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Stephen J. Kirkland
kirkland@math.uregina.ca

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GIRTH AND SUBDOMINANT EIGENVALUES FOR STOCHASTIC MATRICES*

S. KIRKLAND†

Abstract. The set $S(g, n)$ of all stochastic matrices of order $n$ whose directed graph has girth $g$ is considered. For any $g$ and $n$, a lower bound is provided on the modulus of a subdominant eigenvalue of such a matrix in terms of $g$ and $n$, and for the cases $g = 1, 2, 3$ the minimum possible modulus of a subdominant eigenvalue for a matrix in $S(g, n)$ is computed. A class of examples for the case $g = 4$ is investigated, and it is shown that if $g > 2n/3$ and $n \geq 27$, then for every matrix in $S(g, n)$, the modulus of the subdominant eigenvalue is at least $(\frac{1}{5})^1/(2\lfloor n/3 \rfloor)$.

Key words. Stochastic matrix, Markov chain, Directed graph, Girth, Subdominant eigenvalue.

AMS subject classifications. 15A18, 15A42, 15A51.

1. Introduction and preliminaries. Suppose that $T$ is an irreducible stochastic matrix. It is well known that the spectral radius of $T$ is 1, and that in fact 1 is an eigenvalue of $T$ (with the all ones vector $1$ as a corresponding eigenvector). Indeed, denoting the directed graph of $T$ by $D$ (see [2]), Perron-Frobenius theory (see [8]) gives more information on the spectrum of $T$, namely that the number of eigenvalues having modulus 1 coincides with the greatest common divisor of the cycle lengths in $D$. In particular, if that greatest common divisor is 1, it follows that the powers of $T$ converge. (This in turn leads to a convergence result for the iterates of a Markov chain with transition matrix $T$.) Denoting the eigenvalues of $T$ by $1 = \lambda_1(T) \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_n(T)|$ (throughout we will use this convention in labeling the eigenvalues of a stochastic matrix), it is not difficult to see that the asymptotic rate of convergence of the powers of $T$ is governed by $|\lambda_2(T)|$. We refer to $\lambda_2(T)$ as a subdominant eigenvalue of $T$.

In light of these observations, it is natural to wonder whether stronger hypotheses on the directed graph $D$ will yield further information on the subdominant eigenvalue(s) of $T$. This sort of question was addressed in [6], where it was shown that if $T$ is a primitive stochastic matrix of order $n$ whose exponent (i.e. the smallest $k \in \mathbb{N}$ so that $T^k$ has all positive entries) is at least \( \frac{n^2 - 2n + 2}{2} \), then $T$ has at least $2\lfloor (n - 4)/4 \rfloor$ eigenvalues with moduli exceeding $\left( \frac{1}{2} \sin \left( \frac{\pi}{n - 1} \right) \right)^{2/(n - 1)}$. Thus a hypothesis on the directed graph $D$ can lead to information about the eigenvalues of $T$.

In this paper, we consider the influence of the girth of $D$ - that is, the length of the shortest cycle in $D$ - on the modulus of the subdominant eigenvalue(s) of $T$. (It is straightforward to see that the girth of $D$ is the smallest $k \in \mathbb{N}$ such that $\text{trace}(T^k) > 0$.) Specifically, let $S(g, n)$ be the set of $n \times n$ stochastic matrices having
digraphs with girth $g$. If $T \in \mathcal{S}(g, n)$, how large can $|\lambda_2(T)|$ be? How small can $|\lambda_2(T)|$ be?

We note that the former question is readily dealt with. If $g \geq 2$, consider the directed graph $G$ on $n$ vertices that consists of a single $g$-cycle, say on vertices $1, \ldots, g$, along with a directed path $n \rightarrow n-1 \rightarrow \ldots \rightarrow g+1 \rightarrow 1$. Letting $A$ be the $(0, 1)$ adjacency matrix of $G$, it is straightforward to determine that $A \in \mathcal{S}(g, n)$, and that the eigenvalues of $A$ consist of the $g$-th roots of unity, along with the eigenvalue 0 of algebraic multiplicity $n-g$. In particular, $|\lambda_2(A)| = 1$, so we find that $\max\{|\lambda_2(T)||T \in \mathcal{S}(g, n)\} = 1$. Similarly, for the case $g = 1$, we note that the identity matrix of order $n$, $I_n$, is an element of $\mathcal{S}(1, n)$, and again we have $\max\{|\lambda_2(T)||T \in \mathcal{S}(1, n)\} = 1$.

The bulk of this paper is devoted to a discussion of how small $|\lambda_2(T)|$ can be if $T \in \mathcal{S}(g, n)$ (and hence, of how quickly the powers of $T$ can converge). To that end, we let $\lambda_2(g, n)$ be given by $\lambda_2(g, n) = \inf\{|\lambda_2(T)||T \in \mathcal{S}(g, n)\}$.

**Remark 1.1.** We begin by discussing the case that $g = 1$. Let $J$ denote the $n \times n$ all ones matrix, and observe that for any $n \geq 2$, the $n \times n$ matrix $\frac{1}{n}J$ has the eigenvalues 1 and 0, the latter with algebraic and geometric multiplicity $n-1$. It follows immediately that that $\lambda_2(1, n) = 0$.

Indeed there are many stochastic matrices yielding this minimum value for $\lambda_2$, of all possible admissible Jordan forms. To see this fact, let $M$ be any nilpotent Jordan matrix of order $n-1$. Let $v_1, \ldots, v_{n-1}$ be an orthonormal basis of the orthogonal complement of $1$ in $\mathbb{R}^n$, and let $V$ be the $n \times (n-1)$ matrix whose columns are $v_1, \ldots, v_{n-1}$. We find readily that for all sufficiently small $\varepsilon > 0$, the matrix $T = \frac{1}{n}J + \varepsilon V MV^T$ is stochastic; further, the Jordan form for $T$ is given by $[1] \oplus M$, so that the Jordan structure of $T$ corresponding to the eigenvalue 0 coincides with that of $M$. Evidently for such a matrix $T$, the powers of $T$ converge in a finite number of iterations; in fact that number of iterations coincides with the size of the largest Jordan block of $M$.

The following elementary result provides a lower bound on $\lambda_2(g, n)$ for $g \geq 2$.

**Theorem 1.1.** Suppose that $g \geq 2$ and that $T \in \mathcal{S}(g, n)$. Then $|\lambda_2(T)| \geq 1/(n-1)^{1/(g-1)}$. Equality holds if and only if $g = 2$ and the eigenvalues of $T$ are 1 (with algebraic multiplicity 1) and $\frac{1}{n-1}$ (with algebraic multiplicity $n-1$). In particular,

$$
\lambda_2(g, n) \geq 1/(n-1)^{1/(g-1)}.
$$

**(1.1)**

**Proof.** Let the eigenvalues of $T$ be $1, \lambda_2, \ldots, \lambda_n$. Since $\text{trace}(T^{g-1}) = 0$, we find that $\sum_{i=2}^{n} \lambda_i^{g-1} = -1$. Hence, $(n-1)|\lambda_2|^{g-1} \geq \sum_{i=2}^{n} |\lambda_i|^{g-1} \geq \sum_{i=2}^{n} \lambda_i^{g-1} | = 1$. The inequality on $|\lambda_2|$ now follows readily.

Now suppose that $|\lambda_2| = 1/(n-1)^{1/(g-1)}$. Inspecting the proof above, we find that $|\lambda_i| = |\lambda_2|$, $i = 3, \ldots, n$, and that since equality holds in the triangle inequality, it must be the case that each of $\lambda_2, \ldots, \lambda_n$ has the same complex argument. Thus $\lambda_i = \lambda_1$ for each $i = 3, \ldots, n$. Since $\text{trace}(T) = 0$, we deduce that $\lambda_2 = -1/(n-1)$; but then $\text{trace}(T^2) = n/(n-1) > 0$, so that $g = 2$. The converse is straightforward.


Remark 1.2. If $T \in \mathcal{S}(2, n)$ and $|\lambda_2(T)| = 1/(n-1)$, it is straightforward to see that the matrix $S = \frac{n-1}{2}T + \frac{1}{2}I_n$ has just two eigenvalues, 1 and 0, the latter with algebraic multiplicity $n-1$. In particular, $S$ is a matrix in $\mathcal{S}(1, n)$ such that $\lambda_2(S) = \lambda_2(1, n) = 0$.

Remark 1.3. From Theorem 1.1, we see that if $\exists \epsilon > 0$ such that $g \geq cn$, then necessarily $\lambda_2(g, n) \geq 1/(n-1)^{\frac{1}{n-1}} - \epsilon$. An application of l'Hôpital’s rule shows that $1/(n-1)^{\frac{1}{n-1}} \to 1$ as $n \to \infty$. Consequently, we find that for each $\epsilon > 0$, and any $\epsilon > 0$, there is a number $N$ such that if $n > N$ and $g \geq cn$, then each matrix $T \in \mathcal{S}(g, n)$ has $|\lambda_2(T)| \geq 1 - \epsilon$.

We close this section with a discussion of $\lambda_2(g, n)$ as a function of $g$ and $n$.

Proposition 1.2. Fix $g$ and $n$ with $2 \leq g \leq n - 1$. Then

a) $\lambda_2(g, n) \geq \lambda_2(g, n + 1)$, and

b) $\lambda_2(g + 1, n) \geq \lambda_2(g, n)$.

Proof. a) Suppose that $T \in \mathcal{S}(g, n)$, and partition off the last row and column of $T$, say $T = \begin{bmatrix} T_1 & x \\ y^T & 0 \end{bmatrix}$ . Now let $S$ be the stochastic matrix of order $n + 1$ given by

$$S = \begin{bmatrix} T_1 & 0 \\ e \times y & 0 \\ y^T & 0 \end{bmatrix} .$$

Note that the digraph of $S$ is formed from that of $T$ by adding the vertex $n + 1$, along with the arcs $i \to n + 1$ for each $i$ such that $i \to n$ in the digraph of $T$, and the arcs $n + 1 \to j$ for each $j$ such that $n \to j$ in the digraph of $T$. It now follows that the girth of the digraph of $S$ is also $g$, so that $S \in \mathcal{S}(g, n + 1)$. Observe also that we can write $S$ as $S = ATB$, where the $(n + 1) \times n$ matrix $A$ is given by

$$A = \begin{bmatrix} I_{n-1} & 0 \\ 0^T & 1 \end{bmatrix} ,$$

while the $n \times (n + 1)$ matrix $B$ is given by $B = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0^T & 0 & \frac{1}{2} \end{bmatrix}$. It is straightforward to see that $BA = I_n$; from this we find that since the matrix $ATB$ and the matrix $TBA$ have the same nonzero eigenvalues, so do $S$ and $T$. In particular, $\lambda_2(S) = \lambda_2(T)$, and we readily find that $\lambda_2(g, n) \geq \lambda_2(g, n + 1)$.

b) Let $\epsilon > 0$ be given, and suppose that $T \in \mathcal{S}(g + 1, n)$ is such that $|\lambda_2(T)| < \lambda_2(g + 1, n) + \epsilon/2$. Without loss of generality, we suppose that the digraph of $T$ contains the cycle $1 \to 2 \to 3 \to \ldots \to g + 1 \to 1$. For each $x \in (0, T_{g, g + 1})$, let $S(x) = T(x) + xe_2(e_1 - e_{g + 1})^T$, where $e_i$ denotes the $i$-th standard unit basis vector. Note that for each $x \in (0, T_{g, g + 1})$, $S(x) \in \mathcal{S}(g, n)$. By the continuity of the spectrum, there is a $\delta > 0$ such that for any $0 < x < \min \{ \delta, T_{g, g + 1} \}$, $|\lambda_2(S(x)) - |\lambda_2(T)| < \epsilon/2$. Hence we find that for $0 < x < \min \{ \delta, T_{g, g + 1} \}$ we have $\lambda_2(g, n) \leq |\lambda_2(S(x))| < |\lambda_2(T)| + \epsilon/2 < \lambda_2(g + 1, n) + \epsilon$. In particular, we find that for each $\epsilon > 0$, $\lambda_2(g, n) \leq \lambda_2(g + 1, n) + \epsilon$, from which we conclude that $\lambda_2(g, n) \leq \lambda_2(g + 1, n)$. □

2. Girths 2 and 3. In this section, we use some elementary techniques to find $\lambda_2(2, n)$ and $\lambda_2(3, n)$. We begin with a discussion of the former.

Theorem 2.1. For any $n \geq 2$, $\lambda_2(2, n) = 1/(n - 1)$.

Proof. From Theorem 1.1, we have $\lambda_2(2, n) \geq 1/(n - 1)$; the result now follows upon observing that the matrix $\frac{1}{n}(J - I) \in \mathcal{S}(2, n)$, and has eigenvalues 1 and $-1/(n - 1)$, the latter with multiplicity $n - 1$. □
Our next result shows that there is just one diagonal matrix that yields the minimum value $\lambda_2(2, n)$.

**Theorem 2.2.** Suppose that $T \in S(2, n)$. Then $T$ is diagonalizable with $|\lambda_2(T)| = 1/(n-1)$ if and only if $T = \frac{1}{n-1}(J-I)$.

**Proof.** Suppose that $T$ is diagonalizable, with $|\lambda_2(T)| = 1/(n-1)$; from Theorem 1.1 we find that the eigenvalue $\lambda_2 = -1/(n-1)$ has algebraic multiplicity $n-1$. Since $T$ is diagonalizable, the dimension of the $\lambda_2$-eigenspace is $n-1$. Let $x^T$ be the left Perron vector for $T$, normalized so that $x^T1 = 1$. It follows that there are right $\lambda_2$-eigenvectors $v_2, \ldots, v_n$ and left $\lambda_2$-eigenvectors $w_2, \ldots, w_n$ so that $T = 1x^T + \frac{1}{n-1}\sum_{i=2}^n v_iw_i^T$ and $I = 1x^T + \sum_{i=2}^n v_iw_i^T$. Substituting, we see that $T = \frac{1}{n-1}(n1x^T-I)$, and since $T$ has trace zero, necessarily, $x^T = \frac{1}{n}1^T$, yielding the desired expression for $T$. The converse is straightforward. $\blacksquare$

Our next example shows that other Jordan forms are possible for matrices yielding the minimum value $\lambda_2(2, n)$.

**Example 2.1.** Consider the polynomial

$$(\lambda + \frac{1}{n-1})^{n-1} = \sum_{j=0}^{n-1} \lambda^j \left( \frac{1}{n-1} \right)^{n-1-j} \binom{n-1}{j} = \lambda^{n-1} + \lambda^{n-2} + \sum_{j=0}^{n-3} \lambda^j \left( \frac{1}{n-1} \right)^{n-1-j} \binom{n-1}{j}.$$  

From the fact that $n-j > \frac{j}{n-1}$ for $j = 1, \ldots, n-2$, it follows readily that

$$(\frac{1}{n-1})^{n-1-j} \binom{n-1}{j} > (\frac{1}{n-1})^{n-3} \binom{n-1}{j}$$  

for each such $j$.

We thus find that $(\lambda - 1)(\lambda + \frac{1}{n-1})^{n-1}$ can be written as $\lambda^n - \sum_{j=2}^n a_j \lambda^{n-j}$, where $a_j > 0$ for $j = 2, \ldots, n$, and $\sum_{j=2}^n a_j = 1$. Consequently, the companion matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & 0 \end{bmatrix}$$  

is in $S(2, n)$, and $\lambda_2(C) = -1/(n-1)$. Note that since any eigenvalue of a companion matrix is geometrically simple, the eigenvalue $-1/(n-1)$ of $C$ has a single Jordan block of size $n-1$.

Next, we compute $\lambda_2(3, n)$ for odd $n$.

**Theorem 2.3.** Suppose that $n \geq 3$ is odd. If $T \in S(3, n)$, then $|\lambda_2(T)| \geq \frac{-n+1}{n-1}$, with equality holding if and only if the eigenvalues of $T$ are $1$ (with algebraic multiplicity one) and $\frac{-1 \pm \sqrt{\frac{n+1}{n-1}}}{n-1}$ (with algebraic multiplicity $(n-1)/2$ each). Further, $\lambda_2(3, n) = \frac{-n+1}{n-1}$.

**Proof.** Suppose that $T \in S(3, n)$, and denote the eigenvalues of $T$ by $1$, and $x_j + iy_j, j = 2, \ldots, n$ (where of course each complex eigenvalue appears with a corresponding complex conjugate). Since $\text{trace}(T^2) = 0$, we have $\sum_{j=2}^n x_j = -1$, while from the fact that $\text{trace}(T^2) = 0$, we have $1 + \sum_{j=2}^n (x_j^2 - y_j^2) = 0$. Consequently,
\[
\sum_{j=2}^n (x_j^2 + y_j^2) = 1 + 2 \sum_{j=2}^n x_j^2 \geq 1 + 2|\sum_{j=2}^n x_j|^2/(n-1) = \frac{n+1}{n-1},
\]
the inequality following from the Cauchy-Schwarz inequality, and the fact that \(\sum_{j=2}^n x_j = -1\). Thus we find that \((n-1)|\lambda|^2 \geq \sum_{j=2}^n (x_j^2 + y_j^2) \geq \frac{n+1}{n-1}\), so that \(|\lambda(T)| \geq \frac{\sqrt{n+1}}{n-1}\). Inspecting the proof above, we see that \(|\lambda(T)| = \frac{\sqrt{n+1}}{n-1}\) if and only if each \(x_j\) is equal to \(-1/(n-1)\), and each \(y_j\) is equal to \(n/(n-1)^2\). The equality characterization now follows.

We claim that for each odd \(n\), the companion matrix for the polynomial \((\lambda - 1)/(\lambda - \frac{n(n-1)}{n+1}\sqrt{(n-1)/2})(\lambda - \frac{n(n-1)}{n+1}\sqrt{(n-1)/2}) = (\lambda - 1)(\lambda^2 + \frac{2}{n-1}\lambda + \frac{n+1}{(n-1)^2})^{(n-1)/2}\) is in fact a nonnegative matrix, from which it will follow that for each odd \(n\), there is a matrix in \(S(3, n)\) having \(-\frac{1+\sqrt{n}}{n-1}\) as a subdominant eigenvalue. In order to prove that this companion matrix is nonnegative, it suffices to show that the coefficients of the polynomial \(q(\lambda) = (\lambda^2 + \frac{2}{n-1}\lambda + \frac{n+1}{(n-1)^2})^{(n-1)/2}\) are increasing with the powers of \(\lambda\).

Note that \(q(\lambda) = \left((\lambda + \frac{1}{n-1})^2 + \frac{n-1}{(n-1)^2}\right)^{(n-1)/2}\). Applying the binomial expansion, and collecting powers of \(\lambda\), we find that
\[
(2.1) q(\lambda) = \sum_{j=0}^{n-1} \binom{(n-1)/2}{j} \frac{n-j}{(n-1)^2} \binom{n-1/2}{j} \frac{2j}{l} \binom{j}{l} \left((n-1)/2\right)_j.
\]
Write \(q(\lambda) = \sum_{j=0}^{n-1} \lambda^j \alpha_j\). We claim that \(\alpha_l \geq \alpha_{l-1}\) for each \(l = 1, \ldots, n-1\), which will yield the desired result. Note that for each such \(l\), the inequality \(\alpha_l \geq \alpha_{l-1}\) is equivalent to \((n-1)\sum_{j=l+1}^{(n-1)/2} \binom{j}{l} \frac{n-j}{(n-1)^2} \binom{n-1/2}{j} \frac{2j}{l} \binom{j}{l} \left((n-1)/2\right)_j \geq \alpha_{l-1}\). Observe that \((n-1)\binom{j}{l} - \binom{j}{l-1} = \frac{2j}{l-1}\binom{j}{l} - \frac{1}{l-2}\binom{j}{l-1}\geq 0\), so in particular, if \(l\) is even (so that \([l/2] = \lfloor (l-1)/2 \rfloor\)) it follows readily that \(\alpha_l \geq \alpha_{l-1}\).

Finally, suppose that \(l\) is odd with \(1 \leq l \leq n-1\) and \(l = 2r+1\). Then \([l/2] = r+1, \lfloor (l-1)/2 \rfloor = r\), and since \(2r+1 \leq n-1\), we find that \(r \leq \frac{n-3}{2}\). In order to show that \(\alpha_l \geq \alpha_{l-1}\), it suffices to show, with the inequalities proven above, that the inequality can be equivalent to \(2\binom{2r}{l} - \frac{2r+1}{n} - \frac{1}{n-1/2} \geq 0\), and since we have \(2\binom{2r}{l} - \frac{2r+1}{n} - \frac{1}{n-1/2} \geq 2\binom{2r}{l} - \frac{2r+1}{n} - \frac{1}{n-1/2} \geq 0\), the desired inequality is thus established. Hence for odd \(l\), we have \(\alpha_l \geq \alpha_{l-1}\), and it now follows that there is a companion matrix \(C \in S(3, n)\) such that \(|\lambda_2(C)| = \frac{n+1}{n-1}\).

**Example 2.2.** Another class of matrices in \(S(3, n)\) yielding the minimum value for \(|\lambda_2|\) arises in the following combinatorial context. A square \((0, 1)\) matrix \(A\) of order \(n\) is called a tournament matrix if it satisfies the equation \(A + A^T = J - I\). From that equation, one readily deduces that there are no cycles of length 2 in the digraph of a tournament matrix, and a standard result in the area asserts that the digraph associated with any tournament matrix either contains a cycle of length 3, or it has no cycles at all. Thus the digraph of any nonnilpotent tournament matrix necessarily has girth 3.

If, in addition, a tournament matrix \(A\) satisfies the identity \(A^T A = A^2 = \frac{n+1}{n-1} I + \frac{n-1}{n+1} J = A A^T\), then \(A\) is known as a doubly regular (or Hadamard) tournament ma-
trix; note that necessarily $n \equiv 3 \mod 4$ in that case. It turns out that doubly regular tournament matrices are co-existent with skew-Hadamard matrices, and so of course the question of whether there is a doubly regular tournament matrix in every admissible order is open, and apparently quite difficult.

In [3] it is shown that if $A$ is a doubly regular tournament matrix, then its eigenvalues consist of $\frac{n-1}{2}$ (of algebraic multiplicity one, and having 1 as a corresponding right eigenvector) and $\frac{-1}{2} + i\sqrt{n}$, each of algebraic multiplicity $(n-1)/2$. Consequently, we find that if $A$ is an $n \times n$ doubly regular tournament matrix, then $T = \frac{2}{n-1} A$ is in $S(3,n)$ and has eigenvalues 1 and $\frac{-1}{2} + i\sqrt{n}$, the latter with algebraic multiplicity $(n-1)/2$ each. From Theorem 2.3, we find that $|\lambda_2(T)| = \lambda_2(3,n)$.

We adapt the technique of the proof of Theorem 2.3 in order to compute $\lambda_2(3,n)$ for even $n$.

**Theorem 2.4.** Suppose that $n \geq 4$ is even. If $T \in S(3,n)$, then $|\lambda_2(T)| \geq \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}$, with equality holding if and only if the eigenvalues of $T$ are 1 (with algebraic multiplicity one), $-2/n$ (also with algebraic multiplicity one) and $\frac{1}{n} \pm \frac{1}{n} \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}$ (with algebraic multiplicity $(n-2)/2$ each). Further, $\lambda_2(3,n) = \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}$.

**Proof.** Suppose that $T \in S(3,n)$. Since $T$ is stochastic, it has 1 as an eigenvalue, and since $n$ is even, there is at least one more real eigenvalue for $T$, say $z$. Let $x_j + iy_j, j = 2, \ldots, n-1$, denote the remaining eigenvalues of $T$. From the fact that $\text{trace}(T) = 0$, we have $1 + z + \sum_{j=2}^{n-1} x_j = 0$, while $\text{trace}(T^2) = 0$ yields $1 + z^2 + \sum_{j=2}^{n-1} (x_j^2 - y_j^2) = 0$. Thus we have $\sum_{j=2}^{n-1} (x_j^2 + y_j^2) = 1 + z^2 + 2 \sum_{j=2}^{n-1} x_j^2$. Consequently, we find that $(n-2)|\lambda|^2 \geq \sum_{j=2}^{n-1} (x_j^2 + y_j^2) = 1 + z^2 + 2 \sum_{j=2}^{n-1} x_j^2 \geq 1 + z^2 + 2(1 + z^2)/(n-2)$, the second inequality following from the Cauchy-Schwarz inequality. The expression $1 + z^2 + 2(1 + z^2)/(n-2)$ is readily seen to be uniquely minimized when $z = -2/n$, with a minimum value of $\frac{n^2 + n + 2}{n^2 - 2n}$. Hence we find that $(n-2)|\lambda|^2 \geq \frac{n^2 + n + 2}{n^2 - 2n}$, and the lower bound on $|\lambda_2|$ follows.

Inspecting the argument above, we see that if $|\lambda_2(T)| = \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}$, then necessarily $z$ must be $-2/n$, each $x_j$ must be $-1/n$, while each $y_j^2$ is equal to $\frac{1}{n} \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}$. The characterization of equality now follows.

We claim that for each even $n$, there is a companion matrix in $S(3,n)$ having $\frac{1}{n} + \frac{i}{n} \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}$ as a subdominant eigenvalue. To see the claim, first consider the polynomial $q(\lambda) = \left(\lambda - \left(\frac{-1}{n} + \frac{i}{n} \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}\right)\right)^{(n-2)/2} \left(\lambda - \left(\frac{1}{n} + \frac{i}{n} \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}\right)\right)^{(n-2)/2} = \left(\lambda + \frac{1}{n} \right)^{(n-2)/2} \left(\lambda + \frac{1}{n} \right)^{(n-2)/2}$ and write it as $q(\lambda) = \sum_{l=0}^{n-2} \lambda^l a_l$, so that $(\lambda+2/n)q(\lambda) = \lambda^{n+1} + \sum_{l=1}^{n-2} \lambda^l (a_{l-1} + 2a_l/n) + 2a_0/n$. As in the proof of Theorem 2.3, it suffices to show that in this last expression, the coefficients of $\lambda^l$ are nondecreasing in $l$. Also as in the proof of that theorem, we find that for each $l = 0, \ldots, n-2$, $a_l = \sum_{j=n/2}^{(n-2)/2-1} \left(\frac{1}{n} \right)^{(n-2)/2-l} \left(\frac{n^2 + n + 2}{n^2 - 2n}\right)^{(n-2)/2-j} \left(\frac{2}{n}\right)^{(n-2)/2-j} \binom{n-2}{j}$. Straightforward computations now reveal that the coefficients of $\lambda^{n-1}, \lambda^{n-2}, \lambda^{n-3}$ and $\lambda^{n-4}$ in the polynomial $(\lambda + \frac{1}{n} + \frac{i}{n} \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}})^{(n-2)/2}$ are nondecreasing in $l$. Since $\lambda_2(3,n) = \sqrt{\frac{n^2 + n + 2}{n^2 - 2n}}$, the characterization of equality now follows.
2/n)q(λ) are 1, 1, and $2n^2 - 3n^2 - \frac{2}{3}$, respectively. We claim that for each $l = 1, \ldots, n - 4$, $a_l \geq a_{l-1}$, which is sufficient to give the desired result.

The claim is equivalent to proving that for each $l = 1, \ldots, n - 4$,

$$n \sum_{j=l/2}^{(n-2)/2} \left( \frac{n-2}{n^2 + n^2 + 2} \right)^j \binom{(n-2)/2}{j} \geq n \sum_{j=[(l-1)/2]}^{(n-2)/2} \left( \frac{n-2}{n^2 + n^2 + 2} \right)^j \binom{(n-2)/2}{j}.$$

Observe that $n \binom{(n-2)/2}{j} - \binom{(n-2)/2}{j} = \frac{2j!}{(l-1)!} (\frac{1}{2} - \frac{1}{2j+1}) \geq 0$, so in particular, if $l$ is even (so that $[(l-1)/2] = [(l-1)/2]$) it follows readily that $a_l \geq a_{l-1}$. Now suppose that $l \geq 1$ is odd, say $l = 2r + 1$, so that $[(l-1)/2] = r + 1$ and $[(l-1)/2] = r$. Note also that since $l \leq n - 4$, in fact $l \leq n - 5$, so that $r \leq (n-6)/2$. In conjunction with the argument above, it suffices to show that $n \left( \frac{n-2}{n^2 + n^2 + 2} \right)^{r+1} \left( \frac{2r+2}{r+1} \right)^{\binom{(n-2)/2}{r}} n \left( \frac{n-2}{n^2 + n^2 + 2} \right)^{r+1} \left( \frac{2r+2}{r+1} \right)^{\binom{(n-2)/2}{r} - \left( \frac{n-2}{n^2 + n^2 + 2} \right)^{r+1} \left( \frac{2r+2}{r+1} \right)^{\binom{(n-2)/2}{r}} n \left( \frac{n-2}{n^2 + n^2 + 2} \right)^{r+1} \left( \frac{2r+2}{r+1} \right)^{\binom{(n-2)/2}{r}} \right) \geq 0$. This last inequality can be seen to be equivalent to $2n(n-2) \frac{n^2 + n^2 + 2}{n^2 + n^2 + 2} = (2r+1) \frac{n^2 + n^2 + 2}{r+1} = \frac{1}{(n-2)/2} \geq 2n(n-2) = (n-5) n^2 + n^2 + 2 \geq 0$, the last since $n \geq 4$. Hence we have $a_l \geq a_{l-1}$ for each $l = 1, \ldots, n - 4$, as desired.

The following result shows that the lower bound of (1.1) on $\lambda_2^2(g, n)$ is of the correct order of magnitude for $g = 3$. Its proof is immediate from Theorems 2.3 and 2.4.

**Corollary 2.5.** $\lim_{n \to -\infty} \lambda_2(3, n) \sqrt{n} - 1 = 1$.

**3. A class of examples for girth 4.** Our object in this section is to identify, for infinitely many $n$, a matrix $T \in S(4, n)$ such that $|\lambda_2(T)|$ is of the same order of magnitude as $1/\sqrt{n - 1}$, the lower bound on $\lambda_2(4, n)$ arising from (1.1). Our approach is to identify a certain sequence of candidate spectra, and then show that each candidate spectrum is attained by an appropriate stochastic matrix.

Fix an integer $p \geq 3$, and let $r = \frac{1}{3p}$. Set $q = 9p^3 + 2p, l = 18p^3 + 9p^2 + p$ and $m = 9p^2 + 3p$. Letting $n = q + l + m + 1$, it follows that $(n-1) r^3 - 2r^2 - 2r - 1 = 0.$

We would like to show that there is a matrix $T \in S(4, n)$ whose eigenvalues are: 1 (with multiplicity 1), $-r$ (with multiplicity $q$), $re^{\pm \pi i/3}$ (each with multiplicity $l/2$) and $re^{\pm 2\pi i/3}$ (each with multiplicity $m/2$).

For each $j \in \mathbb{N}$, let

$$s_j = 1 + q(-r)^j + (l/2)(re^{\pi i/3})^j + (l/2)(re^{-\pi i/3})^j + (m/2)(re^{2\pi i/3})^j + (m/2)(re^{-2\pi i/3})^j.$$  

(Observe that if we could find the desired matrix $T$, then $s_j$ would just be the trace of $T^j$.) We find readily that $s_1 = s_2 = s_3 = 0$, while $s_4 = 1 - r^2, s_5 = 1 - r^4$, and $s_6 = 1 + r^3 + 2r^2 + 2r^3$. Finally, note that for any $j \in \mathbb{N}, s_{j+6} = 1 - r^6(s_j - 1)$.

Write the polynomial

$$(\lambda - 1)(\lambda + r)^3(\lambda - re^{\pi i/3})^3(\lambda - re^{-\pi i/3})^3(\lambda - re^{2\pi i/3})^3(\lambda - re^{-2\pi i/3})^3.$$  

as $\lambda^n + \sum_{j=0}^{n-1} a_j \lambda^j$. Let $C_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$ be the asso-
cated companion matrix, let \( M_n = \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ s_1 & n-1 & 0 & \cdots & 0 \\ s_2 & s_1 & n-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & \cdots & s_1 & 1 \end{bmatrix} \), and let

\[ A_n = \begin{bmatrix} s_1 & n-1 & 0 & \cdots & 0 \\ s_2 & s_1 & n-2 & \cdots & 0 \\ s_3 & s_2 & s_1 & n-3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_n & s_{n-1} & \cdots & s_2 & s_1 \end{bmatrix}. \]

Following an idea from [7], we note that from the Newton identities, it follows that \( C_n M_n = A_n \), so that \( M_n^{-1} C_n M_n = M_n^{-1} A_n \). In particular, \( C_n \) is similar to \( M_n^{-1} A_n \). Much of our goal in this section is to show that \( M_n^{-1} A_n \) is an irreducible nonnegative matrix. Since any irreducible nonnegative matrix with Perron value 1 is diagonally similar to a stochastic matrix, we will then conclude that there is a matrix \( T \in S(4, n) \) such that \( |\lambda_2(T)| = r \).

Throughout the remainder of this section, we take the parameters \( p, n, r \) and the sequence \( \{s_j\} \) to be as defined above. In particular, we will rely on the facts that \( p \geq 3, r \leq 1/9 \) and \((n-1)r^3 - 2r^2 - 2r - 1 = 0\).

We begin with some technical results. In what follows, we use \( 0_k \) denote the \( k \)-vector of zeros.

**Lemma 3.1.** Suppose that \( k \in \mathbb{N} \) with \( 7 \leq k \leq n \). Then

\[
M_k 1 = (k - 3 - r^2) 1 + (3 + r^2) e_1 + (2 + r^2) e_2 + (1 + r^2) e_3 + r^2 e_4 + r^3 \begin{bmatrix} 0_6 \\ v \end{bmatrix},
\]

where \( ||v||_\infty = 1 + r + 2r^2 \).

**Proof.** Evidently the first four entries of \( M_k 1 \) are \( k, k-1, k-2 \) and \( k-3 \), respectively. For \( j \geq 5 \), the \( j \)-th entry of \( M_k 1 \) is \( k-3+t_j \), where \( t_j = \sum_{i=1}^{j} (s_i-1) \). We have \( t_5 = -r^2, t_6 = -r^2 - r^4, t_7 = -r^2 - r^4 - 2r^5, t_8 = -r^2 - r^4 - 2r^5 - 2r^6, \) and \( t_9 = -r^2 - r^4 - 2r^5 - 3r^6 \). In particular, for \( 4 \leq j \leq 9 \), note that \( -r \leq t_j r^{-2} \leq 1 + r + 2r^2 \), with equality holding in the upper bound for \( j = 6 \). Also, for each \( 4 \leq j \leq 9 \) and \( i \in \mathbb{N} \), we have \( t_{j+i} = t_j \frac{r^{-6}}{1-r^6} + t_j r^{6i} \). We find that for such \( i \) and \( j \), \( 0 \leq \frac{t_{j+i} r^{-2}}{1-r^6} \leq \frac{1}{3}(t_9(1-r^6) + r^2 + r^6 t_6) \), and \( t_j = \frac{1}{3}(t_9(1-r^6) + r^2 + r^6 t_6) \). An interesting computation shows that the rightmost member is equal to \( 1 + r + 2r^2 + \frac{1}{1-r^6}(-3r^3 - 2r^5 + 2r^6 + 2r^7 + 4r^9 + r^{11} - r^{12} - r^{13} - 2r^{14}) \). Since \( r \leq 1/9 \), it follows that this last quantity is strictly less than \( 1 + r + 2r^2 \). Consequently, for any \( j \geq 4 \), we have \( \frac{t_{j+i} r^{-2}}{1-r^6} \leq 1 + r + 2r^2 \), with equality holding for \( j = 6 \). The result now follows. \( \Box \)

**Proposition 3.2.** For each \( 1 \leq k \leq n \), we have

a) the offdiagonal entries of \( M_k^{-1} \) are nonpositive, so that \( M_k^{-1} \) is an M-matrix,
b) \( M_k^{-1} \mathbf{1} \geq \frac{1}{k+1} \mathbf{1} \), and

c) \( M_k^{-1} \begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix} \) is a positive vector.

\[ \text{Proof:} \] We proceed by extended induction on \( k \) using a single induction proof for all three statements. Note that each of a), b) and c) is easily established for \( k = 1, \ldots, 6 \). Suppose now that a), b) and c) hold for natural numbers up to and including \( k - 1 \geq 6 \).

First, we consider statement a). We have

\[ M_k^{-1} = \frac{1}{k-3-r^2} \begin{bmatrix} 1 - (3+r^2)M_k^{-1}e_1 - (2+r^2)M_k^{-1}e_2 - (1+r^2)M_k^{-1}e_3 - r^2M_k^{-1}e_4 + r^3M_k^{-1} \left[ \frac{0}{v} \right] \end{bmatrix} \]

for some vector \( v \) with \( ||v||_\infty = 1 + r + 2r^2 \). The first four entries of \( M_k^{-1} \mathbf{1} \) are \( 1/k, 1/(k-1), 1/(k-2) \) and \( 1/(k-3) \), respectively, so it remains only to show that \( M_k^{-1} \mathbf{1} \geq \frac{1}{k+1} \mathbf{1} \) in positions after the fourth.

Let \( \text{trunc}_4(M_k^{-1} \mathbf{1}) \) denote the vector formed from \( M_k^{-1} \mathbf{1} \) by deleting its first four entries. Noting that the entries of \( M_k^{-1}e_1, M_k^{-1}e_2, M_k^{-1}e_3, \) and \( M_k^{-1}e_4 \) are nonpositive after the fourth position, it follows that \( \text{trunc}_4(M_k^{-1} \mathbf{1}) \geq \frac{1}{k-3-r^2} \mathbf{1} + \frac{r^3}{k-3-r^2} \left[ \frac{0}{M_k^{-1}v} \right] \).

From part b) of the induction hypothesis, \( M_{k-6}^{-1} \mathbf{1} \) is a positive vector, and from part a) of the induction hypothesis, \( M_{k-6}^{-1} \mathbf{1} \) is an M-matrix. Note that \( M_{k-6}^{-1} \) has diagonal entries \( 1/(k-6), 1/(k-7), \ldots, 1/2, 1 \). Letting \( u_i \) be the \( i \)-th row sum of \( M_{k-6}^{-1} \), it follows that \( ||e_i^T M_{k-6}^{-1} \mathbf{1}||_1 = 1/(k-5+i) + (1/(k-5+i) - u_i) \leq 2/(k-5+i) \leq 2 \). Letting \( ||\bullet||_\infty \) denote the absolute row sum norm (induced by the infinity norm for vectors), we conclude that \( ||M_{k-6}^{-1} \mathbf{1}||_\infty \leq 2. \) Hence \( M_{k-6}^{-1}v \geq -2 ||v||_\infty \mathbf{1} = -2(1+r+2r^2) \mathbf{1} \). As
a result, we have $\frac{1}{k-3-r^2}1 + \frac{r^3}{k-3-r^2} \left[ \frac{0_2}{M_{k-6} v} \right] \geq \frac{1}{k-3-r^2}1 - 2(1 + r + 2r^2) \frac{r^3}{k-3-r^2}1 = \frac{1-2r^3(1+r+2r^2)}{k-3-r^2}1$

Since $(k-1)r^3 \leq 2r^2 + 2r + 1$, we have

$$\frac{1-2r^3(1+r+2r^2)}{k-3-r^2} \geq \frac{1-2(1+r+2r^2)(1+2r+2r^2)/(k-1)}{k-3-r^2} \geq \frac{k-3.8325}{(k-1)(k-3)},$$

the last inequality following from the fact that $r \leq 1/9$. Since $k \geq 7$, we find readily that $\frac{k-3.8325}{(k-1)(k-3)} \geq 1$. Putting the inequalities together, we have $M_k^{-1} 1 \geq \frac{1}{k-1} 1$, which completes the proof of the induction step for statement b).

Finally, we consider statement c). We have

$$\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix} = 1 + \begin{bmatrix} 0_6 \\ s_{10} - 1 \\ \vdots \\ s_{k+3} - 1 \end{bmatrix}.$$

Recall that for $4 \leq j \leq 9$ and $i \in \mathbb{N}$,

$$s_{j+6i} - 1 = r^{6i}(s_j - 1),$$

so that $\frac{|s_{j+6i} - 1|}{r^6} \leq \frac{|s_j - 1|}{r^6} \leq 1$. Hence

$$\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix} = 1 - r^2 e_1 - r^4 e_2 + r^3(1 + 2r + 2r^2) e_3 - r^6(e_4 + e_5 + e_6) + r^8 \begin{bmatrix} 0_6 \\ v \end{bmatrix},$$

where $\|v\|_\infty \leq 1$. Thus we have

$$M_k^{-1} \begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix} = M_k^{-1} 1 - M_k^{-1}(r^2 e_1 + r^4 e_2 + r^6(e_4 + e_5 + e_6)) +$$

$$r^3(1 + 2r + 2r^2)M_k^{-1} e_3 + r^8 \begin{bmatrix} 0_6 \\ M_k^{-1} v \end{bmatrix}.$$
Thus, it is sufficient to show that $M_k^{-1}c_3 = \begin{pmatrix} \cdots \end{pmatrix}$ below the sixth position are all nonpositive, that $M_k^{-1}c_3 = \begin{pmatrix} 0_6 & \cdots \end{pmatrix}$, and that the infinity norm of $\begin{pmatrix} \cdots \end{pmatrix}$ is bounded above by $s_6 = 1 + r^4(1 + 2r + 2r^2).$

Let $\text{trunc}_6 \begin{pmatrix} M_k^{-1} & \cdots \end{pmatrix}$ denote the vector formed from $M_k^{-1}$ by deleting its first six entries, and define $\text{trunc}_6(M_k^{-1}1)$ similarly. From the considerations above, we find that

$$\text{trunc}_6 \begin{pmatrix} M_k^{-1} & \cdots \end{pmatrix} \geq \text{trunc}_6(M_k^{-1}1) - \frac{r^3(1 + 2r + 2r^2)}{k - 2} M_k^{-1} \begin{pmatrix} \cdots \end{pmatrix} + r^8 M_k^{-1}v.$$ 

As above, since $M_k^{-1}$ is an M-matrix, we find that $||M_k^{-1}||_\infty \leq 2.$ Applying b), and using the bound on the norm of $M_k^{-1},$ we have

$$\text{trunc}_6(M_k^{-1}1) - \frac{r^3(1 + 2r + 2r^2)}{k - 2} M_k^{-1} \begin{pmatrix} \cdots \end{pmatrix} + r^8 M_k^{-1}v \geq \frac{1}{k + 1} - \frac{r^3(1 + 2r + 2r^2)(2 + 2r^3 + 4r^4 + 4r^5)}{k - 2} 1 - 2r^8 1.$$ 

Thus, it is sufficient to show that $\frac{1}{k + 1} - \frac{r^3(1 + 2r + 2r^2)(2 + 2r^3 + 4r^4 + 4r^5)}{k - 2} - 2r^8 > 0.$

Since $r^3 \leq \frac{2r^2 - 2r + 1}{k - 2},$ it follows that $\frac{1}{k + 1} - \frac{r^3(1 + 2r + 2r^2)(2 + 2r^3 + 4r^4 + 4r^5)}{k - 2} - 2r^8 \geq \frac{1}{k + 1} - \frac{2(1 + 2r + 2r^2)^2(k - 1)(k - 1)}{(k - 1)^2(k - 2)} - \frac{2r^2(1 + 2r + 2r^2)^2}{(k - 1)^2(k - 2)}.\text{ Now using the fact that } r \leq 1/9,\text{ it eventually follows that } \frac{1}{k + 1} - \frac{2(1 + 2r + 2r^2)^2(k - 1)(k - 1)}{(k - 1)^2(k - 2)} - \frac{2r^2(1 + 2r + 2r^2)^2}{(k - 1)^2(k - 2)} \geq \frac{k^3 - 6.5k^2 + 1.5k - 6.2}{(k + 1)(k - 2)(k - 1)^2}.\text{ This last is positive, since } k \geq 7.\text{ This completes the proof of the induction step for statement c).}$
The preceding results lead to the following.

**Theorem 3.3.** $M^{-1}_n A_n$ is an irreducible nonnegative matrix.

**Proof.** We claim that for each $4 \leq k \leq n$, $M^{-1}_k A_k$ is irreducible and nonnegative. The statement clearly holds if $k = 4$, and we proceed by induction. Suppose that the claim holds for some $4 \leq k \leq n - 1$. Note that $M^{-1}_{k+1} = \begin{bmatrix} k+1 & 0^T \\ s & M_k \end{bmatrix}$, where $s = \begin{bmatrix} s_1 \\ \vdots \\ s_k \end{bmatrix}$. We also have $A_{k+1} = \begin{bmatrix} 0 & k c e^T \\ \sigma & A_k \end{bmatrix}$, where $\sigma = \begin{bmatrix} s_2 \\ \vdots \\ s_{k+1} \end{bmatrix}$. It then follows that $M^{-1}_{k+1} A_{k+1} = \begin{bmatrix} 0 & \frac{1}{\sigma} k c e^T A_k \\ \frac{1}{\sigma} k c e^T M_k^{-1} & M_k^{-1} A_k - \frac{1}{\sigma} k c e^T \end{bmatrix}$.

From the induction hypothesis, $M^{-1}_k A_k e_j \geq 0$ for each $1 \leq j \leq k$. Note also that $M^{-1}_k A_k e_1 = M^{-1}_k s \geq 0$, so that the first column of $M^{-1}_k A_k - \frac{1}{\sigma} k c e^T$ is just $\frac{1}{\sigma} k c e^T$, which is nonnegative, and has the same zero-nonzero pattern as the first column of $M^{-1}_k A_k$. Thus the $(2,2)$ block of $M^{-1}_{k+1} A_{k+1}$ is nonnegative and irreducible by the induction hypothesis, while the $(1,2)$ block is a nonnegative nonzero vector. Further, from Proposition 3.2 it follows that $M^{-1}_k \sigma$ is also nonnegative and nonzero.

Hence $M^{-1}_{k+1} A_{k+1}$ is both nonnegative and irreducible, completing the induction step.

Here is the main result of this section; it follows from Theorem 3.3.

**Theorem 4.4.** For infinitely many $n$, $\lambda_2(4,n) \leq r$, where $r$ is the positive root of the equation $(n-1)r^3 - 2r^2 - 2r - 1 = 0$.

**Remark 3.1.** Let $f(x) = (n-1)x^3 - 2x^2 - 2x - 1$. A straightforward computation shows that for all sufficiently large $n$, $f((n-1)^{-2} + (n-1)^{1/3}) > 0$. It now follows that for all sufficiently large $n$, the positive root $r$ for the function $f$ satisfies $r < (n-1)^{-2} + (n-1)^{1/3}$.

The following is immediate from Theorem 1.1, Theorem 4.4 and Remark 3.1.

**Corollary 3.5.** $\liminf_{n \to \infty} \lambda_2(4,n) \sqrt{n} - 1 = 1$.

**4. Bounds for large girth.** At least part of the motivation for the study of $\lambda_2(g,n)$ is to develop some insight when $g$ is large relative to $n$. As noted in Remark 1.3, if both $n$ and $g$ are large, then we expect $\lambda_2(g,n)$ to be close to 1, so that any primitive matrix in $S(g,n)$ will give rise to a sequence of powers that converges only very slowly. The purpose of this section is to quantify these notions more precisely.

To that end, we focus on the case that $g > 2n/3$.

The following result is useful. Its proof appears in [4] and (essentially) in [6] as well.

**Lemma 4.1.** Suppose that $g > n/2$ and that $T \in S(g,n)$. Then the characteristic polynomial for $T$ has the form $\lambda^n - \sum_{j=g}^{n} a_j \lambda^{n-j}$, where $a_j \geq 0$, $j = g, \ldots, n$ and $\sum_{j=g}^{n} a_j = 1$.

Our next result appears in [5].

**Lemma 4.2.** Suppose that $g > 2n/3$ and that $T \in S(g,n)$. Then $T$ has an eigenvalue of the form $p e^{i \theta}$, where $\theta \in [2\pi/n, 2\pi/g]$, and where $\rho \geq r(\theta)$, where $r(\theta)$ is the (unique) positive solution to the equation $r^g \sin(n \theta) - r^n \sin(g \theta) = \sin((n-g) \theta)$. 


Remark 4.1. It is shown in [5] that there is a one-to-one correspondence between the family of complex numbers $r(\theta)e^{i\theta}, \theta \in [2\pi/n, 2\pi/g]$, and a family of roots of the polynomial $\lambda^n - \alpha \lambda^{n-g} - (1 - \alpha), \alpha \in [0, 1]$ Specifically, [5] shows that for each $\alpha \in [0, 1]$, there is a $\theta \in [2\pi/n, 2\pi/g]$ such that $r(\theta)e^{i\theta}$ is a root of $\lambda^n - \alpha \lambda^{n-g} - (1 - \alpha)$, and conversely that for each $\theta \in [2\pi/n, 2\pi/g]$, there is an $\alpha \in [0, 1]$ such that $\lambda^n - \alpha \lambda^{n-g} - (1 - \alpha)$ has $r(\theta)e^{i\theta}$ as a root. As $\alpha$ runs from 0 to 1, $\theta$ runs from $2\pi/n$ to $2\pi/g$, while $r(\theta)e^{i\theta}$ interpolates between $e^{2\pi i/n}$ and $e^{2\pi i/g}$.

The following result produces lower bounds on $\lambda_2(g, n)$ for $g > 2n/3$ and for $g \geq 3(n + 3)/4$.

**Theorem 4.3.** a) Suppose that $n \geq 27$ and that $g > 2n/3$. Then $\lambda_2(g, n) \geq (\frac{4}{3})^{1/(3n)}$, where $l(n) = 2\lceil\frac{n}{4}\rceil + 1$ if $n \equiv 0, 1 \pmod{3}$, and $l(n) = 2\lceil\frac{n}{3}\rceil$ if $n \equiv 2 \pmod{3}$. 

b) If $n \geq 3(n + 3)/4$, then $\lambda_2(g, n) \geq (\frac{2\sqrt{2} - 1}{3})^{1/(3n)}$.

**Proof.** a) Let $k = \lceil\frac{n}{4}\rceil$, so that $n = 3k + i$, for some $0 \leq i \leq 2$. Since $g > 2n/3$, it follows that $g \geq 2k + 1$ if $i = 0, 1$, and $g \geq 2k + 2$ if $i = 2$. Let $j_0 = 1, j_1 = 1$ and $j_2 = 2$. From Proposition 1.2 b), we find that $\lambda_2(g, n) \geq \lambda_2(2k + j_i, 3k + i)$. From Lemma 4.2 it follows that for each $T \in S(2k + j_i, 3k + i)$, there is a $\theta \in [2\pi/(3k + i), 2\pi/(2k + j_i)]$ such that $|\lambda_2(T)| \geq r$, where $r$ is the positive solution to the equation $r^{2k+j_i}\sin((3k+i)\theta) - r^{3k+i}\sin((2k+j_i)\theta) = \sin((k+i-j)\theta)$. Evidently for such an $r$ we have $r^{2k+j_i}\sin((3k+i)\theta) - r^{(2k+j_i)\theta} \geq \sin((k+i-j)\theta)$, and it now follows that $\lambda_2(g, n)^{2k+j_i} \geq \min\left\{\frac{\sin((k+i-j)\theta)}{\sin((3k+i)\theta) - \sin((2k+j_i)\theta)} \mid \theta \in [2\pi/(3k+i), 2\pi/(2k+j_i)]\right\}$. In order to establish the desired inequality, it suffices to show that for each $\theta \in [2\pi/(3k+i), 2\pi/(2k+j_i)]$, $\sin((k+i-j)\theta) \geq \sin((3k+i)\theta) - \sin((2k+j_i)\theta)$.

To that end, set $t = (k + i - j)\theta$, so that $t \in [2\pi/(3k+i), 2\pi/(2k+j_i)]$, $\pi - \frac{\pi}{3(n+3)} < t \leq \frac{\pi}{3(n+3)}$. Set $b_i = \frac{3t - 2i}{k+i-j}$; we find that $(3k+i)\theta = 3t + b_i t$ and that $(2k+j_i)\theta = 2t + b_i t$. We claim that for each $t \in [2\pi/3 - 2\pi/(3k+i), \pi - \pi/(2k+j_i)]$, $5\sin(t) \geq \sin(3t + b_i t) - \sin(2t + b_i t)$. Let $\cos(t) = x$, so that $-1 < x < 0$. Our claim is equivalent to proving that

$$5(1 - 4x^2 - 2x - 1) \cos(b_i t) \sqrt{1 - x^2} \geq (x - 1)(4x^2 + 2x - 1) \sin(b_i t).$$

From the hypothesis, it follows that $k \geq 9$, so we find that $\sin(b_i t), \cos(b_i t) \geq 0$. First, we note that if $-1 < x \leq -\frac{1+\sqrt{5}}{2}$, then we have $4x^2 - 2x - 1 > 4x^2 + 2x - 1 > 0$, so that the left side of (4.1) is positive while the right side is nonpositive.

Next, note that if $-\frac{1+\sqrt{5}}{2} < x \leq \frac{1-\sqrt{5}}{2}$, then $4x^2 - 2x - 1 \geq 0 > 4x^2 + 2x - 1$. It then follows that $(5 - 4x^2 - 2x - 1) \cos(b_i t) \sqrt{1 - x^2} \geq (x - 1)(4x^2 + 2x - 1) \sin(b_i t), \quad (x - 1)(4x^2 + 2x - 1) \equiv f(x)$, while $(x - 1)(4x^2 + 2x - 1) \sin(b_i t) \leq (x - 1)(4x^2 + 2x - 1) \equiv g(x)$. For $\frac{1-\sqrt{5}}{4} < x \leq \frac{1-\sqrt{5}}{2}$, we find readily that $f(x)$ is an increasing function of $x$, so that in particular, $f(x) \geq \sqrt{\frac{5-\sqrt{5}}{2}} \left(\frac{3-\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right) \approx 1.0368312\ldots$ on that interval. A straightforward computation also reveals that $g(x)$ is increasing on the interval $[-\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}]$, and is maximized on $[-1, 0]$ at $x = \frac{1-\sqrt{5}}{6}$, with $g\left(\frac{1-\sqrt{5}}{6}\right) = \left(\frac{5-\sqrt{5}}{6}\right) \left(4\left(\frac{1-\sqrt{5}}{6}\right) + \frac{1-\sqrt{5}}{6} - 1\right) \approx 1.63$. Since $\frac{1-\sqrt{5}}{6} > -\frac{7}{8}$, we find from these considerations that for $-\frac{1+\sqrt{5}}{2} < x \leq -\frac{7}{8}$ we have $g(x) \leq g\left(-\frac{7}{8}\right) \approx 0.748 < ...$
1.036. On the other hand, if \(-0.7 < x \leq \frac{1}{4}\sqrt{5}\), then \(f(x) \geq f(-0.7) \approx 1.88 > 1.63\). It now follows that for each \(-\frac{1}{4}\sqrt{5} < x \leq \frac{1}{4}\sqrt{5}\), \(f(x) \geq g(x)\).

Finally, if \(-\frac{1}{4}\sqrt{5} < x < 0\), the left side of (4.1) is easily seen to exceed \(5\sqrt{1 - \left(\frac{1}{4}\sqrt{5}\right)^2}\), which in turn exceeds the maximum value for \(g(x)\) on \([-1, 0]\). We conclude that (4.1) holds, as desired.

b) Let \(k = \left\lfloor \frac{n}{4} \right\rfloor\), so that \(n = 4k + i\) for some \(i = 0, 1, 2, 3\). Since \(g \geq 3(n + 3)/4\), then we have \(g \geq 3k + (9 + 3i)/4\). If \(i = 0\), then \(g \geq 3k\), while if \(i = 1, 2, 3\), then \(g \geq 3k + 3\). Consequently, we have \(\lambda_2(g, n) \geq \lambda_3(3k, 4k)\) if \(i = 0\), and \(\lambda_2(g, n) \geq \lambda_2(3(k + 1), 4(k + 1))\) if \(i = 1, 2, 3\), or equivalently, \(\lambda_2(g, n) \geq \lambda_2(\left\lfloor \frac{n}{4} \right\rfloor, 4\left\lfloor \frac{n}{4} \right\rfloor)\).

Set \(j = \left\lfloor \frac{n}{4} \right\rfloor\). From Lemma 4.2, we find that \(\lambda_3(3j, 4j) \geq \min\{\sin((3j)\theta) - \sin((3\theta)\theta)\} \in \{2\pi/(4j), 2\pi/(3j)\}\). We claim that \(\min\{\frac{\sin((3j)\theta)}{\sin((3\theta)\theta)}\} \in \{2\pi/(4j), 2\pi/(3j)\}\) from which the result will follow.

To see the claim, let \(x = \cos(j\theta)\) and note that \(x \in [-1/2, 0]\). Further, we have \(\sin((3j)\theta) - \sin((3\theta)\theta) = \sin((3\theta)\theta) (8x^3 - 4x^2 - 4x + 1)\). Consequently, \(\min\{\frac{\sin((3j)\theta)}{\sin((3\theta)\theta)}\} \in \{2\pi/(4j), 2\pi/(3j)\}\) is \(\leq \frac{\pi}{4j}\) on \([-1/2, 0]\). The claim now follows from a standard calculus computation.

**Remark 4.2.** Note that \(\frac{4\pi}{3} \approx 0.6130718\ldots\).

**Remark 4.3.** We note that Theorem 4.3 provides an estimate on \(r\theta/\theta\) for the case that \(g > 2n/3\); that estimate is a clear improvement on that of [6], which proves a lower bound of \((\frac{1}{n} \sin[\pi/(n - 1)])^{2/(n-1)}\) on that quantity.

Our final result considers the case that \(n \to \infty\), while \(n - g\) is fixed. In the proof, we use the notation \(O(\frac{1}{n^4})\) to denote a sequence \(s_n\) with the property that \(n^b s_n\) is a bounded sequence.

**Theorem 4.4.** Suppose that \(i \geq 1\) is fixed. Then \(\lambda_2(n - i, n) \geq 1 - \frac{i^2}{2n^2} + O\left(\frac{1}{n^4}\right)\).

**Proof.** From Lemma 4.2, we find that for \(n > 3i\) we have

\[
\lambda_2(n - i, n) \geq \left(\frac{\sin((i\theta))}{\sin((n\theta) - \sin((n-i)\theta))} \in [2\pi/n, 2\pi/(n-i)]\right) \frac{1}{n}.
\]

Let \(\theta_0\) be a critical point of the function \(\frac{\sin((i\theta))}{\sin((n\theta) - \sin((n-i)\theta))}\) on the interval \([2\pi/n, 2\pi/(n-i)]\). Then we have

\[
sin(i\theta_0) (n \cos(n\theta_0) - (n-i) \cos((n-i)\theta_0)) = i \cos(i\theta_0) (\sin(n\theta_0) - \sin((n-i)\theta_0)).
\]

Let \(\theta_0 = \frac{an}{n} + \frac{\pi}{n}\) where \(a = O(1)\). We then have \(n\theta_0 = 2\pi + \frac{an}{n}\). Expanding the equation above for \(\theta_0\) to terms in \(O\left(\frac{1}{n^4}\right)\), we have

\[
i \left(1 - \frac{2\pi a}{2n}\right) \left(\frac{\sin(an)}{n} + \frac{(2i-a)\pi}{2n}\right) = i \left(1 - \frac{2\pi a}{2n}\right) + O\left(\frac{1}{n^4}\right).
\]

Collecting terms and simplifying eventually yields \(\frac{(2i-a)^2-a^2}{2n^2} = O\left(\frac{1}{n^4}\right)\) from which we conclude that \(a = i + O\left(\frac{1}{n}\right)\).

Next, we write \(\theta_0 = \frac{2\pi}{n} + \frac{\pi}{n} + \frac{b}{n}\), where \(b = O(1)\). As above, we find that \(n\theta_0 = 2\pi + \frac{\pi}{n} + \frac{b}{n}\), \((n-i)\theta_0 = 2\pi - \left(\frac{\pi}{n} + \frac{(i-b)\pi}{n} + \frac{b}{n}\right)\) and \(i\theta_0 = \frac{2\pi}{n} + \frac{\pi}{n} + \frac{b}{n}\).
From this it follows that

\[ \frac{\sin(i\theta_0)}{\sin(n\theta_0) - \sin((n - i)\theta_0)} = \frac{2\sin(\frac{\pi}{n}) + \frac{\pi i}{n^2} + \frac{\pi i}{n^2} - \frac{4\pi^3 i^3}{3n^3} + O(\frac{1}{n^4})}{2\sin(\frac{\pi}{n}) + \frac{\pi i}{n^2} + \frac{\pi i}{n^2} - \frac{4\pi^3 i^3}{3n^3} + O(\frac{1}{n^4})} = 1 - \frac{\pi^2 i^2}{2n^2} + O\left(\frac{1}{n^3}\right). \]

(4.2)

Thus we have \( \lambda_2(n - i, n) \geq \left(1 - \frac{\pi^2 i^2}{2n^2} + O\left(\frac{1}{n^3}\right)\right)^{\frac{1}{i\theta}} = 1 - \frac{\pi^2 i^2}{2n^2} + O\left(\frac{1}{n^3}\right) \), as desired. □

**Remark 4.4.** Suppose that we have a matrix \( T \in \mathcal{S}(n - 1, n) \). Then the characteristic polynomial of \( T \) is given by \( p_\alpha(\lambda) \equiv \lambda^n - \alpha \lambda - (1 - \alpha) \), for some \( \alpha \in [0, 1] \). Conversely, for each \( \alpha \in [0, 1] \), there is a matrix \( T \in \mathcal{S}(n - 1, n) \) whose characteristic polynomial is \( p_\alpha \), namely the companion matrix of that polynomial. Thus we see that the eigenvalues of matrices in \( \mathcal{S}(n - 1, n) \) are in one-to-one correspondence with the roots of polynomials of the form \( p_\alpha, \alpha \in [0, 1] \). For such a polynomial, we say that a root \( \lambda \) is a **subdominant root** if \( \lambda \neq 1 \) and \( \lambda \) has maximum modulus among the roots of the polynomial that are distinct from 1. In particular, we find that discussing the subdominant roots of the polynomials \( p_\alpha, \alpha \in [0, 1] \) is equivalent to discussing the subdominant eigenvalues of the matrices in \( \mathcal{S}(n - 1, n) \).

Fix a value of \( n \geq 4 \). It follows from Corollary 2.1 of [5] that for each \( \alpha \in [0, 1] \), there is precisely one root of \( p_\alpha \) whose argument lies in \([2\pi/n, 2\pi/(n - 1)] \) (including multiplicities). Denote that root by \( \sigma(\alpha) \). Evidently an analogous statement holds for the interval \([2\pi - 2\pi/(n - 1), 2\pi - 2\pi/n]\), and we claim that in fact \( \sigma(\alpha) \) and \( \sigma(\alpha) \) are subdominant roots for \( p_\alpha \).

To see the claim, first suppose that \( \alpha \in (0, 1) \), and that \( z_1 \) and \( z_2 \) are two roots of \( p_\alpha \) of equal moduli. Writing \( z_1 = A e^{i\theta_1}, z_2 = A e^{i\theta_2} \), and substituting each into the equation \( p_\alpha(\lambda) = 0 \), we find that \( A^2 \rho^2 + |\alpha e^{i\theta_1} + 1 - \alpha|^2 = |\alpha e^{i\theta_2} + 1 - \alpha|^2 \). It follows that \( \alpha^2 \rho^2 + (1 - \alpha)^2 + 2(1 - \alpha)\rho \cos(\theta_1) = \alpha^2 \rho^2 + (1 - \alpha)^2 + 2(1 - \alpha)\rho \cos(\theta_2) \), from which we conclude that \( \cos(\theta_1) = \cos(\theta_2) \). Consequently, we find that for each \( \alpha \in (0, 1) \), if \( z_1 \) and \( z_2 \) are roots of \( p_\alpha \) that have equal moduli, then either \( z_1 = z_2 \) or \( z_1 = -\overline{z_2} \).

For each \( \alpha \in [0, 1] \), denote the roots of \( p_\alpha \) that are distinct from 1 and whose argument fall outside of \([2\pi/n, 2\pi/(n - 1)] \cup [2\pi - 2\pi/(n - 1), 2\pi - 2\pi/n]\) by \( \gamma_1(\alpha), \ldots, \gamma_{n-3}(\alpha) \), labeled in nondecreasing order according to their arguments. Suppose that \( \exists \alpha_1, \alpha_2 \in (0, 1) \) such that \( |\sigma(\alpha_1)| > \max\{|\gamma_i(\alpha_1)| | i = 1, \ldots, n - 3\} \) and \( |\sigma(\alpha_2)| < \min\{|\gamma_i(\alpha_2)| | i = 1, \ldots, n - 3\} \). From the continuity of the roots of \( p_\alpha \) in the parameter \( \alpha \), and the intermediate value theorem, we find that \( \exists \alpha_3 \in (0, 1) \) such that \( |\sigma(\alpha_3)| = \max\{|\gamma_i(\alpha_3)| | i = 1, \ldots, n - 3\} \). Hence for some i we have either \( \gamma_i(\alpha_3) = \sigma(\alpha_3) \) or \( \gamma_i(\alpha_3) = -\sigma(\alpha_3) \), a contradiction since the argument of \( \gamma_i \) falls outside of \([2\pi/n, 2\pi/(n - 1)] \cup [2\pi - 2\pi/(n - 1), 2\pi - 2\pi/n]\). Consequently, we find that one of the following alternatives must hold: either \( |\sigma(\alpha)| > \max\{|\gamma_i(\alpha)| | i = 1, \ldots, n - 3\} \) for all \( \alpha \in (0, 1) \), or \( |\sigma(\alpha)| < \min\{|\gamma_i(\alpha)| | i = 1, \ldots, n - 3\} \) for all \( \alpha \in (0, 1) \).

Next, we claim that for all sufficiently small \( \alpha > 0, \sigma(\alpha) \) is a subdominant eigenvalue of \( p_\alpha \). To see this, observe that at \( \alpha = 0 \), the roots of \( p_\alpha \) that are distinct from 1 are given by \( e^{2\pi i j/n}, 1 \leq j \leq n - 1 \). Note that since these roots are distinct, there is a neighbourhood of \( \alpha = 0 \) on which each root of \( p_\alpha \) is a differentiable function of \( \alpha \).

Fix an index \( l \) such that either \( 1 \leq l < (n - 2)/2 \) or \( (n - 2)/2 < l \leq n - 3 \) and
consider $\gamma_l(\alpha)$. We write $\gamma_l(\alpha) = \rho e^{i\theta}$, where on the right hand side, the explicit dependence on $\alpha$ is suppressed. Considering the real and imaginary parts of the equation $p_\alpha(\rho e^{i\theta}) = 0$, we find that for each $0 < \alpha \leq 1$ we have

\begin{equation}
\rho^n \cos(n\theta) - 1 = \alpha(\rho \cos(\theta) - 1) \quad (4.3)
\end{equation}

and

\begin{equation}
\rho^{n-1} \sin(n\theta) = \alpha \sin(\theta). \quad (4.4)
\end{equation}

In particular, crossmultiplying (4.3) and (4.4), canceling the common factor of $\alpha$, and simplifying, we find that for each $0 < \alpha \leq 1$, we have

\begin{equation}
\rho^{n-1} \sin(n\theta) - \rho^n \sin((n-1)\theta) = \sin(\theta). \quad (4.5)
\end{equation}

(Observe that in fact (4.5) also holds when $\alpha = 0$, since then $\rho = 1$ and $\theta = \frac{2\pi(n+1)}{n}$.)

Differentiating (4.4) with respect to $\alpha$ and evaluating at $\alpha = 0$, it follows that $\frac{d\rho}{d\alpha}|_{\alpha=0} = \frac{\sin(2\pi(l+1)/n)}{n}$. Differentiating (4.5) with respect to $\alpha$ (via the chain rule) and evaluating at $\alpha = 0$ then yields $\frac{d\sigma}{d\alpha}|_{\alpha=0} = -\frac{1-\cos(2\pi l/n)}{n}$. Similar arguments show that if $l = (n-2)/2$, then $\frac{d\rho}{d\sigma}|_{\alpha=0} = \frac{-2}{n}$, and that $\frac{d\sigma}{d\alpha}|_{\alpha=0} = \frac{1-\cos(2\pi/n)}{n}$.

We conclude that for all sufficiently small $\alpha > 0, |\sigma(\alpha)| = 1 - \alpha \left(\frac{1-\cos(2\pi/n)}{n}\right) + O(\alpha^2) > 1 - \alpha \left(\frac{1-\cos(2\pi(l+1)/n)}{n}\right) + O(\alpha^2) = |\gamma_l(\alpha)|, l = 1 \ldots, n - 3$. Hence, for such $\alpha$, $\sigma$ (and $\overline{\sigma}$) are subdominant roots of $p_\alpha$. From the considerations above, we conclude that for each $\alpha \in [0,1], |\sigma(\alpha)|$ is a subdominant root of $p_\alpha$, as claimed.

From the claim, it now follows that $\lambda_2(n-1,n) = \min\{||\sigma(\alpha)|| | \alpha \in [0,1]\} = \min\{|r(\theta)|r(\theta)^{n-1} \sin(n\theta) - r(\theta)^n \sin((n-1)\theta) = \sin(\theta), r(\theta) > 0, \theta \in [\pi/n, 2\pi/(n-1)]\}$. Arguing as in Theorem 4.4, there is a $\theta_0 \in [2\pi/n, 2\pi/(n-1)]$ such that $\frac{\sin(\theta_0)}{\sin((n-1)\theta_0)} = 1 - \frac{s^2}{2n} + O\left(\frac{1}{n}\right)$, which yields $\left(1 - \frac{s^2}{2n} + O\left(\frac{1}{n}\right)\right)^{1/n} \geq r(\theta_0) \geq \lambda_2(n-1,n)$. Applying Theorem 4.4, we find that $\left(1 - \frac{s^2}{2n} + O\left(\frac{1}{n}\right)\right)^{1/n} \geq \lambda_2(n-1,n) \geq \left(1 - \frac{s^2}{2n} + O\left(\frac{1}{n}\right)\right)^{(1/n)^{-1}}$. But since both the upper and lower bounds on $\lambda_2(n-1,n)$ can be written as $1 - \frac{s^2}{2n} + O\left(\frac{1}{n}\right)$, we conclude that $\lambda_2(n-1,n) = 1 - \frac{s^2}{2n} + O\left(\frac{1}{n}\right)$.

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Girth and Subdominant Eigenvalues


