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A BRUHAT ORDER FOR THE CLASS OF (0, 1)-MATRICES WITH ROW SUM VECTOR R AND COLUMN SUM VECTOR S

RICHARD A. BRUALDI† AND SUK-GEUN HWANG‡

Abstract. Generalizing the Bruhat order for permutations (so for permutation matrices), a Bruhat order is defined for the class of m by n (0, 1)-matrices with a given row and column sum vector. An algorithm is given for constructing a minimal matrix (with respect to the Bruhat order) in such a class. This algorithm simplifies in the case that the row and column sums are all equal to a constant k. When k = 2 or k = 3, all minimal matrices are determined. Examples are presented that suggest such a determination might be very difficult for k ≥ 4.

Key words. Bruhat order, Row sum and column sum vectors, Interchanges, Minimal matrix.

AMS subject classifications. 05B20, 06A07, 15A36.

1. Introduction. Let \( R = (r_1, r_2, \ldots, r_m) \) and \( S = (s_1, s_2, \ldots, s_n) \) be nonincreasing, positive integral vectors, so that

\[ r_1 \geq r_2 \geq \cdots \geq r_m > 0 \quad \text{and} \quad s_1 \geq s_2 \geq \cdots \geq s_n > 0. \]

Then \( A(R, S) \) denotes the class of all m by n (0, 1)-matrices with row sum vector \( R \) and column sum vector \( S \).

The row and column sum vectors \( R \) and \( S \) of a (0, 1)-matrix are partitions of the same integer \( t \) (its number of 1’s). Let \( R^* = (r_1^*, r_2^*, \ldots, r_n^*) \) denote the conjugate of \( R \) (with trailing 0’s included to get an n-tuple). The class \( A(R, R^*) \) is nonempty, and it contains a unique matrix, the perfectly nested matrix \( \bar{A} \) with all 1’s left justified. Let \( R \) and \( S \) be proposed row and column sum monotone vectors of a (0, 1)-matrix that satisfy (1.1). The Gale-Ryser Theorem (see e.g., [4]) asserts that \( A(R, S) \) is nonempty if and only if \( S \) is majorized by \( R^* \) (written \( S \preceq R^* \)), that is,

\[ s_1 + \cdots + s_k \leq r_1^* + \cdots + r_k^* \quad (k = 1, 2, \ldots, n) \]

with equality for \( k = n \). If \( A(R, S) \neq \emptyset \), then every matrix in \( A(R, S) \) can be obtained from the perfectly nested matrix \( \bar{A} \) with row and column sum vectors \( R \) and \( R^* \), respectively, by shifting 1’s in rows to the right. Ryser also proved that given matrices \( A \) and \( B \) in \( A(R, S) \) then \( B \) can be gotten from \( A \) by a sequence of interchanges

\[ L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

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which replace a submatrix equal to \(L_2\) by \(I_2\), or the other way around.

There is a well-known order on the symmetric group \(S_n\) (more generally, on Coxeter groups) of permutations of \(\{1, 2, \ldots, n\}\) called the Bruhat order, given by:

If \(\tau\) and \(\pi\) are permutations, then \(\pi \leq_B \tau\) (in the Bruhat order) provided \(\pi\) can be gotten from \(\tau\) by a sequence of transformations of the form:

- If \(a_i > a_j\), then \(a_1 \cdots a_i \cdots a_j \cdots a_n\) is replaced with \(a_1 \cdots a_j \cdots a_i \cdots a_n\).

Thus if \(n = 3\), 123 is the unique minimal element and 321 is the unique maximal element in the Bruhat order on \(S_3\).

As usual, the permutations in \(S_n\) can be identified with the permutation matrices of order \(n\), where the permutation \(\tau\) corresponds to the permutation matrix \(P = [p_{ij}]\) with \(p_{ij} = 1\) if and only if \(j = \tau(i)\). If \(P\) and \(Q\) are permutation matrices of order \(n\) corresponding to permutations \(\tau\) and \(\pi\), then we write \(P \leq_B Q\) whenever \(\tau \leq_B \pi\). The reduction in the Bruhat order, interpreted for permutation matrices, is that of one-sided interchanges:

\[
L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

For \(n = 3\), the minimal permutation (matrix) in the Bruhat order is

\[
I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

and the maximal permutation matrix is

\[
D_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

There are equivalent ways to define the Bruhat order on \(S_n\). One is in terms of the Gale order (see e.g., [1]) on subsets of size \(k\) of \(\{1, 2, \ldots, n\}\). Let \(k\) be an integer with \(1 \leq k \leq n\), and let \(X = \{a_1, a_2, \ldots, a_k\}\) and \(Y = \{b_1, b_2, \ldots, b_k\}\) be subsets of \(\{1, 2, \ldots, n\}\) of size \(k\) where \(a_1 < a_2 < \cdots < a_k\) and \(b_1 < b_2 < \cdots < b_k\). Then in the Gale order, \(X \leq_G Y\) if and only if \(a_1 < b_1, a_2 < b_2, \ldots, a_k < b_k\). For \(\tau = i_1i_2\ldots i_n \in S_n\), let \(\tau[k] = \{i_1, i_2, \ldots, i_k\}\). Then it is straightforward to check that, if also \(\pi \in S_n\), then

\[
\tau \leq_B \pi \quad \text{if and only if} \quad \tau[k] \leq_G \pi[k] \quad (k = 1, 2, \ldots, n).
\]

For an \(m\) by \(n\) matrix \(A = [a_{ij}]\), let \(\Sigma_A\) denote the \(m\) by \(n\) matrix whose \((k, l)\)-entry equals

\[
\sigma_{kl}(A) = \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} \quad (1 \leq k \leq m; 1 \leq l \leq n),
\]
the sum of the entries in the leading $k$ by $l$ submatrix of $A$. Using the Gale order, one easily checks that for permutation matrices $P$ and $Q$ of order $n$, $P \preceq_B Q$ if and only if $\Sigma_P \geq \Sigma_Q$, where this latter order is entrywise order.

The Bruhat order on permutation matrices can be extended to the classes $A(R, S)$. For $A_1$ and $A_2$ in $A(R, S)$ we define $A_1 \preceq_B A_2$ provided, in the entrywise order, $\Sigma_{A_1} \geq \Sigma_{A_2}$. It is immediate that if $A_1$ and $A_2$ are matrices in $A(R, S)$ and $A_1$ is obtained from $A_2$ by a sequence of one-sided interchanges

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then $A_1 \preceq_B A_2$. This observation gives the following corollary.

**Corollary 1.1.** Let $A$ be a matrix in $A(R, S)$ that is minimal in the Bruhat order. Then no submatrix of $A$ equals $L_2$.

**Example.** Let $R = S = (2, 2, 2, 2, 2)$. Then

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Sigma_A = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 5 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Sigma_B = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix},$$

are both minimal elements of $A(R, S)$ in the Bruhat order.

Let $A$ be a matrix in $A(R, S)$ which is minimal in the Bruhat order. Let $A^c = J_{m,n} - A$ be the complement of $A$. Here $J_{m,n}$ is the $m$ by $n$ matrix of all 1’s (abbreviated to $J_n$ when $m = n$), and thus $A^c$ has 1’s exactly where $A$ has 0’s. Let $R^c$ and $S^c$ be, respectively, the row and column sum vectors of $A^c$. Since $R$ and $S$ are monotone nonincreasing, $R^c$ and $S^c$ are monotone nondecreasing. Since $\Sigma_{A^c} = \Sigma_{J_{m,n}} - \Sigma_A$, it follows that, after reordering rows and columns to get monotone nonincreasing vectors $\tilde{R}^c = (n - r_1, \ldots, n - r_l)$ and $\tilde{S}^c = (m - s_1, \ldots, m - s_1)$, the resulting matrix $\tilde{A}$ is a maximal matrix in the class $A(\tilde{R}^c, \tilde{S}^c)$.

**Example.** Let $R = S = (2, 2, 2, 2, 2)$. Then $\tilde{R}^c = \tilde{S}^c = (3, 3, 3, 3, 3)$. A matrix in $A(\tilde{R}^c, \tilde{S}^c)$ that is minimal in the Bruhat order is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$
Thus the matrix
\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
is a matrix in \(A(R, S)\) that is maximal in the Bruhat order.

2. An Algorithm for a Minimal Matrix. In this section we give an algorithm that, starting from the perfectly nested matrix in \(A(R, R^*)\), constructs a matrix in \(A(R, S)\) that is minimal in the Bruhat order. From the above discussion, it follows that we also get an algorithm for constructing a matrix in \(A(R, S)\) that is maximal in the Bruhat order.

I. Algorithm to Construct a Minimal Matrix in the Bruhat Order

Let \(R = (r_1, r_2, \ldots, r_m)\) and \(S = (s_1, s_2, \ldots, s_n)\) be monotone nonincreasing positive integral vectors with \(S \preceq R^*\). Let \(\bar{A}\) be the unique matrix in \(A(R, R^*)\).

1. Rewrite \(R\) by grouping together its components of equal value:
   \[R = (a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots, a_k)\]
   where \(a_1 > a_2 > \cdots > a_k\), and the number of \(a_i\)'s equals \(p_i\), \((i = 1, 2, \ldots, k)\).

2. Determine nonnegative integers \(x_1, x_2, \ldots, x_k\) satisfying \(x_1 + x_2 + \cdots + x_k = s_n\) where \(x_k, x_{k-1}, \ldots, x_1\) are maximized in turn in this order subject to \((s_1, s_2, \ldots, s_{n-1}) \preceq R_1^*\) where \(R_1 = R(x_1, x_2, \ldots, x_k)\) is the vector
   \[
   \left(\frac{x_1}{p_1}, \frac{a_1 - 1}{x_1}, \ldots, \frac{a_k - 1}{x_k}, \ldots, \frac{a_k}{p_k}, \frac{a_k}{x_k}, \ldots, \frac{a_1}{x_1}, \frac{a_1}{p_1}\right).
   \]

3. Shift \(s_n = x_1 + x_2 + \cdots + x_k\)'s to the last column as specified by those rows whose sums have been diminished by 1: thus the last column consists of \(p_1 - x_1\) 0's followed by \(x_1\)'s, \(\ldots, p_k - x_k\) 0's followed by \(x_k\)'s.

4. Proceed recursively and return to step 1, with \(R\) replaced with \(R_1\) and \(S\) replaced with \(S_1 = (s_1, s_2, \ldots, s_{n-1})\)

Example. Let \(R = (4, 4, 3, 3, 2, 2)\), \(S = (4, 4, 3, 3, 3, 1)\). Then \(R^* = (6, 6, 4, 2, 0, 0)\). Starting with the matrix \(\bar{A}\) in \(A(R, R^*)\) and applying the algorithm, we get:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\rightarrow
\]
A matrix in its class has no submatrix equal to \( L_2 \), and it is straightforward to verify that it is a minimal matrix in its class \( A(R,S) \).

**Theorem 2.1.** Let \( R \) and \( S \) be positive, monotone vectors such that \( A(R,S) \) is nonempty. Then algorithm I constructs a matrix \( A = [a_{ij}] \) in \( A(R,S) \) that is minimal in the Bruhat order.

**Proof.** We prove the theorem by induction on \( n \). If \( n = 1 \), there is a unique matrix in \( A(R,S) \), and the theorem holds trivially. Assume that \( n > 1 \). Let \( R_1 \) be defined as in the algorithm. Let \( P = [p_{ij}] \) be a matrix in \( A(R,S) \) such that \( P \preceq_B A \). Let \( u = (u_1, u_2, \ldots, u_m)^T \) and \( v = (v_1, v_2, \ldots, v_m)^T \) be, respectively, the last columns of \( A \) and \( P \). First suppose that \( u = v \). Then the matrices \( A' \) and \( P' \) obtained by deleting the last column of \( A \) and \( P \), respectively, belong to the same class \( A(R',S') \), and \( P' \preceq_B A' \). Since \( A' \) is constructed by algorithm I, it now follows from the inductive assumption that \( P' = A' \) and hence \( P = A \).

Now suppose that \( u \neq v \). We may assume that the last column of \( P \) consists of \( p_1 = y_1 \) 0’s followed by \( y_1 \)'s, \( \ldots \), \( p_k = y_k \) 0’s followed by \( y_k \) 1’s where \( y_1, y_2, \ldots, y_k \) are nonnegative integers satisfying \( y_1 + y_2 + \cdots + y_k = s_n \). Otherwise, the last column of \( P \) contains a 1 above a 0 in two rows with equal sums, and \( P \) contains a submatrix equal to \( L_2 \). A one-sided interchange then replaces \( P \) with \( Q \) where \( Q \preceq_B P \preceq_B A \).

The row sum vector \( R_{(y_1,y_2,\ldots,y_k)} \) of the matrix \( P' \) obtained by deleting the last column of \( P \) is nonincreasing. Since \( P \in A(R,S) \), \((s_1, s_2, \ldots, s_{n-1}) \preceq R_{(y_1,y_2,\ldots,y_k)}^T \).

The choice of \( x_1, x_2, \ldots, x_k \) implies that

\[
y_1 + \cdots + y_j \leq x_1 + \cdots + x_j \quad (j = 1, 2, \ldots, k)
\]

with equality for \( j = k \). Let \( q \) be the smallest integer such that \( u_q = v_q \). Then it follows from (2.1) that \( u_q = 0 \) and \( v_q = 1 \). We calculate that

\[
\sum_{i=1}^{q} \sum_{j=1}^{n-1} p_{ij} = r_1 + \cdots + r_q - \sum_{j=1}^{q-1} v_j - 1
\]

\[
= r_1 + \cdots + r_q - \sum_{j=1}^{q-1} u_j - 1
\]

\[
= r_1 + \cdots + r_q - \sum_{j=1}^{q} u_j - 1
\]
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\[ = \sum_{i=1}^{q} \sum_{j=1}^{n-1} a_{ij} - 1, \]

contradicting that \( P \preceq_B A \). The theorem now follows. \( \square \)

We now consider classes \( A \) with constant row and column sums. Let \( k \) be an integer with \( 1 \leq k \leq n \), let \( K = (k, k, \ldots, k) \), the \( n \)-vector of \( k \)'s, and let \( R = S = K \). We denote the corresponding class \( \mathcal{A}(R, S) \) by \( \mathcal{A}(n, k) \). In case \( k = 1 \), this gives the class of permutation matrices of order \( n \). Our algorithm for constructing a minimal matrix in \( \mathcal{A}(K, K) \) simplifies in this case.

**II. Algorithm to Construct a Minimal Matrix in the Bruhat order for \( \mathcal{A}(n, k) \)**

1. Let \( n = qk + r \) where \( 0 \leq r < k \).
2. If \( r = 0 \), then \( A = J_k \oplus \cdots \oplus J_k \), \( (q J_k \)'s) \) is a minimal matrix.
3. Else, \( r \neq 0 \).
   (a) If \( q \geq 2 \), let \( A = X \oplus J_k \oplus \cdots \oplus J_k \), \( (q - 1 J_k \)'s, \( X \) has order \( k + r \)),
   and let \( n \leftarrow k + r \).
   (b) Else, \( q = 1 \), and let
   \[ A = \begin{bmatrix} J_{t,k} & O_{k,r} \\ X & J_{k,r} \end{bmatrix}, \]
   \( \) (\( X \) has order \( k \)), and let \( n \leftarrow k \) and \( k \leftarrow k - r \).
   (c) Proceed recursively with the current values of \( n \) and \( k \) to determine \( X \).

**Example.** Let \( n = 18 \) and \( k = 11 \). The algorithm constructs the following minimal matrix in \( \mathcal{A}(K, K) \).

\[
\begin{bmatrix}
J_{7,11} & O_7 \\
J_{3,4} & O_3 \\
I_4 & J_{4,3} \\
O_{4,7} & J_4
\end{bmatrix}.
\]

Here we first construct (with \( 18 = 1 \cdot 11 + 7 \)),

\[
\begin{bmatrix}
J_{7,11} & O_7 \\
X & J_{11,7}
\end{bmatrix}.
\]

Then to construct the matrix \( X \) of order 11 with \( k = 11 - 7 = 4 \) (and \( 11 = 2 \cdot 4 + 3 \)), we construct

\[
\begin{bmatrix}
Y & O_{7,4} \\
O_{4,7} & J_4
\end{bmatrix}.
\]

Then to construct the matrix \( Y \) of order \( 4 + 3 = 7 \) with \( k = 4 \) (and \( 7 = 1 \cdot 4 + 3 \)), we construct

\[
\begin{bmatrix}
J_{3,4} & O_3 \\
Z & J_{4,3}
\end{bmatrix}.
\]
Finally, to construct the matrix \( Z \) of order 4 with \( k = 4 - 3 = 1 \) (and \( 4 = 4 \cdot 1 + 0 \)), we construct

\[
Z = I_1 \oplus I_1 \oplus I_1 \oplus I_1 = I_4.
\]

3. Minimal Matrices in \( A(n, 2) \) and \( A(n, 3) \). In this section we characterize the minimal matrices in the classes \( A(n, 2) \) and \( A(n, 3) \). Clearly, if \( A \) is minimal, so is its transpose \( A^T \). We first record a useful lemma.

**Lemma 3.1.** Let \( k \) and \( n \) be positive integers with \( n \geq k \), and let \( A = [a_{ij}] \) be a matrix in \( A(n, k) \). Assume that \( A \) is minimal in the Bruhat order. Let \( p \) and \( q \) be integers with \( 1 \leq p < q \leq n \), and let \( r \) be an integer with \( 0 \leq r < n \). If

\[
a_{1p} + a_{2p} + \cdots + a_{rp} = a_{1q} + a_{2q} + \cdots + a_{rq},
\]

then \((a_{r+1,p}, a_{r+1,q}) \neq (0, 1)\). (If \( r = 0 \), then both sides of (3.1) are interpreted as 0.)

**Proof.** Assume that (3.1) holds and \((a_{r+1,p}, a_{r+1,q}) = (0, 1)\). Since \( A \) has \( k \) 1’s in each column, there exists an integer \( s \) with \( r+1 < s \leq n \) such that \((a_{sp}, a_{sq}) = (1,0)\). Hence \( A \) has a submatrix of order 2 equal to \( L_2 \), and \( A \) cannot be minimal in the Bruhat order. \( \blacksquare \)

The minimal matrices in \( A(n, 2) \) are easily determined. Let \( F_n \) denote the matrix of order \( n \) with 0’s in positions \((1,n), (2,n-2), \dotsc, (n,1)\) and 0’s elsewhere.

**Theorem 3.2.** Let \( n \) be an integer with \( n \geq 2 \). Then a matrix in \( A(n, 2) \) is a minimal matrix in the Bruhat order if and only if it is the direct sum of matrices equal to \( J_2 \) and \( F_3 \).

**Proof.** Let \( A = [a_{ij}] \) be a minimal matrix in \( A(n, 2) \). It follows from several applications of Lemma 3.1 (the case \( r = 0 \)) to \( A \) and its transpose that \( A \) has the form

\[
\begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
1 & a_{22} & & & \\
0 & & & & \\
\vdots & & & & \\
0 & & & & 
\end{bmatrix}
\]

If \( a_{22} = 1 \), then \( A = J_2 \oplus A' \) where \( A' \) is a minimal matrix in \( A(n-2, 2) \). Suppose that \( a_{22} = 0 \). There exists \( i,j \geq 3 \) such that \( a_{ij} = a_{i2} = 1 \). Since \( A \) cannot have a submatrix equal to \( L_2 \), \( a_{ij} = 1 \), and then it follows that \( i = j = 3 \). Hence \( A = F_3 \oplus A' \) where \( A' \) is a minimal matrix in \( A(n-3, 2) \). The theorem now follows by induction on \( n \). \( \blacksquare \)

Let

\[
V = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 
\end{bmatrix}
\]
For $i \geq 1$, let $U_i$ be the matrix in $A(i+6,3)$ of the form

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & \ldots \\
1 & 1 & 1 & 0 & \ldots \\
1 & 1 & 0 & 1 & \ldots \\
0 & 0 & 1 & 1 & 1 \\
& \ddots & \ddots & \ddots & \ddots \\
& & 1 & 1 & 0 \\
& & 0 & 1 & 1 \\
& & 0 & 1 & 1 \\
& & 0 & 1 & 1
\end{bmatrix}
$$

Thus

$$
U_1 = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

**Theorem 3.3.** Let $n$ be an integer with $n \geq 3$. Then a matrix in $A(n,3)$ is a minimal matrix in the Bruhat order if and only if it is the direct sum of matrices equal to $J_3$, $F_4$, $V$, $V^T$ and $U_i$ ($i \geq 1$).

**Proof.** Let $A = [a_{ij}]$ be a minimal matrix in $A(n,3)$. Then $A$ has the form

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & \cdots & 0 \\
1 & a_{22} & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & \cdots & \cdots \\
\vdots & & \ddots & \ddots & \cdots \\
0 & & & & 1 & \cdots \\
0 & & & & & \cdots 
\end{bmatrix}
$$

First suppose that $a_{22} = 0$. Then Lemma 3.1 implies that $a_{23} = a_{32} = 0$. Since each row of $A$ has three 1’s, there exist $l > j > 3$ such that $a_{2j} = a_{2l} = 1$. Since each column of $A$ has three 1’s, there exist $k > i > 3$ such that $a_{ik} = a_{i2} = 1$, and there exist $q > p > 3$ such that $a_{kj} = a_{k3} = 1$. Since column $j$ contains only three 1’s, we must have, by Lemma 3.1, that $p = i$ and $q = k$. But then row $i$ has at least four 1’s, a contradiction. Therefore we have $a_{22} = 1$.

We now focus on $a_{23}$.

Case I: Assume that $a_{23} = 0$. Let the third 1 in row 2 occur in column $j \geq 3$. There exist integers $k > i \geq 3$ such that $a_{k3} = a_{i3} = 1$. Since $A$ is minimal, we must have $a_{kj} = a_{ij} = 1$. If $j > 4$, then $a_{24} = 0$, contradicting Lemma 3.1. Hence $j = 4$. Using Lemma 3.1 and a little thought, we see that $k = i + 1$ and $i \in \{3,4\}$. Thus the
submatrix $A[(i, i + 1), \{3, 4\}]$ at the intersection of rows $i$ and $i + 1$ and columns 3 and 4 equals $J_2$, and this submatrix intersects row 3 or row 4. We now consider two subcases according to the value of $a_{32}$.

First suppose that $a_{32} = 1$. Since row 3 has only three 1’s, we see that $i = 4$, and applying Lemma 3.1 we see that $a_{35} = 1$. Thus $A$ has the form

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & 1 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 1 & 1 & \cdots & \\
0 & 0 & 1 & 1 & \cdots & \\
0 & 0 & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & 0 & 
\end{bmatrix}.$$  

Applying Lemma 3.1 to $A^T$, we see that $a_{45} = a_{55} = 1$. Hence $A = V^T \oplus A'$ for some $A'$.

Now suppose that $a_{32} = 0$. Recall that $i \in \{3, 4\}$. Suppose that $i = 4$. Since each column contains only three 1’s, we have $a_{33} = a_{34} = 0$. Applying Lemma 3.1, we get that $a_{35} = a_{36} = 1$. Since $A$ cannot have a submatrix equal to $L_2$, we conclude that $a_{45} = a_{55} = 1$, giving four 1’s in row 4. Therefore we must have $i = 3$. Now $A$ has the form

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & a_{42} & 1 & 1 
\end{bmatrix}.$$  

Since $a_{12} + a_{22} + a_{32} = a_{13} + a_{23} + a_{33}$ and $a_{43} = 1$, we have, from Lemma 3.1, that $a_{42} = 1$, and $A = F_4 \oplus A'$ for some $A'$.

Case II: Assume that $a_{23} = 1$.

First suppose that $a_{32} = 0$, and so by Lemma 3.1, $a_{33} = 0$. Since rows 1 and 2 contain only 0’s beyond column 3, and since row 4 contains three 1’s, it again follows from Lemma 3.1 that $a_{34} = a_{35} = 1$. Since $a_{41} = 0$ for all $i \geq 4$, applying Lemma 3.1 to $A^T$, we have $a_{42} = 1$, and to avoid $L_2$, we also have $a_{44} = a_{45} = 1$, and so $a_{43} = 0$. Since $a_{41} = a_{42} = 0$ for all $i \geq 5$, we have $a_{53} = 1$ by Lemma 3.1 applied to $A^T$, and using Lemma 3.1 again we see that $a_{54} = a_{55} = 1$. Therefore, $A = V \oplus A'$ for some matrix $A'$.

We now suppose that $a_{32} = 1$ so that $A$ begins with the form

$$\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & a_{33} 
\end{bmatrix}.$$
If $a_{33} = 1$, then $A = J_3 \oplus A'$ for some matrix $A'$. Now assume that $a_{33} = 0$. By Lemma 3.1 we must have $a_{34} = 1$ and, by considering $A^T$, $a_{43} = 1$. Hence also $a_{44} = 1$. It follows also from Lemma 3.1, using the fact that row 4 contains a 1 in some column $k$ with $k \geq 5$, that $a_{45} = a_{54} = 1$. Suppose that $a_{55} = 0$. Then $a_{56} = a_{57} = 1$ by Lemma 3.1, and by symmetry, $a_{65} = a_{75} = 1$, implying also that $a_{66} = a_{67} = a_{76} = a_{77} = 1$. Hence $A$ has the form

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}.$$ 

Therefore, $A = U_1 \oplus A'$ for some matrix $A'$. Now suppose that $a_{55} = 1$. Then $a_{56} = a_{57} = 1$. If $a_{66} = 0$, then arguing as above we see that $A = U_2 \oplus A'$ for some matrix $A'$. Otherwise we continue and eventually see that $A = U_i \oplus A'$ for some integer $i$ and matrix $A'$.

It would be interesting to characterize all minimal matrices in the Bruhat order for $k \geq 4$ as done for $k = 2$ and $k = 3$. To do this would require a characterization, for all $k \leq n$, of all minimal matrices in $\mathcal{A}(n,k)$ which cannot be expressed as a nontrivial direct sum. But even for $k = 4$, this appears difficult. For example, the following matrices are minimal matrices in $\mathcal{A}(n,4)$ for an appropriate $n$ that cannot be expressed as a nontrivial direct sum.

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}.$$
In fact, there are many more that can be constructed.

We conclude this note with a conjecture. By Corollary 1.1, a minimal matrix in \( A(R, S) \) has no submatrix equal to \( L_2 \). We conjecture that the converse holds.

**Conjecture.** A matrix in \( A(R, S) \) that does not have \( L_2 \) as a submatrix is minimal in the Bruhat order on \( A(R, S) \).

**REFERENCES**


