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NON-EXISTENCE OF $5 \times 5$ FULL RAY-NONSINGULAR MATRICES

CHI-KWONG LI$^1$, THOMAS MILLIGAN$^1$, AND BRYAN L. SHADER$^1$

Abstract. An $n \times n$ complex matrix is full ray-nonsingular if it has no zero entries and every matrix obtained by changing the magnitudes of its entries is nonsingular. It is shown that a $5 \times 5$ full ray-nonsingular matrix does not exist. This, combined with earlier results, shows that there exists an $n \times n$ full ray-nonsingular matrix if and only if $n \leq 4$.

Key words. Ray-patterns, Ray-nonsingular matrices.

AMS subject classifications. 15A48, 15A57.

1. Introduction. A complex matrix is a ray-pattern matrix if each of its nonzero entries has modulus 1. A ray-pattern matrix is full if each of its entries is nonzero. An $n \times n$ complex matrix $A$ is ray-nonsingular if $A \circ X$ is nonsingular for all (entry-wise) positive matrices $X$, where $A \circ X$ denotes the Schur (entry-wise) product. Ray-nonsingular matrices with real entries are simply sign-nonsingular matrices; see [2] and its references. In [2], the authors posed the following question: for which $n$ does there exist an $n \times n$ full ray-nonsingular matrix? It is not hard to construct examples of $n \times n$ full ray-nonsingular matrices for $n \leq 4$; see [1, 2]. In [1], the authors showed that there are no full $n \times n$ ray-nonsingular matrices for $n \geq 6$. The question of whether there are $5 \times 5$ full ray-nonsingular matrices remains open. In this paper, we show that there is no $5 \times 5$ full ray-nonsingular matrix. As a result, we have the following complete answer for the question raised in [2]:

Main Theorem. There is an $n \times n$ full ray-nonsingular matrix if and only if $n \leq 4$.

The proof of the main theorem is quite detailed. In section 2, we recall some known results and outline our strategy for the proof. The key to the proof is an understanding of $3 \times 3$ full ray-patterns that are not ray-nonsingular. These are studied in section 3. The proof of the main theorem is given in section 4.

2. Preliminary results and basic strategies of proof. We first recall some terminology from [1]. A nonzero, diagonal ray-pattern matrix is a called a complex signing. A complex signing is strict if each diagonal entry is nonzero. A $(1, -1)$-signing is a diagonal matrix with diagonal entries in $\{1, -1\}$. A vector $v$ is balanced if zero is in the relative interior of the convex hull its entries (viewed as points of the complex plane). Furthermore, it is strongly balanced if its entries take on at least three distinct values. A ray-pattern vector $v$ is generic if for all $i < j$, $v_i \neq \pm v_j$.

Let $A$ denote the entry-wise conjugate of $A$. Consider the relation on the set of ray-patterns defined by $A \sim B$ if and only if there exist matrices $P$ and $Q$, each a

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product of permutation matrices and strict complex signings, such that $B = P \hat{A} Q$ where $A = A', A^t$ or $A$. Clearly, $\sim$ is an equivalence relation, and we have the following observation.

**Lemma 2.1.** Suppose $A$ and $B$ are ray-pattern matrices with $A \sim B$. Then $A$ is ray-nonsingular if and only if $B$ is ray-nonsingular.

We say that the matrix $A$ is **strongly balanceable** if $A \sim B$ for some $B$ each of whose columns is strongly balanced. The following three lemmas from [1] will be useful in establishing the nonexistence of a $5 \times 5$ full ray-nonsingular matrix.

**Lemma 2.2.** [1, Lemma 3.7] Let $A$ be an $n \times n$ full ray-pattern. If $A$ has an $m \times m$ strongly balanceable submatrix with $m \geq 3$, then $A$ is not ray-nonsingular.

In section 3, we establish sufficient conditions for a $3 \times 3$ full ray-pattern to be strongly balanceable.

**Lemma 2.3.** [1, Theorem 4.3] Let $A = (a_{jk})$ be a $5 \times 5$ full ray-pattern. If $a_{jk} \in \{1, -1, i, -i\}$ for all $j$ and $k$, then $A$ is not ray-nonsingular.

**Lemma 2.4.** [1, Proposition 4.4] Let $A$ be a $5 \times 5$ full ray-pattern with first column consisting of all 1’s and each remaining column generic. Then $A$ is not ray-nonsingular.

Note that Lemma 2.4 implies that if $A$ is a $5 \times 5$ full ray-nonsingular matrix, then each row and column of $A$ intersects a $2 \times 2$ submatrix of the form

$$\begin{pmatrix} x & y \\ z & \pm yz/x \end{pmatrix}.$$ 

We now give a basic outline of our strategy for proving the main theorem. The proof will be by contradiction. Thus, we will assume to the contrary that there is a $5 \times 5$ full ray-nonsingular matrix $A$. We then use the results of section 3 (that give sufficient conditions for a $3 \times 3$ full ray-pattern to be strongly balanceable) and Lemmas 2.1–2.4 to show that, up to $\sim$-equivalence, the leading $3 \times 3$ submatrix of $A$ has one of the following forms:

(a) \[
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & e^{i\beta} \\ 1 & e^{i\alpha} & -1 \end{pmatrix},
\]

(d) \[
\begin{pmatrix} 1 & 1 & e^{i\beta} \\ 1 & 1 & -1 \\ 1 & e^{i\alpha} & -1 \end{pmatrix}, \quad \text{or} \quad (e) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \end{pmatrix}.
\]

Next, for each of these cases, we use Lemma 2.2 and the results of section 3 to conclude that either

(i) all entries of $A$ belong to $\{1, -1, i, -i\}$, or

(ii) all entries of $A$ belong to $\{e^{2\pi i/3}, e^{4\pi i/3}\}$ arranged in certain patterns.

Finally, we obtain a contradiction by showing that if $A$ satisfies (i) or (ii), then $A$ is not ray-nonsingular.
3. Sufficient conditions for $3 \times 3$ patterns to be strongly balanceable.

One of the keys to our proof of the main theorem is Lemma 2.2 which implies that no $3 \times 3$ submatrix of a $5 \times 5$, full ray-nonsingular matrix is strongly balanceable. In this section, we give sufficient conditions for a $3 \times 3$ full ray-pattern to be strongly balanceable.

By Lemma 2.1, we may restrict our attention to ray-patterns of the form

$$B = \begin{bmatrix} 1 & e^{i\alpha_2} & 1 \\ 1 & e^{i\alpha_3} & e^{i\beta_3} \\ 1 & e^{i\beta_2} & e^{i\beta_3} \end{bmatrix}.$$ \hspace{1cm} (3.1)

As the function $e^{ix}$, $x$ real, is $2\pi$-periodic, we may assume that each of $\alpha_2$, $\alpha_3$, $\beta_2$ and $\beta_3$ lies in the interval $(-\pi, \pi]$. For convenience we partition $(-\pi, \pi]$ into the following sets:

$$\mathcal{P} = (0, \pi), \quad \mathcal{N} = (-\pi, 0), \quad \{0\}, \quad \{\pi\}.$$

We first determine the strict signings $S$ for which the vector $(1, 1, 1)^S$ is strongly balanced. Note that for each $\theta \in (-\pi, \pi]$, the vector $(1, 1, 1)(e^{i\theta}S)$ is strongly balanced if and only if the vector $(1, 1, 1)$ is strongly balanced. Hence, it suffices to determine the $S$ whose leading diagonal entry is 1.

**Lemma 3.1.** Let $S = \text{diag}(1, e^{ix}, e^{iy})$ be a strict signing with $x, y \in (-\pi, \pi]$. Then $(1, 1, 1)^S$ is strongly balanced if and only if $x \in \mathcal{P}$ and $-\pi < y < x - \pi$, or $x \in \mathcal{N}$ and $\pi + x < y < \pi$.

**Proof.** Note that $(1, 1, 1)^S$ is strongly balanced if and only if no two of 1, $e^{ix}$ and $e^{iy}$ are equal or opposite, and the convex hull, $H$, of $\{1, e^{ix}, e^{iy}\}$ contains the origin. Thus, $(1, 1, 1)^S$ is not strongly balanced if $x = 0$, $x = \pi$, $y = 0$, $y = \pi$ or $x = y \pm \pi$. If $x \in \mathcal{P}$, then it is easy to verify that $H$ contains the origin if and only if $-\pi < y < x - \pi$. If $x \in \mathcal{N}$, then it is easy to verify that $H$ contains the origin if and only if $\pi + x < y < \pi$. The lemma now follows.

The shaded regions without their boundaries given in Figure 1, represent the region of the Cartesian plane determined by the inequalities in Lemma 3.1.

![Figure 1](image1.png)

![Figure 2](image2.png)
Next, we investigate a general vector \( z = (1, e^{i\alpha}, e^{i\beta}) \), and let \( R(\alpha, \beta) \) be the region of the Cartesian plane consisting of the points \((x, y)\) such that \( z \text{ diag}(1, e^{ix}, e^{iy}) \) is strongly balanced and \( x, y \in (-\pi, \pi] \). Thus \( R(0, 0) \) is the region described in Lemma 3.1, and illustrated in Figure 1. Let \( D = \text{diag}(1, e^{i\alpha}, e^{i\beta}) \). Note that \( S \) is a strict signing such that \( zS \) is strongly balanced if and only if \( DS \) is a strict signing such that \((1, 1, 1)DS \) is strongly balanced. It follows that \( R(\alpha, \beta) \) can be obtained from \( R(0, 0) \) by identifying opposite edges of the square \([-\pi, \pi] \times [-\pi, \pi] \) or torus, and then translating the shaded region in Figure 1 by \((-\alpha, -\beta)\). For example, \( R(\alpha, \beta) \) (where \( \alpha \in \mathcal{P} \) and \( \beta < \alpha \)) is presented in Figure 2.

Note that \( R(0, 0) \cap R(\alpha, \beta) \) represents the points \((x, y)\) in the plane such that \(-\pi \leq x, y < \pi \) and both rows of

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & e^{i\alpha} & e^{i\beta}
\end{bmatrix}
\text{diag}(1, e^{ix}, e^{iy})
\]

are strongly balanced. It is a tedious, but straightforward, to determine the regions \( R(0, 0) \cap R(\alpha, \beta) \). We do this as follows. First partition the vectors of the form \( z = [1, e^{i\alpha}, e^{i\beta}] \) according to the locations and relationships between \( \alpha \) and \( \beta \) as given by the 24 types described in Table 1. The sets \( R(0, 0) \cap R(\alpha, \beta) \) for each of these 24 types are the shaded regions without the boundaries illustrated in the Appendix.

### Table 1. Types for the vector \([1, e^{i\alpha}, e^{i\beta}]\).

<table>
<thead>
<tr>
<th>Type</th>
<th>(\alpha) in (\beta) in</th>
<th>Conditions</th>
<th>Class</th>
<th>(\alpha) in (\beta) in</th>
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<tr>
<td>1</td>
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<td>(\mathcal{P})</td>
<td>(\alpha &gt; \beta)</td>
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<td>(\beta - \alpha &lt; \pi)</td>
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<td>(\mathcal{P})</td>
<td>(\alpha = \beta)</td>
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<td>(\alpha = \beta)</td>
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<td>(\mathcal{N})</td>
<td>(\alpha - \beta = \pi)</td>
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<td>12</td>
<td>(\mathcal{N})</td>
<td>(\mathcal{P})</td>
<td>(\beta - \alpha = \pi)</td>
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</table>

We finally turn our attention to studying the strong balanceability of the matrix \( B \) in (3.1). Note that \( B \) is strongly balanceable if and only if \( R(0, 0) \cap R(\alpha_2, \beta_2) \cap R(\alpha_3, \beta_3) \neq \emptyset \), or equivalently if and only if

\[
(R(0, 0) \cap R(\alpha_2, \beta_2)) \cap (R(0, 0) \cap R(\alpha_3, \beta_3)) \neq \emptyset.
\]

If the second (or third) row has form (C9), i.e. is \([1, 1, 1]\), then trivially, this intersection corresponds to the solution set of the first and third (or second) row. Also, if the second (or third) row has form (C10)-(C12), then the solution set is empty and
so intersection is trivially empty. Thus, we need only consider those cases when the second and third rows are of one of the first 20 types listed on Table 1 that is, for each pair of these 20 types we need to study the intersection of the pair of corresponding regions listed in the Appendix.

The results of this straight-forward but tedious study are summarized in Table 2 below. The rows and columns of Table 2 are indexed by the 20 classes other than (C9)-(C12). An entry of ‘1’ indicates that the pair of specified regions always has nonempty intersection. For example, the fact that there is a ‘1’ in the row indexed by 9 and column indexed by 1, implies that every $3 \times 3$ matrix whose first row is $[1, 1, 1]$, whose second row is of type (9), and whose third row is type (2), is strongly balanceable, and hence not ray-nonsingular.

For some pairs of types the regions intersect only under certain conditions on $\alpha_1, \alpha_2, \beta_1, \beta_2$. For example, consider a matrix $B$ whose rows are $u_1 = [1, 1, 1]$, $u_2 = [1, e^{i\alpha_1}, e^{i\beta_1}]$ of type (C1) and $u_3 = [1, e^{i\alpha_2}, e^{i\beta_2}]$ of type (C5). From the figures in the Appendix, we see that the corresponding solution sets have empty intersection if and only if $\pi - \beta_2 \leq \alpha_1$. Table 3 lists other conditions, derived from an analysis of the regions in Figure 1, for certain pairs to have empty intersection. We include only those pairs relevant to our discussion.

Table 2. Types of pairs whose solution sets always have nonempty intersection

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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C7</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>C8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3. Necessary and sufficient conditions for empty intersection of the two solution sets.

<table>
<thead>
<tr>
<th>Type of $[1 \ e^{i\alpha_1} \ e^{i\beta_1}]$</th>
<th>Type of $[1 \ e^{i\alpha_2} \ e^{i\beta_2}]$</th>
<th>Condition on $\alpha_1, \alpha_2, \beta_1, \beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>$\alpha_2 \leq \alpha_1$</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>$\alpha_2 \leq \alpha_1$</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>$\alpha_1 \leq \alpha_2$</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>$\beta_2 \leq \beta_1$</td>
</tr>
<tr>
<td>6</td>
<td>C1</td>
<td>$\alpha_1 \leq \alpha_2$</td>
</tr>
<tr>
<td>6</td>
<td>C5</td>
<td>$\beta_1 - \beta_2 + 2\pi \leq \alpha_1$</td>
</tr>
<tr>
<td>6</td>
<td>C6</td>
<td>$\beta_2 \leq \beta_1$</td>
</tr>
<tr>
<td>9</td>
<td>11</td>
<td>$\alpha_2 \leq \alpha_1$</td>
</tr>
<tr>
<td>9</td>
<td>C7</td>
<td>$\beta_2 \leq \alpha_1 = \beta_1$</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>$\alpha_1 \leq \alpha_2$</td>
</tr>
<tr>
<td>10</td>
<td>C1</td>
<td>$\beta_1 + \pi = \alpha_1 + \pi \leq \alpha_2$</td>
</tr>
<tr>
<td>10</td>
<td>C5</td>
<td>$\beta_1 + \pi = \alpha_1 + \pi \leq \beta_2$</td>
</tr>
<tr>
<td>10</td>
<td>C8</td>
<td>$\beta_1 = \alpha_1 \leq \beta_2$</td>
</tr>
<tr>
<td>11</td>
<td>C6</td>
<td>$\beta_2 \leq \beta_1 = \alpha_1 - \pi$</td>
</tr>
<tr>
<td>12</td>
<td>C5</td>
<td>$\alpha_1 + \pi = \beta_1 \leq \beta_2$</td>
</tr>
<tr>
<td>C1</td>
<td>C5</td>
<td>$\pi \leq \alpha_1 + \beta_2$</td>
</tr>
<tr>
<td>C3</td>
<td>C5</td>
<td>$\pi \leq \alpha_1 + \beta_2$</td>
</tr>
<tr>
<td>C5</td>
<td>C7</td>
<td>$\beta_2 \leq \beta_1$</td>
</tr>
<tr>
<td>C5</td>
<td>C8</td>
<td>$\beta_2 + \pi \leq \beta_1$</td>
</tr>
<tr>
<td>C6</td>
<td>C8</td>
<td>$\beta_1 \leq \beta_2$</td>
</tr>
</tbody>
</table>

We conclude this section by illustrating how to use Tables 2 and 3 to determine information about the columns of certain matrices. This will allow the reader to get a feel for how these arguments work while also providing information needed later.

**Example 3.2.** Let $B$ be a $3 \times 3$ matrix which is not strongly balanceable and whose rows are

$$
\begin{align*}
  u_1 &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \\
  u_2 &= \begin{bmatrix} 1 & 1 & e^{i\beta_1} \end{bmatrix}, \\
  u_3 &= \begin{bmatrix} 1 & -1 & e^{i\beta_2} \end{bmatrix},
\end{align*}
$$

where $\{e^{i\beta_1}, e^{i\beta_2}\} \cap \{\pm 1\} = \emptyset$.

If we assume that $\beta_1 \in P$, then $u_2$ has type (C5) and $u_3$ has type (C7) or (C8). By Table 3, if $u_3$ has type (C7) then $\beta_2 \leq \beta_1$, if $u_3$ has type (C8), then $\beta_2 + \pi \leq \beta_1$. If we are interested in the vector $v = \begin{bmatrix} 1 & e^{i\beta_1} & e^{i\beta_2} \end{bmatrix}$, then $v$ has one of the following types: (1), (6), (9) or (11).

If $\beta_1 \in N$, then we may apply the above reasoning to $\bar{B}$, and thereby conclude that $v$ has one of the following types: (4), (8), (10), (12).

Therefore, $v$ has type (1), (4), (6), (8), (9), (10), (11) or (12).

A similar analysis gives the following.

**Example 3.3.** Let $B$ be a $3 \times 3$ matrix which is not strongly balanceable and whose rows are

$$
\begin{align*}
  u_1 &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \\
  u_2 &= \begin{bmatrix} 1 & -1 & e^{i\beta_1} \end{bmatrix}, \\
  u_3 &= \begin{bmatrix} 1 & 1 & e^{i\beta_2} \end{bmatrix},
\end{align*}
$$
where \( \{e^{i\beta_1}, e^{i\beta_2}\} \cap \{ \pm 1 \} = \emptyset \). Then \([1 e^{i\beta_1}, e^{i\beta_2}]\) has one of the following types: (2), (3), (6) or (8)–(12).

4. Proof of main theorem. Throughout the remainder of the paper we will let \( A \) denote a \( 5 \times 5 \) full ray nonsingular matrix. We say that \( A \) is in standard form if each entry in row and column 1 is equal to 1. By Lemma 2.1, there is no loss of generality in assuming that \( A \) is in standard form. We first show that \( A \) has a \( 3 \times 3 \) submatrix that is \( \sim \)-equivalent to one of several special forms.

Proposition 4.1. If \( A \) is a \( 5 \times 5 \) full ray-nonsingular matrix, then \( A \) has a \( 3 \times 3 \) submatrix that is \( \sim \)-equivalent to a matrix of one of the following forms:

\[
\begin{align*}
(a) & \quad \begin{bmatrix} 1 & 1 & 1 & e^{i\alpha} & e^{i\beta} \\
1 & 1 & 1 & e^{i\alpha} & 1 \\
1 & e^{i\alpha} & e^{i\beta} & 1 & 1 \\
1 & 1 & e^{i\alpha} & e^{i\beta} & 1 \\
1 & 1 & 1 & e^{i\alpha} & 1 \\
\end{bmatrix}, \\
(b) & \quad \begin{bmatrix} 1 & 1 & 1 & e^{i\alpha} & e^{i\beta} \\
1 & 1 & 1 & 1 & 1 \\
1 & e^{i\alpha} & e^{i\beta} & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
\end{bmatrix}, \\
(c) & \quad \begin{bmatrix} 1 & 1 & 1 & e^{i\alpha} & e^{i\beta} \\
1 & 1 & 1 & 1 & 1 \\
1 & e^{i\alpha} & e^{i\beta} & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
\end{bmatrix}, \\
(d) & \quad \begin{bmatrix} 1 & 1 & 1 & e^{i\alpha} & e^{i\beta} \\
1 & 1 & 1 & e^{-i\alpha} & e^{-i\beta} \\
1 & e^{i\alpha} & e^{i\beta} & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
\end{bmatrix}, \\
(e) & \quad \begin{bmatrix} 1 & 1 & 1 & e^{i\alpha} & e^{i\beta} \\
1 & 1 & 1 & 1 & 1 \\
1 & e^{i\alpha} & e^{i\beta} & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
1 & e^{i\alpha} & 1 & 1 & 1 \\
\end{bmatrix}.
\end{align*}
\]

Proof. Let \( A \) be a \( 5 \times 5 \) full ray-nonsingular matrix in standard form. By Lemma 2.4, each row and column of \( A \) intersects a \( 2 \times 2 \) submatrix of the form 

\[
\begin{bmatrix} x & y \\
z & \pm \frac{y}{x} \\
\end{bmatrix}.
\]

By Lemma 2.1, we may assume that the \( 2 \times 2 \) submatrix intersecting the first row is \( A[[1, 2], [1, 2]] \), and that \( a_{jk} = 1 \) whenever \( j = 1 \) or \( k = 1 \). Then \( a_{22} = \pm 1 \). Let \( a_{jk} = e^{i\alpha jk} \) and \( u_j = [1, e^{i\alpha j}, e^{i\beta j}] \) for \( j, k = 1, 2, 3, 4, 5 \).

We claim that one of the following conditions holds:

\[
e^{i\alpha j k} = \pm 1 \text{ or } e^{i\beta j k} = \pm 1 \text{ for some } j \in \{3, 4, 5\},
\]

\[
e^{i\alpha j k} = \pm 1 \text{ for some } j \in \{3, 4, 5\},
\]

\[
e^{i\beta j k} = \pm e^{i\alpha j k} \text{ for some } j \in \{3, 4, 5\}.
\]

Suppose to the contrary that none of these conditions hold. Then \( u_3, u_4 \) and \( u_5 \) do not have types (C1)–(C12) nor (9)–(12). Also, \( u_2 \) can only have type (C5)–(C8). In fact, since \( A \sim \overline{A} \), we may assume without loss of generality that \( x_{23} \in P \), and therefore \( u_4 \) has either type (C5) or (C7).

First consider the case that \( u_2 \) has type (C5). Because the matrix with rows \( u_1, u_2 \) and \( u_j \) is not strongly balanceable, Table 2 implies that each \( u_j \) \( (j = 3, 4, 5) \) has type

\[
(1), (4), (6) \text{ or } (8).
\]

Note that if two vectors, say \( u_j \) and \( u_k \), have the same type, then the matrix with rows \( u_1, u_j \) and \( u_k \) is strongly balanceable by Table 2. Also, from Table 2, any collection
of three distinct rows of types (1), (4), (6) or (8) contains two rows whose solutions sets intersect, and we have the contradiction that $A$ contains a strongly balanceable $3 \times 3$ submatrix. Thus, $u_2$ does not have type (C5).

Next consider the remaining case that $u_2$ has type (C7). Because the matrix with rows $u_1$, $u_2$ and $u_j$ is not strongly balanceable, Table 2 implies that each $u_j$ ($j = 3, 4, 5$) has type (2), (5) or (6).

As no type can be repeated, we can assume that $u_3$ has type (2), $u_4$ has type (5) and $u_5$ has type (6). But then the matrix with rows $u_1, u_3$ and $u_5$ is strongly balanceable by Table 2. Thus, $u_2$ does not have type (C7) and we have a contradiction.

Therefore we have shown that at least one of the three conditions in (4.1) holds. If $e^{ix_2} = \pm 1$ or $e^{ix_j} = \pm 1$ for some $j \in \{3, 4, 5\}$, then $A$ has a $3 \times 3$ submatrix equivalent to a matrix of form (a) or (b). If for some $j \in \{3, 4, 5\}$ we have $e^{ix_j} = \pm 1$, then $A$ has a $3 \times 3$ submatrix equivalent to a matrix of form (c), (d) or (e). Suppose, for some $j \in \{3, 4, 5\}$, that $e^{ix_j} = \pm e^{ix_3}$. Then

$$\begin{bmatrix}
1 & 1 & 1 \\
1 & \pm 1 & e^{ix_3} \\
1 & e^{ix_j} & \pm e^{ix_3}
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 \\
1 & \pm 1 & e^{ix_3} \\
1 & e^{ix_j} & \pm 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 \\
1 & \pm 1 & \pm e^{ix_3} \\
1 & e^{ix_j} & \pm 1
\end{bmatrix}. $$

In other words, $A$ has a $3 \times 3$ submatrix equivalent to a matrix of form (c), (d) or (e).

In the following subsections we show the presence of each type of $3 \times 3$ submatrix in Proposition 4.1 leads to a contradiction.

4.1. Form (a). In this section, we show that a full 5 by 5 full ray-nonsingular matrix does not have a $3 \times 3$ submatrix with form (a).

**Proposition 4.2.** Let $A$ be a $5 \times 5$ full ray pattern whose leading submatrix has form (a). Then $A$ is not ray-nonsingular.

**Proof.** Assume to the contrary that $A$ is ray-nonsingular. Without loss of generality we may assume that $A$ is in standard form.

Let $u_1, u_2, u_3, u_4, u_5$ be the rows of $A[\{1, 2, 3, 4, 5\}, \{1, 2, 3\}]$. Thus, $u_1 = u_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Let

$$u_3 = \begin{bmatrix} 1 & e^{i\alpha_3} & e^{i\beta_3} \end{bmatrix}, \quad u_4 = \begin{bmatrix} 1 & e^{i\alpha_4} & e^{i\beta_4} \end{bmatrix}, \quad u_5 = \begin{bmatrix} 1 & e^{i\alpha_5} & e^{i\beta_5} \end{bmatrix}. $$

Note that since $u_1 = u_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $u_i$ ($i \in \{3, 4, 5\}$) is not of the type (1)–(12) nor (C1)–(C9); otherwise the matrix with rows $u_1, u_2, u_i$ is strongly balanceable. So $u_i$ ($i \in \{3, 4, 5\}$) has type (C10), (C11) or (C12). If $u_3 = u_4 = u_5$, then the matrix with rows $u_3, u_4, u_5$ is strongly balanceable. Thus, $u_3, u_4, u_5$ are not all equal. Thus, by $\sim$-equivalence, we may assume that $A$ is one of the following two matrices.

$$B_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & a_{24} & a_{25} & 1 \\
1 & 1 & -1 & a_{34} & a_{35} \\
1 & 1 & -1 & a_{44} & a_{45} \\
1 & -1 & 1 & a_{54} & a_{55}
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & a_{24} & a_{25} \\
1 & 1 & -1 & a_{34} & a_{35} \\
1 & -1 & 1 & a_{44} & a_{45} \\
1 & -1 & 1 & a_{54} & a_{55}
\end{bmatrix}. $$
In both cases, $A$ has the $3 \times 3$ submatrices

\[
\begin{bmatrix}
1 & 1 & a_{2j} \\
1 & 1 & a_{3j} \\
1 & 1 & a_{5j}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & a_{2j} \\
1 & 1 & a_{5j}
\end{bmatrix}
\]

for $j = 4, 5$. Since the transpose of neither of these matrices is strongly balanceable, $a_{ij} = \pm 1$ for $i = 2, 3, 5$ and $j = 4, 5$. If $A = B_1$, then $A$ has the $3 \times 3$ submatrices

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & a_{2j} \\
1 & 1 & a_{4j}
\end{bmatrix},
\]

and so $a_{4j} = \pm 1$ for $j = 4, 5$. If $A = B_2$, then $A$ has the $3 \times 3$ submatrices

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & a_{2j} \\
-1 & -1 & a_{4j}
\end{bmatrix}\sim
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & a_{2j} \\
1 & 1 & -a_{4j}
\end{bmatrix}.
\]

Thus, $a_{4j} = \pm 1$ for $j = 4, 5$. And therefore, $a_{kj} = \pm 1$ for all $k, j$, and by Lemma 2.3, $A$ is not ray-nonsingular—a contradiction.

Therefore, no $3 \times 3$ submatrix of $A$ is equivalent to (a).

4.2. Form (b). In this section, we show that the existence of a submatrix of $A$ with form (b) implies that all entries of $A$ are in $\{\pm 1\}$, and thereby contradict Lemma 2.3.

**Proposition 4.3.** Let $A$ be a $5 \times 5$ full ray pattern whose leading $3 \times 3$ principle submatrix has form (b). Then $A$ is not ray-nonsingular.

**Proof.** Suppose to the contrary that $A$ is ray-nonsingular. Without loss of generality we may assume that $A$ is in standard form.

Let $u_1, \ldots, u_5$ be the rows of $A[\{1, 2, 3, 4, 5\}, \{1, 2, 3\}]$. Thus $u_1 = [1 \ 1 \ 1]$ and $u_2 = [1 \ -1 \ 1]$. Let

\[
u_3 = [1 \ e^{ix_3} \ e^{iy_3}] , \quad u_4 = [1 \ e^{ix_4} \ e^{iy_4}] , \quad u_5 = [1 \ e^{ix_5} \ e^{iy_5}] .
\]

By Proposition 4.2, no $e^{iy_j}$ ($j = 3, 4, 5$) is equal to $1$, and not all $e^{iy_j}$ ($j = 3, 4, 5$) equal $-1$. Thus, there exists a $j \in \{3, 4, 5\}$, such that $e^{iy_j} \neq \pm 1$ and, since $A \sim \overline{A}$, we may assume without loss of generality that $y_j \in \mathcal{P}$.

Since for each $j \in \{3, 4, 5\}$ the matrix with rows $u_1, u_2, u_j$ is not strongly balanceable and since no submatrix of $A$ is $\sim$-equivalent to a matrix of form (a), Example 3.3 implies that each $u_j$ ($j = 3, 4, 5$) has one of the following types:

(2), (3), (6), (8)–(12), (C3)–(C8), (C10) or (C12).

In particular, we see that $u_3$ has one of the following types:

(2), (8), (9), (12), (C5) or (C7).
Next, let $v_1, \ldots, v_5$ be the rows of the matrix obtained from 

$$A[{1,2, \ldots, 5}, \{1,2,3\}]$$

by multiplying its second column by $-1$. Then 

$$v_1 = [1 -1 1], v_2 = [1 1 1], v_3 = [1 -e^{ix_3} e^{iy_3}], v_4 = [1 -e^{ix_4} e^{iy_4}], v_5 = [1 -e^{ix_5} e^{iy_5}].$$

Note that $u_3$ has type (2), (9), (C5) if and only if $v_3$ has type (8), (12), (C7), respectively. Thus, we may assume without loss of generality that $u_3$ has type (2), (9) or (C5).

Now, we consider several subcases.

**Case A: Either $u_4$ or $u_5$ has type (C10) or (C12).** We may assume that $u_5$ has type (C10) or (C12); otherwise we permute the fourth and fifth rows of $A$.

**Subcase A.i:** $u_5$ has type (C10); i.e. $u_5 = [1 1 -1]$. 

Recall that $u_3$ has type (2), (9) or (C5) while $u_4$ has one of the types: (2), (3), (6), (8), (9)–(12), (C3)–(C8), (C10) or (C12).

Consider the matrix

$$\begin{pmatrix} u_1 \\ u_5 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ e^{ix_3} & e^{iy_3} \end{pmatrix}$$

Proposition 4.2 implies that $u_3$ does not have type (C5). It follows from Example 3.2 that $u_3$ does not have type (2). Hence $u_3$ has type (9).

Since the matrix

$$\begin{pmatrix} u_1 \\ u_3 \\ u_4 \end{pmatrix}$$

is not strongly balanceable, Table 2 implies that $u_4$ is not of type (2), (3), (9), (C4), (C5), or (C8). Since this matrix is not equivalent to a matrix of form (a), $u_4$ is not of type (10) or (C12).

Since

$$\begin{pmatrix} u_1 \\ u_4 \\ u_5 \end{pmatrix}$$

is not $\sim$-equivalent to a matrix of form (a), $u_4$ does not have type (C6) or (C10).

Note that

$$\begin{pmatrix} u_2 \\ u_5 \\ u_4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & e^{ix_4} \\ 1 & -1 & e^{iy_4} \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -e^{ix_4} \\ 1 & -1 & e^{iy_4} \end{pmatrix}. $$
From Table 2, we see that if $e^{ix_j}e^{iy_j} \neq \pm 1$, then the sign of the imaginary parts of $-e^{ix_j}$ and $e^{iy_j}$ do not agree. In other words, $x_j \in P$ implies $y_j \in P$ and $x_j \in N$ implies $y_j \in N$. Thus, $u_4$ does not have type (6), (8), (11) or (12).

If $u_4$ has type (C3), then $v_3$ has type (12) and $v_4$ has type (C4), and so (by Table 2) the matrix with rows $v_2$, $v_3$, $v_4$ is strongly balanceable. Thus, $u_4$ does not have type (C3).

If $u_4$ has type (C7), then by Table 2 the matrix with rows $u_5$, $u_3$, $u_4$ is strongly balanceable because it is equivalent to

\[
\begin{bmatrix}
1 & 1 & -1 \\
1 & e^{ix_3} & e^{iy_3} \\
1 & -1 & e^{iy_4}
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 1 \\
1 & e^{ix_3} & -e^{ix_3} \\
1 & -1 & -e^{iy_4}
\end{bmatrix}
\]

which has rows of type (11) and (C8).

Hence, for each possible type of $u_4$ we obtain a contradiction. Therefore, subcase A.i, does not occur.

**Subcase A.ii: $u_5$ has type (C12); i.e. $u_5 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$**

Recall that $u_5$ has type (2), (9) or (C5) while $u_4$ has one of the following types: (2), (3), (6), (8)–(12), (C3)–(C8), (C10) or (C12). If $u_3$ has type (2) or (9), then

\[
\begin{bmatrix}
u_1 \\
v_3 \\
v_5
\end{bmatrix}
= \begin{bmatrix} 1 & 1 & 1 \\
1 & e^{ix_3} & e^{iy_3} \\
1 & -1 & -1
\end{bmatrix}
\sim
\begin{bmatrix} 1 & 1 & 1 \\
1 & e^{ix_3} & 1 \\
1 & e^{iy_3} & 1
\end{bmatrix}
\]

is strongly balanceable (by Table 2) because the last two rows both have type (C3). Hence $u_5$ has type (C5).

Note that if $u_4$ has type (C10) then we are back to Case A.i., and if it has type (C12) then we have a $3 \times 3$ submatrix of form (a) and we contradict Proposition 4.2. Table 2 applied to the matrix with rows $u_1$, $u_3$, $u_4$ implies that $u_4$ does not have type (2), (3), (9), (11), (C4) or (C5).

The type of $u_4$ is not (6) or (8); otherwise $\begin{bmatrix} 1 & -e^{ix_4} & -e^{iy_4} \end{bmatrix}$ has either type (5) or (7),

\[
\begin{bmatrix}
u_5 \\
u_2 \\
u_4
\end{bmatrix}
= \begin{bmatrix} 1 & -1 & -1 \\
1 & -1 & 1 \\
1 & e^{ix_4} & e^{iy_4}
\end{bmatrix}
\sim
\begin{bmatrix} 1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -e^{ix_4} & -e^{iy_4}
\end{bmatrix}
\sim
\begin{bmatrix} 1 & 1 & 1 \\
1 & 1 & -e^{ix_4} \\
1 & -1 & -e^{iy_4}
\end{bmatrix},
\]

and Example 3.2 leads to the contradiction that the last matrix is strongly balanceable by Table 2.

The type of $u_4$ is not (10); otherwise

\[
\begin{bmatrix}
u_1 \\
u_4 \\
u_5
\end{bmatrix}
= \begin{bmatrix} 1 & 1 & 1 \\
1 & e^{ix_4} & e^{ix_4} \\
1 & -1 & -1
\end{bmatrix}
\sim
\begin{bmatrix} 1 & 1 & 1 \\
1 & e^{ix_4} & 1 \\
1 & e^{iy_4} & -1
\end{bmatrix},
\]

which has second and third row of type (C4), and the matrix is strongly balanceable by Table 2.
The type of \( u_4 \) is not (12); otherwise
\[
\begin{bmatrix}
  u_5 \\
u_3 \\
u_4
\end{bmatrix} = \begin{bmatrix}
  1 & -1 & -1 \\
  1 & 1 & e^{iy_3} \\
  1 & -e^{iy_4} & e^{iy_4}
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & 1 \\
  1 & -1 & -e^{iy_3} \\
  1 & e^{iy_4} & -e^{iy_4}
\end{bmatrix},
\]
where \( y_3, y_4 \in \mathcal{P} \). This matrix has second row of type (C8) and third row of type (11), and hence is strongly balanceable by Table 2.

The type of \( u_4 \) is not (C3); otherwise
\[
\begin{bmatrix}
u_2 \\
u_3 \\
u_4
\end{bmatrix} = \begin{bmatrix}
  1 & -1 & 1 \\
  1 & 1 & e^{iy_3} \\
  1 & e^{ix_4} & -1
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & 1 \\
  1 & -1 & e^{iy_3} \\
  1 & -e^{ix_4} & -1
\end{bmatrix},
\]
where \( y_3, x_4 \in \mathcal{P} \). But then the second row has type (C7) and the third row has type (C4), and the matrix is strongly balanceable by Table 2.

If \( u_4 \) has type (C6), then the matrix with rows \( u_1, u_3, u_4 \) is equivalent to
\[
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & e^{iy_3} \\
  1 & 1 & e^{iy_4}
\end{bmatrix}
\]
and we contradict Proposition 4.2. Likewise, if \( u_4 \) has type (C7) or (C8), then
\[
\begin{bmatrix}
  u_2 \\
u_4 \\
u_5
\end{bmatrix} = \begin{bmatrix}
  1 & -1 & 1 \\
  1 & -1 & e^{iy_3} \\
  1 & -1 & -1
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & e^{iy_3} \\
  1 & 1 & -1
\end{bmatrix},
\]
and we contradict Proposition 4.2.

Therefore, we conclude that Subcase A.ii does not occur. Moreover, Case A does not occur.

**Case B:** \( u_4 \) and \( u_5 \) have neither type (C10) nor (C12).

Recall from the beginning of the proof that \( u_3 \) has type (2), (9) or (C5). Also, for \( j = 4, 5 \), \( u_j \) has type (2), (3), (6), (8) – (12), (C3), (C4), (C6), (C7) or (C8).

**Subcase B.i:** \( \{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} = \emptyset \).

Since \( \{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} = \emptyset \), \( u_3 \) does not have type (C5), and \( u_j \) (\( j = 4, 5 \)), does not have type (C6), (C7), or (C8).

First suppose \( u_3 \) has the type (2). Table 2 applied to the matrices with rows \( u_1, u_3, u_4 \), and rows \( u_1, u_3, u_5 \) implies that \( u_j \) (\( j = 4, 5 \)) has type (8) or (12). Since the matrix with rows \( u_1, u_4, u_5 \) is not strongly balanceable, by Table 2, \( u_4 \) and \( u_5 \) do not have the same type. We may assume that \( u_4 \) has type (8) and \( u_5 \) has type (12), but then, by Table 2, the matrix with rows \( u_1, u_4, u_5 \) is strongly balanceable. Hence \( u_3 \) does not have type (2).

Next suppose \( u_3 \) has type (9). By Table 2, applied to the matrix with rows \( u_1, u_3, u_j \), the vector \( u_j \) (\( j = 4, 5 \)) has type (6), (8), (10), (11), (12) or (C3). Note that \( v_3 \) has type (12), and Table 2 applied to the matrix with rows \( v_1, v_3, v_j \) (\( j = 4, 5 \))
implies that $v_j$ has type $(2), (4), (6), (9), (10), (11), (C2), (C3), (C5), (C8)$; that is, $u_j$ has type $(8), (5), (3), (12), (11), (10), (C1), (C4), (C7)$ or $(C6)$. Upon comparison of the two list of possibilities for the type of $u_j$, we conclude that $u_j$ ($j = 4, 5$) has type $(8), (10), (11)$ or $(12)$.

If $u_j$ has type $(8)$, then $v_j$ has type $(2)$ and we are back to case handled in the second paragraph of this subcase. Hence $u_j$ does not have type $(8)$.

If $u_j$ has type $(10)$, then the matrix with rows $u_1, u_3, u_j$ is $\sim$-equivalent to a matrix of type $(a)$, contrary to Proposition 4.2. Hence $u_j$ does not have type $(10)$.

Now $u_j$ must have the type $(11)$ or $(12)$. By Table 2, $u_4$ and $u_5$ do not have the same type. We may assume that $u_4$ has type $(11)$ and $u_5$ has type $(12)$. But then

$$
\begin{bmatrix}
  u_2 \\
  u_4 \\
  u_5
\end{bmatrix} = \begin{bmatrix}
  1 & -1 & 1 \\
  1 & -e^{ix_4} & e^{ix_4} \\
  1 & -e^{ix_5} & e^{ix_5}
\end{bmatrix} \sim \begin{bmatrix}
  1 & e^{-ix_4} & e^{-ix_5} \\
  1 & 1 & 1 \\
  1 & 1 & 1
\end{bmatrix},
$$

and we contradict Proposition 4.2.

Thus, we conclude that Subcase B.i does not occur.

**Subcase B.ii.** $\{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} \neq \emptyset$.

We know that $u_3$ has type $(2), (9)$ or $(C5)$. We claim that without loss of generality we may assume that $u_3$ has type $(C5)$.

To see this, suppose $u_3$ has type $(2)$ or $(9)$. Since $\{e^{ix_3}, e^{ix_4}, e^{ix_5}\} \cap \{\pm 1\} \neq \emptyset$ and since neither $u_4$ nor $u_5$ have type $(C10)$ nor $(C12)$, there exists $j \in \{4, 5\}$ such that $e^{ix_j} = \pm 1$ while $e^{ix_k} \neq \pm 1$. Now, interchange rows $3$ and $j$ and if $e^{ix_j} = -1$, multiply the second column by $-1$ and interchange the first two rows. We may assume $y_3 \in P$ since $A \sim \mathcal{T}$. Note that this new third row has type $(C5)$. Therefore, we may assume that $u_3$ has type $(C5)$.

Recall that $u_j$ ($j = 4, 5$) has type $(2), (3), (6), (8)-(12), (C3), (C4), (C6), (C7)$ or $(C8)$. By Table 2 applied to the matrix with rows $u_1, u_3$ and $u_j$, the vector $u_j$ does not have type $(2), (3), (9), (11)$, or $(C4)$. By considering the matrix with rows $u_1, u_3, u_j$, we see that by Proposition 4.2, $e^{ix_j} \neq 1$ for $j = 4, 5$; thus, $u_j$ does not have type $(C6)$. Hence $u_j$ can only have one of the following types: $(6), (8), (10), (12), (C3), (C7)$ or $(C8)$.

But $v_3$ has type $(C7)$ and Table 2 applied to the matrix with rows $v_2, v_3$ and $v_j$, implies that has one of the types $(8), (4), (3), (12), (10), (C2), (C4), (C7), (C8)$, or $(C6)$. By comparing the two lists of possibilities for the type of $u_j$, we conclude that $u_j$ ($j = 4, 5$) has type $(8), (10), (12), (C7)$ or $(C8)$.

Let $\{j, k\} = \{4, 5\}$. Suppose $u_j$ has type $(8)$. Then $u_k$ does not have type $(8), (12)$ or $(C7)$ else, by Table 2, the matrix with rows $u_1, u_j, u_k$ is strongly balanceable. Note that $v_j$ has type $(2)$. If $u_k$ has type $(10)$ or $(C8)$, then $v_k$ has type $(11)$ or $(C6)$ and so the matrix with rows $v_1, v_j, v_k$ is strongly balanceable. Therefore, $u_j$ does not have type $(8)$.

Next suppose $u_j$ has type $(C7)$. But then the matrix with rows $u_1, u_j$ and $u_k$ is strongly balanceable for $u_k$ of any type but $(C8)$. However, if $u_k$ has type $(C8)$, then
the matrix formed by rows \( u_2, u_j, u_k \) is equivalent to
\[
\begin{bmatrix}
  u_2 \\
  u_j \\
  u_k
\end{bmatrix} = \begin{bmatrix}
  1 & -1 & 1 \\
  1 & -1 & e^{ij} \\
  1 & -1 & e^{ik}
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & e^{ij} e^{ik}
\end{bmatrix},
\]
which contradicts Proposition 4.2. Thus, \( u_j \) \((j = 4, 5)\) does not have type \((C7)\). Therefore, \( u_j \) and \( u_k \) will have one of the following types: \((10)\), \((12)\) or \((C8)\).

They will not both have the same type, else the matrix with rows \( u_1, u_j, u_k \) is strongly balanceable. We now examine the restrictions on the entries in each of the possible combinations of types by considering the following subcases.

**Subcase B.ii.a** \( u_4 \) and \( u_5 \) have types \((C8)\) and \((10)\) respectively.

In other words, there exist \( \alpha, \beta, \gamma \in \mathcal{P} \) such that
\[
u_3 = [ 1 \; 1 \; e^{i\alpha} ], \quad u_4 = [ 1 \; -1 \; e^{i(\beta - \pi)} ], \quad u_5 = [ 1 \; e^{i(\gamma - \pi)} \; e^{i(\gamma - \pi)} ].
\]

By Table 3, we find conditions on these angles such that there are no 3 \( \times \) 3 submatrices that are strongly balanceable. Because the matrix with rows \( u_1, u_3, u_4 \) is not strongly balanceable,

\[
\gamma \leq \beta.
\]

Because the matrix with rows \( v_2, v_4, v_5 \) is not strongly balanceable,

\[
\gamma \geq \beta.
\]

Equations (4.2) and (4.3) imply \( \gamma = \beta \). Also, because the matrix with rows \( u_1, u_3, u_4 \) is not strongly balanceable,

\[
\gamma = \beta \leq \alpha.
\]

Suppose \( \gamma = \beta < \alpha \). For \( j = 3, 4, 5 \), let \( \hat{u}_j \) be such that
\[
\begin{bmatrix}
  u_3 \\
  u_4 \\
  u_5
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & e^{i\alpha} \\
  1 & -1 & e^{i(\beta - \pi)} \\
  1 & e^{i(\beta - \pi)} & e^{i(\beta - \pi)}
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & e^{i(\beta - \alpha - \pi)} \\
  1 & -1 & e^{i(\beta - \alpha - \pi)} \\
  1 & e^{i(\beta - \alpha - \pi)} & e^{i(\beta - \alpha - \pi)}
\end{bmatrix} = \begin{bmatrix}
  \hat{u}_3 \\
  \hat{u}_4 \\
  \hat{u}_5
\end{bmatrix}.
\]

Since 0 < \( \beta < \beta + (\pi - \alpha) < \alpha + (\pi - \alpha) = \pi \) and \( e^{i(\beta + \pi - \alpha)} = e^{i(\beta - \alpha - \pi)} \), \( \hat{u}_4 \) has type \((C7)\) and \( \hat{u}_5 \) has type \((8)\), and hence, by Table 2, the matrix with rows \( \hat{u}_3, \hat{u}_4, u_5 \) is strongly balanceable. Therefore

\[
\gamma = \beta = \alpha.
\]

**Subcase B.ii.b** \( u_4 \) and \( u_5 \) have types \((C8)\) and \((12)\) respectively.

In other words, there exist \( \alpha, \beta, \gamma \in \mathcal{P} \) such that
\[
u_3 = [ 1 \; 1 \; e^{i\alpha} ], \quad u_4 = [ 1 \; -1 \; e^{i(\beta - \pi)} ], \quad u_5 = [ 1 \; -e^{i\gamma} \; e^{i\gamma} ].
\]
The matrix formed by rows $u_1, \ldots, u_5$ is equivalent, by complex conjugation, $(1, -1)$-signings and row permutation, to

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & e^{i\alpha} \\
1 & -1 & e^{i(\beta - \pi)} \\
1 & e^{i\gamma} & e^{i\gamma}
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -1 & 1 \\
1 & 1 & e^{-i\alpha} \\
1 & 1 & e^{-i(\beta - \pi)} \\
1 & e^{-i\gamma} & e^{-i\gamma} \\
1 & e^{-i\gamma} & e^{-i\gamma}
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & e^{i(\pi - \beta)} \\
1 & e^{i\gamma} & e^{i\gamma} \\
1 & e^{i\gamma} & e^{i\gamma}
\end{bmatrix}.
\]

Note that the third, fourth and fifth rows of this matrix have types (C5), (C8) and (10) respectively. Thus, by Case B.ii.a.,

\[\alpha = \beta = \gamma.\]

Subcase B.ii.c. $u_4$ and $u_5$ have types (10) and (12) respectively. Therefore, there exists $\alpha, \beta, \gamma \in \mathbb{P}$ such that

\[
u_3 = \begin{bmatrix} 1 & 1 & e^{i\alpha} \end{bmatrix}, \quad u_4 = \begin{bmatrix} 1 & e^{i(\beta - \pi)} & e^{i\beta} \end{bmatrix}, \quad u_5 = \begin{bmatrix} 1 & e^{i(\gamma - \pi)} & e^{i(\gamma - \pi)} \end{bmatrix}.
\]

Once again, we use Table 3 to find necessary conditions on these angles for there to be no $3 \times 3$ submatrices that are strongly balanceable. Because the matrix with rows $u_1, u_3, u_4$ is not strongly balanceable,

\[\beta \leq \alpha.\]

Because the matrix with rows $v_1, v_3, v_4$ is not strongly balanceable,

\[\beta \geq \alpha.\]

Therefore, $\beta = \alpha$. Also, because the matrix with rows $u_1, u_4, u_5$ is not strongly balanceable,

\[\gamma \leq \beta = \alpha.\]

Suppose $\gamma < \beta = \alpha$. For $j = 3, 4, 5$, let $\hat{u}_j$ be such that

\[
\begin{bmatrix}
u_3 \\
u_4 \\
u_5
\end{bmatrix}
= \begin{bmatrix} 1 & 1 & e^{i\alpha} \\
1 & e^{i(\alpha - \pi)} & e^{i\alpha} \\
1 & e^{i(\gamma - \pi)} & e^{i(\gamma - \pi)}
\end{bmatrix}
\sim
\begin{bmatrix}
1 & e^{i(\pi - \gamma)} & e^{i(\alpha + \pi - \gamma)} \\
1 & e^{i(\alpha - \gamma)} & e^{i(\alpha + \pi - \gamma)} \\
1 & 1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
\hat{u}_3 \\
\hat{u}_4 \\
\hat{u}_5
\end{bmatrix}.
\]

Because $\gamma < \alpha$, thus $0 < (\alpha - \gamma) < \pi - \gamma < \pi$. Also, $e^{i(\alpha - \gamma + \pi)} = e^{i(\alpha - \gamma - \pi)}$. So $\hat{u}_4$ has type (11) and $\hat{u}_3$ has type (6), and therefore the matrix with rows $\hat{u}_5, \hat{u}_3, \hat{u}_4$ is strongly balanceable. Hence

\[\gamma = \beta = \alpha.\]

We now summarize the implications of the analysis in subcases B.ii.a, B.ii.b, and B.ii.c. Let $\alpha = y_3$. Then $\alpha \in \mathbb{P}$ and $A[\{1, 2, 3\}, \{1, 2, 3\}]$ has the form

\[A_1 = \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & e^{i\alpha}
\end{bmatrix}.
\]
Let
\[ b_1 = \begin{bmatrix} 1 & -1 & -e^{i\alpha} \end{bmatrix}, \]
\[ b_2 = \begin{bmatrix} 1 & -e^{i\alpha} & -e^{i\alpha} \end{bmatrix}, \]
\[ b_3 = \begin{bmatrix} 1 & -e^{i\alpha} & e^{i\alpha} \end{bmatrix}. \]
Because \( A_1^t = A_1 \), the constraints found in Subcases B.ii.a–B.ii.c imply that \( A \) has the form
\[ A = \begin{bmatrix} A_1 & u_4 v_j & v_j \vspace{1pt} \end{bmatrix}, \]
where \( u_4, u_5, v_4, v_5 \in \{ b_1, b_2, b_3 \} \), \( u_4 \neq u_5 \) and \( v_4 \neq v_5 \).
We will use this to show that each entry of \( A \) is in \( \{ 1, -1, i, -i \} \). We do this by analyzing the \( 4 \times 4 \) submatrices
\[ \begin{bmatrix} A_1 & v_j \vspace{1pt} \end{bmatrix}, \]
and show that \( e^{iz_{kj}} \in \{ \pm 1, \pm e^{i\alpha} \} \) and that \( e^{i\alpha} = i \).
First suppose \( v_j = b_1 \). Note that
\[ \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & e^{i\alpha} & -e^{i\alpha} \\
1 & r & s & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 \\
1 & 1 & e^{i\alpha} & -e^{i\alpha} \\
\end{bmatrix}, \]
\[ \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -e^{i\alpha} & e^{i\alpha} \\
1 & r & s & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -e^{i\alpha} & e^{i\alpha} \\
1 & r & s & t \end{bmatrix}. \]
Therefore, \( r, r^2 \in \{ \pm 1, \pm e^{i\alpha} \} \). Let \( u_k = [1, r, s] \) and \( e^{iz_{kj}} = t \). Note that if \( u_k \in \{ b_2, b_3 \} \), then \( r = -e^{i\alpha} \). But this implies \( -e^{i\alpha} = e^{i\alpha} \), i.e. \( e^{i\alpha} = i \). And thus \( t \in \{ \pm 1, \pm i \} \). If \( u_k = b_1 \), then \( r = -1 \) and \( t \in \{ \pm 1, \pm e^{i\alpha} \} \).
Next suppose \( v_j = b_2 \). Note that
\[ \begin{bmatrix} 1 & 1 & 1 & e^{i\alpha} & -e^{-i\alpha} \\
1 & -1 & -1 & e^{-i\alpha} \\
1 & e^{i\alpha} & -e^{i\alpha} \\
1 & r & s & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & e^{-i\alpha} & -e^{-i\alpha} \\
1 & -1 & e^{-i\alpha} & 1 \\
1 & 1 & 1 & 1 \\
1 & r & s & t \end{bmatrix}, \]
\[ \sim \begin{bmatrix} 1 & 1 & 1 & e^{i\alpha} & -e^{-i\alpha} \\
1 & -1 & 1 & e^{-i\alpha} \\
1 & -e^{-i\alpha} & e^{i\alpha} \\
1 & r & -te^{-i\alpha} & se^{-i\alpha} \end{bmatrix}. \]
Therefore, \( r, -te^{-i\alpha} \in \{ \pm 1, \pm e^{-i\alpha} \} \). Let \( u_k = [1, r, s] \) and \( e^{i\alpha j} t = t \). Note that if \( u_k \in \{ b_2, b_3 \} \), then \( r = -e^{i\alpha} \). But then \( -e^{i\alpha} = e^{-i\alpha} \), i.e. \( e^{i\alpha} = i \). And thus \( t \in \{ \pm 1, \pm i \} \). If \( u_k = b_1 \), then \( r = -1 \) and \( t \in \{ \pm 1, \pm e^{i\alpha} \} \).

Finally suppose \( v_j = b_3 \). Note that

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -e^{i\alpha} \\
1 & e^{i\alpha} & e^{i\alpha} & r \\
1 & r & s & t
\end{bmatrix} \sim
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & e^{i\alpha} \\
1 & e^{i\alpha} & e^{i\alpha} & \sigma \\
1 & \sigma & e^{-i\alpha} & \sigma e^{-i\alpha}
\end{bmatrix} \sim
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & e^{i\alpha} \\
1 & 1 & 1 & e^{i\alpha} \\
1 & r & e^{i\alpha} & \sigma e^{-i\alpha}
\end{bmatrix}.
\]

So \( r, \sigma e^{i\alpha} \in \{ \pm 1, \pm e^{i\alpha} \} \), i.e. \( t \in \{ \pm r, \pm re^{i\alpha} \} \). Let \( u_k = [1, r, s] \). If \( u_k = b_1 \) then \( r = -1 \) and so \( e^{i\alpha j} t = t \in \{ \pm 1, \pm e^{i\alpha} \} \). If \( u_k \in \{ b_2, b_3 \} \), or in other words, \( r = -e^{i\alpha} \), then \( t \in \{ \pm e^{i\alpha}, \pm e^{i2\alpha} \} \).

Note that there are 3 choices for the two vectors \( v_j \), therefore, at least one, say \( v_j \) is in \( \{ b_1, b_2 \} \). Similarly, there are two vectors \( u_k \), thus at least one of them, say \( u_4 \) is in \( \{ b_2, b_3 \} \). Therefore, \( e^{i\alpha} = i \) and so \( e^{i2\alpha} = -1 \) and \( e^{i\alpha j} \in \{ \pm 1, \pm i \} \) for all \( j, k = 4, 5 \). By Lemma 2.3, \( A \) is not ray-nonsingular. \( \square \)

### 4.3. Form (c).

In this section we show that \( A \) does not have a submatrix of form (c).

**Proposition 4.3.** Let \( A \) be a \( 5 \times 5 \) full ray pattern whose leading \( 3 \times 3 \) principal submatrix has form (c). Then \( A \) is not ray-nonsingular.

**Proof.** Suppose to the contrary that \( A \) is ray-nonsingular. Without loss of generality we may assume that \( A \) is in standard form. By Lemma 2.2, no \( 3 \times 3 \) submatrix of \( A \) is strongly balanceable.

Let

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5
\end{bmatrix} =
\begin{bmatrix}1 & 1 & 1 \\
1 & -1 & e^{i\alpha} \\
1 & e^{i\beta} & -1 \\
1 & e^{i\alpha} & e^{i\gamma} \\
1 & e^{i\alpha} & e^{i\gamma}
\end{bmatrix}.
\]

Propositions 4.2 and 4.3 imply that \( e^{i\alpha}, e^{i\beta}, e^{i\gamma}, e^{i\delta} \notin \{ \pm 1 \} \) for \( j = 4, 5 \). Furthermore, since \( A \sim A \), we may assume that \( \alpha \in \mathcal{P} \). Therefore, \( u_2 \) has type (C7). Since the matrix with rows \( u_1, u_2 \) and \( u_3 \) is not strongly balanceable, by Table 2, \( u_3 \) has type (C3), i.e. \( \beta \in \mathcal{P} \). Table 2 applied to the matrix with rows \( u_1, u_3 \) and \( u_j \), and the matrix with rows \( u_1, u_2, u_j \) implies that for \( j = 4, 5 \), \( u_j \) has type (9). But this means that rows \( u_4 \) and \( u_5 \) both have type (9) and therefore the matrix with rows \( u_1, u_4 \) and \( u_5 \) is strongly balanceable, which is the desired contradiction. \( \square \)
4.4. Form (d). In this section we show that $A$ does not have a submatrix of form (d).

**Proposition 4.5.** Let $A$ be a $5 \times 5$ full ray pattern whose leading $3 \times 3$ principal submatrix has form (d). Then $A$ is not ray-nonsingular.

**Proof.** Suppose to the contrary that $A$ is ray-nonsingular. Without loss of generality we may assume that $A$ is in standard form. By Lemma 2.2, no $3 \times 3$ submatrix of $A$ is strongly balanceable.

Let

$$
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5 
\end{bmatrix}
= 
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & e^{i\alpha} \\
  1 & e^{i\beta} & -1 \\
  1 & e^{ix_4} & e^{iy_4} \\
  1 & e^{ix_5} & e^{iy_5}
\end{bmatrix}.
$$

By Propositions 4.2 and 4.3, $e^{i\alpha}, e^{i\beta}, e^{ix_4}, e^{iy_4} \not\in \{\pm 1\}$ for $j = 4, 5$. Furthermore, we may assume that $\alpha \in \mathcal{P}$ as $A \sim \overline{A}$. Therefore, $u_2$ has type (C5). By Table 2, applied to the matrix with rows $u_1, u_2$ and $u_3$, $u_3$ has type (C3), i.e. $\beta \in \mathcal{P}$. As in the proof of Proposition 4.4, $u_4$ has type (C3), and so for $j = 4, 5$, $u_j$ has type (1), (8), or (12).

Since the matrix with rows $u_1, u_2$ and $u_j$ is also not strongly balanceable, $u_j$ can only have type (1), (8) or (12). By Table 2, we see that the pairwise intersections of the solution sets are non-empty; thus, the matrix with rows $u_1, u_4$ and $u_5$ is strongly balanceable—a contradiction.

4.5. Form (e). In this section we show that $A$ does not have any $3 \times 3$ submatrix that is $\sim$-equivalent to a matrix of form (e).

**Proposition 4.6.** Let $A$ be a $5 \times 5$ full ray-pattern whose leading principle submatrix has the form (e). Then $A$ is not ray-nonsingular.

**Proof.** Suppose to the contrary that $A$ is ray-nonsingular. Without loss of generality we may assume that $A$ is in standard form. By Lemma 2.2, no $3 \times 3$ submatrix of $A$ is strongly balanceable, and by Propositions 4.2-4.5, no $3 \times 3$ submatrix of $A$ is equivalent to a matrix of form (a), (b), (c), or (d). We will show that this implies that each entry of $A$ lies in $\{1, e^{\pm 2\pi/3}\}$ and that $A$ has a $4 \times 4$ strongly balanceable submatrix (which contradicts Lemma 2.2).

Suppose $u_1, \ldots, u_5$ are the five rows of $[a_{ij}]_{1 \leq i \leq 5, 1 \leq j \leq 3}$. Then

$$
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix} = 
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & e^{i\beta} \\
  1 & e^{i\alpha} & 1
\end{bmatrix}.
$$

Let

$$
u_4 = [1 \ e^{ix_4} \ e^{iy_4}], \quad u_5 = [1 \ e^{ix_5} \ e^{iy_5}].$$

Note that $e^{i\alpha}, e^{i\beta}, e^{ix_j}, e^{iy_j} \not\in \{\pm 1\}$ for $j = 4, 5$; otherwise we contradict Proposition 4.2 or 4.3. Furthermore, we may assume that $\alpha \in \mathcal{P}$, otherwise replace $A$ with $\overline{A}$. 

Therefore, \( u_3 \) has type (C1). Because the matrix with rows \( u_1, u_2, u_3 \) is not strongly balanceable, by Table 2, \( u_2 \) has type (C5), i.e. \( \beta \in \mathcal{P} \). We also know that

\[
\pi \leq \alpha + \beta
\]

by Table 3. Because the matrix with rows \( u_1, u_2, u_j \), for \( j = 4, 5 \), is not strongly balanceable, \( u_j \) has types (1), (4), (6), (8), (10) or (12). Because the matrix with rows \( u_1, u_3, u_j \) is also not strongly balanceable, \( u_j \) has one of the following types:

(6), (8) or (10).

We now consider the three cases where \( u_j \) has type (6), (8) and (10) and examine the matrices with rows \( u_l, u_k, u_j \) where \( l, k \in \{1, 2, 3\} \), to find bounds on \( x_j \) and \( y_j \) dependent on \( \alpha \) and \( \beta \). These bounds are found by using Table 3 for the given matrices.

**Case A:** \( u_j \) has type (6), i.e., \( x_j \in \mathcal{P}, y_j \in \mathcal{N} \) and \( x_j - y_j > \pi \).

Table 3 applied to \( u_1, u_2, u_j \) gives

\[
0 < y_j - \beta + 2\pi \leq x_j.
\]

Table 3 applied to \( u_1, u_3, u_j \) gives

\[
x_j \leq \alpha.
\]

Note that

\[
\begin{bmatrix}
u_2 \\ u_3 \\ u_j
\end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ e^{-i\beta} \\ e^{-i\alpha} \\ e^{-i\gamma}
\end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ e^{-i\beta} \\ e^{-i\alpha} \\ e^{-i\gamma}
\end{bmatrix}.
\]

The second row of the second matrix has type (6) or (11) because \( \alpha + \beta \geq \pi \) by (4.4). Because \( e^{i(x_j - \beta)} = e^{i(y_j - \beta + 2\pi)} \) and (4.5) holds, the third row has either type (1) or (9). By Table 3,

\[
\alpha \leq x_j.
\]

Equations (4.6) and (4.7) imply

\[
\alpha = x_j.
\]

Also, equation (4.5) implies

\[
\alpha + \beta \geq y_j + 2\pi > \pi.
\]

**Case B:** \( u_j \) has type (8), i.e., \( x_j \in \mathcal{N}, y_j \in \mathcal{P} \) and \( y_j - x_j > \pi \).

Note that the following matrices are equivalent.

\[
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} & 1 \\ 1 & e^{i\alpha} & 1 & 1 \\ 1 & e^{i\gamma} & e^{i\gamma} & 1 \\ \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} \\ 1 & e^{i\beta} & 1 \\ 1 & e^{i\gamma} & e^{i\gamma} \end{bmatrix}.
\]
Using the same argument in Case A, we have

\[(4.10)\quad y_j = \beta\] and

\[(4.11)\quad \alpha + \beta \geq x_j + 2\pi > \pi.\]

**Case C:** \(u_j\) has type (10), i.e. \(x_j = y_j \in \mathcal{N}\). Because the matrix with rows \(u_1, u_2, u_j\) is not strongly balanceable,

\[(4.12)\quad \beta \geq x_j + \pi.\]

Also, because the matrix with rows \(u_1, u_3, u_j\) is not strongly balanceable,

\[(4.13)\quad \alpha \geq x_j + \pi.\]

We now use the above information to further determine the structure of \(u_1, \ldots, u_5\). We have the following three cases.

**Case A':** Assume \(u_4\) and \(u_5\) have types (6) and (8) respectively. Then (4.8)–(4.11) imply that \(x_4 = \alpha, y_5 = \beta\) and \(\alpha + \beta \geq \gamma + 2\pi > \pi\) for \(\gamma \in \{y_4, x_5\}\).

Suppose that \(\alpha + \beta > y_4 + 2\pi\). Then the following matrices are equivalent.

\[
\begin{bmatrix}
  u_2 \\
  u_4 \\
  u_5
\end{bmatrix} = 
\begin{bmatrix}
  1 & 1 & e^{i\beta} \\
  1 & e^{i\alpha} & e^{iy_4} \\
  1 & e^{i\alpha} & e^{iy_5}
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & e^{i\alpha} & e^{i(y_4 - \beta)} \\
  1 & e^{i\alpha} & 1
\end{bmatrix}.
\]

But the second row has type (1) and the third row has type (C2) and, by Table 2, the matrix is strongly balanceable. Therefore,

\[(4.14)\quad \alpha + \beta = y_4 + 2\pi.\]

Similarly, using the matrix with rows \(u_3, u_4, u_5\), we can show that

\[(4.15)\quad x_4 = \alpha + \beta = x_5 + 2\pi.\]

Therefore,

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5
\end{bmatrix} = 
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & e^{i\beta} \\
  1 & e^{i\alpha} & 1 \\
  1 & e^{i\alpha} & e^{i(\alpha + \beta)} \\
  1 & e^{i(\alpha + \beta)} & e^{i\beta}
\end{bmatrix}.
\]

**Case B':** Assume \(u_4\) and \(u_5\) have types (6) and (10) respectively. Then by (4.8), (4.9), (4.12) and (4.13) we have

\[x_4 = \alpha, \quad \alpha + \beta \geq y_4 + 2\pi > \pi,\]
$$x_5 = y_5 \in \mathcal{N} \quad \text{and} \quad x_5 + \pi \leq \alpha, \beta.$$  

Table 3 applied to $u_1, u_4, u_5$ implies

$$y_4 \geq x_5.$$  

(4.16)

Note that

$$\begin{bmatrix} 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & e^{iy_4} \\ 1 & e^{ix_5} & e^{ix_5} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i(y_4-\beta)} \\ 1 & e^{ix_5} & e^{i(x_5-\beta)} \end{bmatrix}.$$  

Label the second row $\hat{u}_4$ and the third row $\hat{u}_5$. By equation (4.9), $\hat{u}_4$ has type (1) when $y_4 + 2\pi < \alpha + \beta$, and has type (9) when $y_4 + 2\pi = \alpha + \beta$. Also, by (4.12) and $x_5 - \alpha > x_5 - \pi$, we see that $\hat{u}_5$ has type (8) when $\beta > x_5 + \pi$, and has type (C4) when $\beta = x_5 + \pi$. Referring to Table 2, we see that $\hat{u}_4$ must have type (9), i.e. $y_4 + 2\pi = \alpha + \beta$, and $\hat{u}_5$ must have type (8), i.e. $\beta > x_5 + \pi$ because this matrix is not strongly balanceable.

Similarly, we note that the matrix

$$\begin{bmatrix} 1 & 1 & e^{i\alpha} \\ 1 & e^{i\alpha} & e^{iy_4} \\ 1 & e^{ix_5} & e^{ix_5} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\alpha} & e^{i(y_4-\alpha)} \\ 1 & e^{ix_5} & e^{i(x_5-\alpha)} \end{bmatrix}.$$  

Again, label the second row $\hat{u}_4$ and the third row $\hat{u}_5$. Note that $\hat{u}_4$ has type (C6) and $\hat{u}_5$ has type (6) or (C8) because $x_5 - \alpha > x_5 - \pi$ and (4.13). But this matrix is not strongly balanceable and so by Table 3,

$$x_5 \geq y_4.$$  

(4.17)

Equations (4.16), (4.17) and the fact that $y_4 + 2\pi = \alpha + \beta$ imply

$$x_5 = y_4 = \alpha + \beta - 2\pi.$$  

(4.18)

So

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \\ 1 & e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} \end{bmatrix}.$$  

Case C'. Assume $u_4$ and $u_5$ have types (8) and (10) respectively. Then

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\beta} \\ 1 & e^{i\alpha} & 1 \\ 1 & e^{i\alpha} & e^{i\beta} \\ 1 & e^{ix_4} & e^{ix_5} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & e^{i\alpha} \\ 1 & e^{i\beta} & 1 \\ 1 & e^{i\beta} & e^{ix_4} \\ 1 & e^{ix_5} & e^{ix_5} \end{bmatrix}.$$
Using the argument in Case B', we see that
\[ x_5 = x_4 = \alpha + \beta - 2\pi \]
and
\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & e^{i\beta} \\
  1 & e^{i\alpha} & 1 \\
  1 & e^{i(\alpha+\beta)} & e^{i\beta} \\
  1 & e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)}
\end{bmatrix}.
\]

We now describe the implications of the analysis in Cases A–C, and A'–C'. Let
\[ c_1 = \begin{bmatrix} 1 & e^{i\alpha} & e^{i(\alpha+\beta)} \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 & e^{i(\alpha+\beta)} & e^{i\beta} \end{bmatrix}, \quad c_3 = \begin{bmatrix} 1 & e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} \end{bmatrix}, \]
\[ c_4 = \begin{bmatrix} 1 & e^{i\beta} & e^{i(\alpha+\beta)} \end{bmatrix}, \quad c_5 = \begin{bmatrix} 1 & e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} \end{bmatrix}. \]

Using both A by A', we see that if \( A_1 \) is the 3 × 3 leading principal submatrix of A, i.e., with rows \( u_1, u_2, u_3 \), then
\[
A = \begin{bmatrix}
  A_1 & u_4 & u_5 \\
  u_4 & e^{i\gamma} & e^{i\gamma} \\
  u_5 & e^{i\gamma} & e^{i\gamma}
\end{bmatrix}
\]
where \( u_4, u_5 \in \{c_1, c_2, c_3\} \) and \( v_4, v_5 \in \{c_3, c_4, c_5\} \). We consider the possible 4 × 4 submatrices for the different values of \( u_j \) and \( v_k \) and determine the possible values of \( e^{iz_kj} \).

First suppose \( u_j = c_3 \). Let \( v_k = [1, e^{i\gamma}, e^{i\delta}] \) and \( z_{kj} = \lambda \). We consider the following submatrix of \( A' \):

\[
\begin{bmatrix}
  1 & 1 & 1 & 1 \\
  1 & e^{i\alpha} & 1 & e^{i(\alpha+\beta)} \\
  1 & e^{i\beta} & e^{i(\alpha+\beta)} & 1 \\
  1 & e^{i\gamma} & e^{i\delta} & e^{i\lambda}
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & 1 & e^{-i\alpha} \\
  1 & 1 & 1 & e^{-i(\alpha+\beta)} \\
  1 & e^{i\beta} & e^{-i\alpha} & 1 \\
  1 & e^{i\gamma} & e^{i(\delta-\alpha)} & e^{-i(\lambda-\alpha-\beta)}
\end{bmatrix}
\sim \begin{bmatrix}
  1 & 1 & 1 & e^{-i(\alpha+\beta)} \\
  1 & 1 & 1 & e^{-i\alpha} \\
  1 & e^{i\beta} & e^{-i\alpha} & 1 \\
  1 & e^{i\gamma} & e^{i(\lambda-\alpha-\beta)} & e^{i(\delta-\alpha)}
\end{bmatrix}.
\]

Applying the arguments in Cases A, B, C, A', B', C' to the right most matrix with \( (\alpha, \beta) \) replaced by \( (\beta, -(\alpha+\beta)) \), we conclude that
\[
\begin{bmatrix}
  1 & e^{i\gamma} \\
  e^{i(\lambda-\alpha-\beta)} & e^{-i\alpha}
\end{bmatrix} \in \left\{ \begin{bmatrix}
  1 & e^{i\beta} \\
  e^{-i\alpha}
\end{bmatrix}, \begin{bmatrix}
  1 & e^{-i\alpha} \\
  e^{-i(\alpha+\beta)}
\end{bmatrix}, \begin{bmatrix}
  1 & e^{-i\alpha} \\
  e^{-i\alpha}
\end{bmatrix} \right\}.
\]
Thus, $e^{i\lambda} \in \{1, e^{i\beta}\}$ and $e^{i\gamma} \in \{e^{i(\beta)}, e^{i(-\alpha)}\}$. Because $-\alpha, (\alpha + \beta) \in \mathcal{N}$ and $\beta \in \mathcal{P}$, if $v_k = c_3$ or $c_5$, then $e^{i\gamma} = e^{i(\alpha + \beta)} = e^{-i\alpha}$, i.e. $e^{i(2\alpha + \beta)} = 1$.

Next suppose $v_j = e$. Let $v_k = [1, e^{i\gamma}, e^{i\beta}]$ and $z_k = \lambda$. We consider the following submatrix of $A'$:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & e^{i\alpha} & e^{i(\alpha + \beta)} \\
1 & e^{i\beta} & 1 & e^{i\beta} \\
1 & e^{i\delta} & e^{i\delta} & e^{i\lambda}
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & e^{-i\alpha} & e^{-i\alpha} & e^{i\beta} \\
1 & e^{i\beta} & 1 & e^{i\beta} \\
e^{-i\delta} & e^{i(\gamma - \delta)} & 1 & e^{i(\lambda - \delta)}
\end{bmatrix}
$$

Applying the arguments in Cases A, B, C, A', B', C' to the rightmost matrix with $(\alpha, \beta)$ replaced by $(-\alpha, -\beta)$, we conclude that

$$
\begin{bmatrix}
e^{-i\delta} \\
e^{i(\lambda - \delta)}
\end{bmatrix}
\in
\left\{ \begin{bmatrix} 1 \\ e^{-i(\alpha + \beta)} \end{bmatrix} , \begin{bmatrix} 1 \\ e^{-i\beta} \end{bmatrix} , \begin{bmatrix} 1 \\ e^{-i(\gamma - \alpha)} \end{bmatrix} \right\}.
$$

Thus, $e^{i\lambda} \in \{e^{i\beta}, e^{i(\lambda - \alpha)}\}$. Also,

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & e^{i\alpha} & e^{i(\alpha + \beta)} \\
1 & e^{i\beta} & 1 & e^{i\beta} \\
1 & e^{i\gamma} & e^{i\delta} & e^{i\lambda}
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & e^{i(\alpha + \beta)} & 1 & e^{i\alpha} \\
1 & e^{-i\beta} & 1 & e^{i(\gamma - \alpha)} \\
e^{-i\gamma} & e^{i(\delta - \gamma)} & 1 & e^{i(\lambda - \gamma)}
\end{bmatrix}
$$

Applying the arguments in A, B, C, A', B', C' to the rightmost matrix with $(\alpha, \beta)$ replaced by $(-\beta, \alpha + \beta)$, we conclude that

$$
\begin{bmatrix}
e^{-i\gamma} \\
e^{i(\lambda - \gamma)}
\end{bmatrix}
\in
\left\{ \begin{bmatrix} 1 \\ e^{-i\beta} \end{bmatrix} , \begin{bmatrix} 1 \\ e^{i\alpha} \end{bmatrix} , \begin{bmatrix} 1 \\ e^{i(\alpha + \beta)} \end{bmatrix} \right\}.
$$

Thus, $e^{i\lambda} \in \{e^{i(\alpha + \beta + \gamma)}, e^{i(\alpha + \gamma)}\}$ and $e^{i\gamma} \in \{e^{i(-\alpha)}, e^{i\beta}\}$. Since $-\alpha \in \mathcal{N}$ and $\beta \in \mathcal{P}$, if $v_k = c_3$ or $c_5$, then $e^{i\gamma} = e^{i(\alpha + \beta)} = e^{-i\alpha}$. In other words, $e^{i(2\alpha + \beta)} = 1$. Furthermore, if $v_k = c_3$, then $e^{i\delta} = e^{i(\alpha + \beta)}$ and therefore, $e^{i\lambda} \in \{e^{i(\alpha + \beta)}, e^{i\beta}\} \cap \{e^{i\beta}, 1\}$. So $e^{i\lambda} = e^{i\beta}$. If $v_k = c_5$, then $e^{i\delta} = e^{i\alpha}$; therefore, $e^{i\lambda} \in \{e^{i\alpha}, 1\} \cap \{e^{i\beta}, 1\}$. So
either $e^{i\alpha} = 1$ or $e^{i\alpha} = e^{i\beta}$. If $v_k = c_4$, then $e^{i\delta} = e^{i(\alpha + \beta)}$ and $e^{i\gamma} = e^{i\beta}$. Hence, $e^{i\lambda} \in \{e^{i(\alpha + \beta)}, e^{i\beta}\} \cap \{e^{i(2\beta + \alpha)}, e^{i(\alpha + \beta)}\}$. Recall that $\alpha, \beta \in P = (0, \pi)$. Thus, $e^{i\lambda} = e^{i(\alpha + \beta)}$.

Finally suppose $u_j = c_1$. Let $v_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{kj} = \lambda$. We consider the following submatrix of $A$:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & e^{i\alpha} & 1 & e^{i\alpha} \\
e^{i\beta} & 1 & e^{i(\alpha + \beta)} & 1 \\
e^{i\gamma} & e^{i\delta} & e^{i\lambda} & 1
\end{bmatrix}
\approx
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & e^{i\beta} & e^{i(\alpha + \beta)} & e^{i(\alpha + \beta)} \\
e^{i\gamma} & e^{i\delta} & e^{i\lambda} & e^{i\lambda}
\end{bmatrix}.
$$

Interchanging the roles of $(\alpha, \gamma)$ and $(\beta, \delta)$, we see that this is similar to the case when $u_j = c_2$. In other words,

$$e^{i\lambda} \in \{e^{i\gamma}, e^{i(\gamma - \beta)}\} \cap \{e^{i(\alpha + \beta + \gamma)}, e^{i(\beta + \gamma)}\} \text{ and } e^{i\delta} \in \{e^{i(-\beta)}, e^{i\alpha}\}.$$  

If $v_k = c_3$ or $c_4$, then $e^{i\delta} = e^{i(\alpha + \beta)}$ and so $e^{i(\alpha + \beta)} = e^{i\beta}$, i.e. $e^{i(2\beta + \alpha)} = 1$. Furthermore, if $v_k = c_3$, then $e^{i\lambda} = e^{i\alpha}$. If $v_k = c_4$, then either $e^{i\lambda} = 1$ or $e^{i\lambda} = e^{i\alpha} = e^{i\beta}$. If $v_k = c_5$, then $e^{i\lambda} = e^{i(\alpha + \beta)}$.

We now turn our attention to the possible forms of $v_j$. First suppose that $v_j = c_5$. Let $u_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{kj} = \lambda$. Interchanging $\alpha$ and $\beta$, and using the transpose of $A$, we see that this is similar to the case when $u_j = c_2$. Thus, $e^{i\lambda} \in \{e^{i\beta}, e^{i(\beta - \delta)}\} \cap \{e^{i(\alpha + \beta + \gamma)}, e^{i(\beta + \gamma)}\}$ and $e^{i\gamma} \in \{e^{i\alpha}, e^{i(-\beta)}\}$. Therefore, if $u_k = c_2$ or $c_3$, then $e^{i\gamma} = e^{i(\alpha + \beta)} = e^{-i\beta}$, i.e. $e^{i(\alpha + 2\beta)} = 1$. Furthermore, if $u_k = c_3$, then $e^{i\lambda} = e^{i\alpha}$. If $u_k = c_2$, then either $e^{i\lambda} = 1$ or $e^{i\lambda} = e^{i\beta} = e^{i\alpha}$. And if $u_k = c_1$, then $e^{i\lambda} = e^{i(\alpha + \beta)}$.

Next suppose that $v_j = c_4$. Let $u_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{kj} = \lambda$. Interchanging $\alpha$ and $\beta$, and using the transpose of $A$, we see that this is similar to the case when $u_j = c_1$. Hence, $e^{i\lambda} \in \{e^{i\beta}, e^{i(\gamma - \alpha)}\} \cap \{e^{i(\alpha + \beta + \gamma)}, e^{i(\alpha + \beta)}\}$ and $e^{i\gamma} \in \{e^{i\alpha}, e^{i(\alpha + \beta)}\}$. Therefore, if $u_k = c_1$ or $c_3$, then $e^{i\delta} = e^{i(\alpha + \beta)} = e^{-i\alpha}$, i.e. $e^{i(2\alpha + \beta)} = 1$. Furthermore, if $u_k = c_3$, then $e^{i\lambda} = e^{i\beta}$. And if $u_k = c_1$, then either $e^{i\lambda} = 1$ or $e^{i\lambda} = e^{i\beta} = e^{i\alpha}$. If $u_k = c_2$, then $e^{i\lambda} = e^{i(\alpha + \beta)}$.

Finally suppose $v_j = c_3$. Let $u_k = [1, e^{i\gamma}, e^{i\delta}]$ and $z_{kj} = \lambda$. Interchanging $\alpha$ and $\beta$, and using the transpose of $A$, we see that this is similar to the case when $u_j = c_3$. Thus, $e^{i\lambda} \in \{e^{i\alpha}, e^{i\beta}\}$ and $e^{i\gamma} \in \{e^{i\alpha}, e^{i(-\beta)}\}$. So, if $u_k = c_2$ or $c_3$, then $e^{i\gamma} = e^{i(\alpha + \beta)} = e^{-i\beta}$. In other words, $e^{i(\alpha + 2\beta)} = 1$.

Note that $\{u_4, u_5\} \cap \{c_2, c_3\} \neq \emptyset$ and also $\{v_4, v_5\} \cap \{c_3, c_5\} \neq \emptyset$. Therefore, $e^{-i\beta} = e^{i(\alpha + \beta)} = e^{-i\alpha}$ and so $\alpha = \beta$ and $e^{i(\alpha + \beta)} = 1$. Let $\omega = e^{i\alpha}$ so that $\omega^2 = e^{i(\alpha + \beta)}$.

We can always assume that if $c_3 \in \{u_4, u_5, v_4, v_5\}$, then $u_5 = c_3$ (since $A \sim A'$ and $A \sim P AQ$ where $P, Q$ are permutation matrices). Also, if $u_j = c_1$, then interchange the second and third row and column to get $u_j = c_2$. Thus, we may assume that the pair of pairs $((u_4, u_5), (v_4, v_5))$ is one of the following: $((c_2, c_3), (c_5, c_3))$, $((c_2, c_3), (c_4, c_3))$, $((c_2, c_3), (c_5, c_4))$, $((c_2, c_1), (c_5, c_4))$. Hence, $A$ is one of the follow-
ing matrices:

\[
B_1 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^2 \\
1 & \omega & \omega & \omega^2 \\
1 & \omega^2 & \omega & x_1 \\
1 & \omega^2 & \omega & y_1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \omega & \omega & \omega^2 \\
1 & \omega & \omega & \omega^2 \\
1 & \omega^2 & \omega & x_2
\end{bmatrix},
\]

\[
B_3 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega \\
1 & \omega & \omega^2 & \omega \\
1 & \omega^2 & \omega & x_3 \\
1 & \omega^2 & \omega & y_3
\end{bmatrix}, \quad B_4 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \omega & \omega & \omega^2 \\
1 & \omega & \omega^2 & \omega \\
1 & \omega^2 & \omega & x_4 \\
1 & \omega^2 & \omega & y_4
\end{bmatrix},
\]

with \(x_i, y_i \in \{1, \omega\}\) for \(i = 1, \ldots, 4\). However, if \(x_1, y_1, x_2, x_3\) or \(y_4 = \omega\), then we contradict Proposition 4.2 because \(A\) has the submatrix

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & \omega^2 & \omega \\
1 & \omega & \omega
\end{bmatrix} \sim \begin{bmatrix}
1 & \omega^2 \\
1 & 1 \\
1 & 1
\end{bmatrix}.
\]

Also, if \(x_4 = \omega\), then we contradict Proposition 4.2 because \(A\) has the submatrix

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{bmatrix} \sim \begin{bmatrix}
1 & \omega \\
1 & 1 \\
1 & 1
\end{bmatrix}.
\]

Thus, \(x_i, y_i = 1\) for all \(i\). But, for each of the four matrices \(B_1, B_2, B_3, B_4\), there exists a \(4 \times 4\) strongly balanceable submatrix (since each row contains each of the entries \(1, \omega, \omega^2\)). To find these submatrices, in each case remove the first row. For \(B_1\) and \(B_4\), remove the first column. For \(B_2\) and \(B_3\), remove the third and second columns, respectively. Thus \(A\) is not ray-nonsingular.

Propositions 4.1-4.6 imply our main result:

**Theorem 4.7.** There does not exist a \(5 \times 5\) full ray-nonsingular matrix.

Combined with the results of [1, 2], we have the the main theorem:

**Main Theorem** There is an \(n \times n\) full ray-nonsingular matrix if and only if \(n \leq 4\).

---

**REFERENCES**


Appendix

Graphical representations of $R(0, 0) \cap R(\alpha, \beta)$.

Form (1)

Form (2)

Form (3)

Form (4)

Form (5)

Form (6)
Form (7)

Form (8)

Form (9)

Form (10)

Form (11)

Form (12)
Non-existence of $5 \times 5$ Full Ray-Nonsingular Matrices

Form (C1)

Form (C2)

Form (C3)

Form (C4)

Form (C5)

Form (C6)
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Form (C7)

Form (C8)

Form (C9)

Forms (C10-12)