domains, ranges, combinations, odd, even, periodicity, symmetry, asymptotes, zeros, upper and lower bounds, and intervals where the function is increasing or decreasing.

APC.2
The student will define and apply the properties of limits of functions. Limits will be evaluated graphically and algebraically. This will include:

a) limits of a constant;
b) limits of a sum, product, and quotient;
c) one-sided limits; and
d) limits at infinity, infinite limits, and non-existent limits.
APC.3
The student will use limits to define continuity and determine where a function is continuous or discontinuous. This will include
a) continuity in terms of limits;
b) continuity at a point and over a closed interval;
c) application of the Intermediate Value Theorem and the Extreme Value Theorem; and
 d) geometric understanding and interpretation of continuity and discontinuity.

### Time Sequence for Lesson Plan #7:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2ish minutes</td>
<td>Homework Turn-In</td>
<td>• Have students turn in homework from previous class period. • Have the students prepare notes.</td>
<td></td>
</tr>
<tr>
<td>5 minutes</td>
<td>Review of Open and Closed Intervals</td>
<td>• Start at the SmartBoard • Review and discuss open and closed intervals.</td>
<td>What are the difference between open and closed intervals? How do we designate open and closed intervals mathematically?</td>
</tr>
<tr>
<td>20 Minutes</td>
<td>Intermediate Value Theorem (IVT)</td>
<td>• Write out and introduce Intermediate Value Theorem • Draw graphical examples with generic points and terms.</td>
<td>Why might this Theorem be useful with functions and limits? Why does the function need to be continuous? What happens if it is not continuous?</td>
</tr>
<tr>
<td>15-20 Minutes</td>
<td>Example using IVT</td>
<td>• Release the class to do this example: ( f(x) = x^3 + 3x - 2 ) Is there a zero on the interval ([0,1])? • Ask them to verify the example with a graph. Is there a zero on the interval ([0,1])? How can you prove this algebraically using the Theorem? How can you represent this graphically? How would you explain this theorem to somebody else?</td>
<td></td>
</tr>
<tr>
<td>2 minutes</td>
<td>Assignment of Homework</td>
<td>• Assign homework on the white board from textbook:</td>
<td></td>
</tr>
</tbody>
</table>
Rest of the Time
• Post-Test

Post-Test

• Pg. 80: 63, 65, 83, 91, 95-98

• Have the students take the Post-Test.
• If they get done early, they can start on their homework.

Accommodations:

Allison will have extra time to complete the Post-Test. She will also be allowed to ask any clarifying questions relating to the problems on the test. I will also print out the SmartBoard notes for her after class.
Appendix G

Post-Test

Answer the following questions using the graph(s) provided:

1. What is the slope of this function at point A and point B?
2. What is the slope of the function at \( x = 0 \)? At \( x = 3 \)? And \( x = 2 \)?

\[
where \ f(x) = (x - 2)^2 - 1
\]
3. What is the slope of the function at $x = 0$? At $x = -1$? At $x = 1$?

$$f(x) = \frac{1}{x}$$
4. What is the slope of the function at $x = -3$? At $x = 1$? At $x = 2$? At $x = 3$?

where $f(x) = \begin{cases} 
  x^2, & \text{when } x \leq 1 \\
  3, & \text{when } 1 < x \leq 2 \\
  x, & \text{when } 2 < x 
\end{cases}$
<table>
<thead>
<tr>
<th>Problem #</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Student writes the correct value for the slope at each point.</td>
<td>Student incorrectly calculates some of the slopes at the points.</td>
<td>Student incorrectly calculates the slope at all of the points.</td>
<td></td>
</tr>
</tbody>
</table>
In Exercises 63–68, find the constant \( a \), or the constants \( a \) and \( b \), such that the function is continuous on the entire real line.

63. \( f(x) = \begin{cases} 3x^2, & x \geq 1 \\ ax - 4, & x < 1 \end{cases} \)

64. \( f(x) = \begin{cases} 3x^3, & x \leq 1 \\ ax + 5, & x > 1 \end{cases} \)

65. \( f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2, & x > 2 \end{cases} \)

66. \( g(x) = \begin{cases} 4\sin x, & x < 0 \\ x, & a - 2x, & x \geq 0 \end{cases} \)

67. \( f(x) = \begin{cases} 2, & x \leq -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases} \)

**Writing** In Exercises 83–86, explain why the function has a zero in the given interval.

<table>
<thead>
<tr>
<th>Function</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>83. ( f(x) = \frac{1}{12}x^4 - x^3 + 4 )</td>
<td>[1, 2]</td>
</tr>
<tr>
<td>84. ( f(x) = x^3 + 5x - 3 )</td>
<td>[0, 1]</td>
</tr>
<tr>
<td>85. ( f(x) = x^2 - 2 - \cos x )</td>
<td>[0, ( \pi )]</td>
</tr>
<tr>
<td>86. ( f(x) = -\frac{5}{x} + \tan\left(\frac{\pi x}{10}\right) )</td>
<td>[1, 4]</td>
</tr>
</tbody>
</table>
In Exercises 87–90, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly “zoom in” on the graph of the function to approximate the zero accurate to two decimal places. Use the zero or root feature of the graphing utility to approximate the zero accurate to four decimal places.

87. \( f(x) = x^3 + x - 1 \)
88. \( f(x) = x^3 + 5x - 3 \)
89. \( g(t) = 2 \cos t - 3t \)
90. \( h(t) = 1 + \sqrt{t} - 3 \tan t \)

In Exercises 91–94, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of \( c \) guaranteed by the theorem.

91. \( f(x) = x^2 + x - 1 \), \([0, 5]\), \( f(c) = 11 \)
92. \( f(x) = x^2 - 6x + 8 \), \([0, 3]\), \( f(c) = 0 \)
93. \( f(x) = x^3 - x^2 + x - 2 \), \([0, 3]\), \( f(c) = 4 \)
94. \( f(x) = \frac{x^2 + x}{x - 1} \), \( f(c) = 6 \)

WRITING ABOUT CONCEPTS

95. State how continuity is destroyed at \( x = c \) for each of the following graphs.

96. Sketch the graph of any function \( f \) such that

\[ \lim_{x \to 3} f(x) = 1 \quad \text{and} \quad \lim_{x \to 3} f(x) = 0. \]

Is the function continuous at \( x = 3 \)? Explain.

97. If the functions \( f \) and \( g \) are continuous for all real \( x \), is \( f + g \) always continuous for all real \( x \)? Is \( f/g \) always continuous for all real \( x \)? If either is not continuous, give an example to verify your conclusion.

98. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at \( x = 4 \)
(b) A function with a removable discontinuity at \( x = -4 \)
(c) A function that has both of the characteristics described in parts (a) and (b)

True or False? In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

99. If \( \lim_{x \to c} f(x) = L \) and \( f(c) = L \), then \( f \) is continuous at \( c \).
100. If \( f(x) = g(x) \) for \( x \neq c \) and \( f(c) \neq g(c) \), then either \( f \) or \( g \) is not continuous at \( c \).
101. A rational function can have infinitely many \( x \)-values at which it is not continuous.
102. The function \( f(x) = |x - 1|/(x - 1) \) is continuous on \((-\infty, \infty)\).

103. Swimming Pool Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine \( f(t) \) in the pool after \( t \) days.

Estimate and interpret \( \lim_{t \to 4} f(t) \) and \( \lim_{t \to 4} f(t) \).

104. Think About It Describe how the functions

\( f(x) = 3 + \lfloor x \rfloor \)

and

\( g(x) = 3 - \lfloor -x \rfloor \)

differ.

105. Telephone Charges A long distance phone service charges $0.40 for the first 10 minutes and $0.05 for each additional minute or fraction thereof. Use the greatest integer function to write the cost \( C \) of a call in terms of the time \( t \) (in minutes). Sketch the graph of this function and discuss its continuity.
Rubric for Problems 97-99 on the Lesson 5 HW:

<table>
<thead>
<tr>
<th>Problem #</th>
<th>1</th>
<th>1/2</th>
<th>0</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>Student correctly states how each graph is not continuous at ( x = c ).</td>
<td>Student incorrectly states how the graphs are not continuous at ( x = c ).</td>
<td>Student does not state how each graph is not continuous at ( x = c ).</td>
<td>Score</td>
</tr>
<tr>
<td>96</td>
<td>Student graphs a function that satisfies the limits and explains why the function is not continuous at ( x = 3 ).</td>
<td>Student either just graphs a function or only explains why the function is not continuous at ( x = 3 ).</td>
<td>Student does not graph or explain the continuity of the function at ( x = 3 ).</td>
<td>Score</td>
</tr>
<tr>
<td>97</td>
<td>Student writes out a correct explanation and gives an example for continuous functions being added and divided.</td>
<td>Student writes out a incorrect explanation or gives an incorrect example for continuous functions being added and divided.</td>
<td>Student does not give an explanation, and does not give an example of continuous functions being added and divided.</td>
<td>Score</td>
</tr>
<tr>
<td>98</td>
<td>Student describes the difference between a removable and nonremovable discontinuity, and gives examples.</td>
<td>Student either describes the difference between a removable and nonremovable discontinuity or just gives example.</td>
<td>Student does not describe the difference between a removable and nonremovable discontinuity, and does not give examples.</td>
<td>Score</td>
</tr>
</tbody>
</table>
Lesson Plan #8: Unit 1 Review

Description:

This lesson is a review day. It will take place on a C-day (45-minute class period).

Concepts:

The majority of the concepts from the previous lesson plans will be covered on this test.

Standards:

All of the standards listed in previous lessons.

Time Sequence for Lesson Plan #8:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole Class Period:</td>
<td>Unit Review</td>
<td>• Collect previous homework.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Hand out Unit Review</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Allow students to work on their Unit Review together and ask questions if need be.</td>
<td></td>
</tr>
</tbody>
</table>

Accommodations:

No accommodations in this lesson plan.
Appendix H

Unit 1 Review

Name: ________________________

Please show all of your work, do not assume that I will follow your thinking. Justify answers in as many different methods (analytic, numeric, graphical) as possible and applicable.

Determine the limits for each of the following:

1. \( \lim_{x \to -1} f(x) = \)

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.01</td>
<td>0.95</td>
</tr>
<tr>
<td>-1.001</td>
<td>0.995</td>
</tr>
<tr>
<td>2.9999</td>
<td>0.9995</td>
</tr>
<tr>
<td>3.0001</td>
<td>1.0005</td>
</tr>
<tr>
<td>3.001</td>
<td>1.005</td>
</tr>
<tr>
<td>3.01</td>
<td>1.05</td>
</tr>
</tbody>
</table>

2. \( \lim_{x \to 1^+} f(x) = \)

Determine the value of the following limits, analytically. Show all work leading to your answer.

3. A. \( \lim_{x \to 0} \frac{x^2 + 2x - 35}{x-5} \)

B. \( \lim_{x \to 0} \frac{1}{x^2} - \frac{1}{x} \)
4. \( \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 3(x+\Delta x) - (x^2 - 3x)}{\Delta x} \)

5. \( \lim_{x \to 0} \frac{\sqrt{x} + 4 - 2}{x} \)

6. A. For \( f(x) = \begin{cases} -x, & x < 7 \\ x - 7, & x \geq 7 \end{cases} \)

   a) \( \lim_{x \to -1^-} f(x) = \)

   b) \( \lim_{x \to -1^+} f(x) = \)

   c) \( \lim_{x \to -1^0} f(x) = \)
Find the following limits. Show all of your work.

7. A. \( \lim_{x \to 3^-} \frac{1-x}{x-3} \)  
    B. \( \lim_{x \to 4^+} \frac{2-x}{x-4} \)

8. A. \( \lim_{x \to 0} \frac{\sin(6x)}{x} \)  
    B. \( \lim_{x \to \pi} \sin \left( \frac{x}{2} \right) \)

Determine if each of the following have any vertical asymptotes. Show all of your work.

9. A. \( f(x) = \frac{3x+7}{x+4} \)  
    B. \( f(x) = \frac{3x^2+5x-2}{x+2} \)
Using the definition of continuity, determine if a discontinuity exists anywhere on the function. If the function is discontinuous, determine if it is removable or none-removable.

10.

![Graph of a function]

11. Write a rational function with a vertical asymptote at $x = -3$, a hole at $x = 0$, and a zero (x-intercept) at $x = 4$. 

![Graph of a rational function]
Lesson Plan #9: Unit 1 Test

Description:
This lesson plan includes 45 minutes of a 1-hour and 25-minute class period (B-day). The students will technically be a lesson into Unit 2 about derivatives. However, this 45-minute period will be for the Unit 1 Test. Then, the rest of the period will be for continuing the introduction of derivatives.

Concepts:
The majority of the concepts from the previous lesson plans will be covered on this test.

Standards:
All of the standards listed in previous lessons.

Time Sequence for Lesson Plan #9:

<table>
<thead>
<tr>
<th>Time and Materials</th>
<th>Activity</th>
<th>Description</th>
<th>Questions</th>
</tr>
</thead>
</table>
| 45ish Minutes:     | End-of-the-Unit Test | • Answer any last questions  
| • Calculators      |          | • Give students the end-of-the-unit test  
| • Unit 1 Test (Appendix F) |          | • If students finish early, they will have homework over derivatives. |           |

Accommodations:
Allison will have extra time to complete the assessment. She will also be allowed to ask any clarifying questions relating to the problems on the assessment, especially pertaining to language.
Appendix I

Unit 1 Test

Name: ____________________________
Period: ________________________

Please show all of your work, do not assume that I will follow your thinking. Justify answers in as many different methods (analytic, numeric, graphical) as possible and applicable.

1. \[ \lim_{x \to 3} f(x) = \]

<table>
<thead>
<tr>
<th>x</th>
<th>2.99</th>
<th>2.999</th>
<th>2.9999</th>
<th>3.0001</th>
<th>3.001</th>
<th>3.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.95</td>
<td>0.995</td>
<td>0.9995</td>
<td>1.0005</td>
<td>1.005</td>
<td>1.05</td>
</tr>
</tbody>
</table>

2. Use the figure to the right to determine the value of the following:
   a) \[ \lim_{x \to b^+} f(x) = \]
   b) \[ \lim_{x \to b^-} f(x) = \]
   c) \[ \lim_{x \to b^+} f(x) = \]
   d) \[ f(b) = \]

Determine the value of the following limits, analytically. Show all work leading to your answer.

3. A. \[ \lim_{x \to 0} \frac{x^2 + 2x - 35}{x + 7} \]
   B. \[ \lim_{x \to 0} \frac{1 + \frac{1}{x^2}}{x} \]
4. \( A. \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 3(x+\Delta x) - (x^2 - 3x)}{\Delta x} \) \hspace{1cm} B. \( \lim_{x \to 0} \frac{\sin(x)}{4x} \)

5. A. For \( f(x) = \begin{cases} -x, & x < -1 \\ x + 1, & x \geq -1 \end{cases} \)

\[ \text{a) } \lim_{x \to -1^-} f(x) = \]
\[ \text{b) } \lim_{x \to -1^+} f(x) = \]
\[ \text{c) } \lim_{x \to -1} f(x) = \]

B. If \( f(x) = \begin{cases} -x, & x < -1 \\ x + a, & x \geq -1 \end{cases} \)

\( \) determine the value of \( a \) so that \( f(x) \) is continuous

Determine the value of the following limits, analytically. Show all work necessary to support your answer

6. A. \( \lim_{x \to 2^-} \frac{5+x}{x+2} \) \hspace{1cm} B. \( \lim_{x \to 3^+} \frac{6-x}{3-x} \)
Determine if the following functions have a vertical asymptote. Show all work/explanations necessary to support your answer.

7. A. \( f(x) = \frac{x^2 - 4x - 5}{x - 5} \) 
   B. \( f(x) = \frac{2x^2 - x - 3}{x + 3} \)

Using the definition of continuity, determine if a discontinuity exits anywhere on the function. If the function is discontinuous, determine if it is removable or none-removable. Show all work necessary to justify your answer.

8. A. 
   B. 

[Graphs shown below]
9. Write a rational function with a vertical asymptote at $x = 6$, a hole at $x = 1$, and a zero (x-intercept) at $x = -1$.

10. Sketch a graph that meets the following criteria:

$$\lim_{x \to 2} f(x) = 4$$

$$f(x) = -2$$

11. Sketch a graph that meets the following criteria:

$$\lim_{x \to 2} f(x) = \text{DNE}$$

$$\lim_{x \to -2} f(x) = 5$$

12. Sketch a graph that meets the following criteria:

$$\lim_{x \to 2^+} f(x) = \text{DNE}$$

$$\lim_{x \to -1} f(x) = 5$$

$$f(0) = 8$$
Unit Test with Points Assigned

UNIT PLAN: LIMITS

Appendix H

Unit 1 Test

Name: ________________________
Period: ________________________

Please show all of your work, do not assume that I will follow your thinking. Justify answers in as many different methods (analytic, numeric, graphical) as possible and applicable.

1. \( \lim_{x \to 3} f(x) = \)

<table>
<thead>
<tr>
<th>x</th>
<th>2.99</th>
<th>2.999</th>
<th>2.9999</th>
<th>3.0001</th>
<th>3.001</th>
<th>3.01</th>
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<td>0.9995</td>
<td>1.0005</td>
<td>1.005</td>
<td>1.05</td>
</tr>
</tbody>
</table>

2. Use the figure to the right to determine the value of the following:

- a) \( \lim_{x \to b^-} f(x) = \)
- b) \( \lim_{x \to b^+} f(x) = \)
- c) \( \lim_{x \to b} f(x) = \)
- d) \( f(b) = \)

Determine the value of the following limits, analytically. Show all work leading to your answer.

3. A. \( \lim_{x \to 0} \frac{x^2+2x-35}{x+7} \)

\( \frac{1}{x+1} - \frac{1}{x} \)
UNIT PLAN: LIMITS

5 pt  4. A. \( \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2-3(x+\Delta x)-(x^2-3x)}{\Delta x} \)  

5 pt  B. \( \lim_{x \to 0} \frac{\sin(x)}{4x} \)

5 pt  5. A. For \( f(x) = \begin{cases} -x, & x < -1 \\ x + 1, & x \geq -1 \end{cases} \)  

a) \( \lim_{x \to -1^-} f(x) = \)  

b) \( \lim_{x \to -1^+} f(x) = \)

c) \( \lim_{x \to -1} f(x) = \)

5 pt  B. If \( f(x) = \begin{cases} -x, & x < -1 \\ x + a, & x \geq -1 \end{cases} \) determine the value of \( a \) so that \( f(x) \) is continuous

5 pt  6. A. \( \lim_{x \to 2^-} \frac{5+x}{x+2} \)  

5 pt  B. \( \lim_{x \to 3^+} \frac{6-x}{3-x} \)
Determine if the following functions have a vertical asymptote. Show all work/explanations necessary to support your answer.

8pt 7. A. \( f(x) = \frac{x^2-4x-5}{x-5} \) 8pt B. \( f(x) = \frac{2x^2-x-3}{x+3} \)

Using the definition of continuity, determine if a discontinuity exists anywhere on the function. If the function is discontinuous, determine if it is removable or non-removable. Show all work necessary to justify your answer.

4pt 8. A. 4pt B.
9. Write a rational function with a vertical asymptote at $x = 6$, a hole at $x = 1$, and a zero (x-intercept) at $x = -1$.

10. Sketch a graph that meets the following criteria:

- $\lim_{x \to 2} f(x) = 4$
- $f(x) = -2$

11. Sketch a graph that meets the following criteria:

- $\lim_{x \to 2} f(x) = \text{DNE}$
- $\lim_{x \to -2} f(x) = 5$

12. Sketch a graph that meets the following criteria:

- $\lim_{x \to -2^+} f(x) = \text{DNE}$
- $\lim_{x \to -1} f(x) = 5$
- $f(0) = 8$