

2005

## Algebraic connectivity of trees with a pendant edge of infinite weight

Abraham Berman

berman@techunix.technion.ac.il

Karl Heinz Foerster

Follow this and additional works at: <http://repository.uwyo.edu/ela>

---

### Recommended Citation

Berman, Abraham and Foerster, Karl Heinz. (2005), "Algebraic connectivity of trees with a pendant edge of infinite weight", *Electronic Journal of Linear Algebra*, Volume 13.

DOI: <https://doi.org/10.13001/1081-3810.1160>

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact [scholcom@uwyo.edu](mailto:scholcom@uwyo.edu).

## ALGEBRAIC CONNECTIVITY OF TREES WITH A PENDANT EDGE OF INFINITE WEIGHT\*

A. BERMAN<sup>†</sup> AND K.-H. FÖRSTER<sup>‡</sup>

**Abstract.** Let  $G$  be a weighted graph. Let  $v$  be a vertex of  $G$  and let  $G_\omega^v$  denote the graph obtained by adding a vertex  $u$  and an edge  $\{v, u\}$  with weight  $\omega$  to  $G$ . Then the algebraic connectivity  $\mu(G_\omega^v)$  of  $G_\omega^v$  is a nondecreasing function of  $\omega$  and is bounded by the algebraic connectivity  $\mu(G)$  of  $G$ . The question of when  $\lim_{\omega \rightarrow \infty} \mu(G_\omega^v)$  is equal to  $\mu(G)$  is considered and answered in the case that  $G$  is a tree.

**Key words.** Weighted graphs, Trees, Laplacian matrix, Algebraic connectivity, Pendant edge.

**AMS subject classifications.** 5C50, 5C40, 15A18.

**1. Introduction.** A *weighted graph on  $n$  vertices* is an undirected simple graph  $G$  on  $n$  vertices such that with each edge  $e$  of  $G$ , there is an associated positive number  $\omega(e)$  which is called the *weight* of  $e$ .

The *Laplacian matrix* of a weighted graph  $G$  on  $n$  vertices is the  $n \times n$  matrix  $L(G) = L = (l_{ij})$ , where for each  $i, j = 1, \dots, n$ ,

$$l_{ij} = \begin{cases} -\omega(e) & \text{if } i \neq j \text{ and } e = \{i, j\}, \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ \sum_{k \neq i} l_{ik} & \text{if } i = j. \end{cases}$$

Clearly  $L$  is a singular  $M$ -matrix and positive semidefinite, so  $\lambda_1(L) = 0$ , where for a symmetric matrix  $A$  we arrange the eigenvalues in nondecreasing order

$$\lambda_1(A) \leq \lambda_2(A) \leq \dots$$

Fiedler [3] showed that  $\lambda_2(L)$  is positive iff  $G$  is connected and called it the *algebraic connectivity* of  $G$ . The algebraic connectivity of  $G$  will be denoted by  $\mu(G)$ .

In this paper  $G$  always denotes a connected weighted graph without loops.

Let  $G$  be a graph with  $n$  vertices. Let  $v$  be a vertex of  $G$  and let  $G_\omega^v$  be the graph with  $n + 1$  vertices obtained by adding to  $G$  a vertex  $u$  and an edge  $e = \{v, u\}$  with weight  $\omega$ .

---

\*Received by the editors 21 July 2003. Accepted for publication 24 June 2005. Handling Editor: Stephen J. Kirkland.

<sup>†</sup>Department of Mathematics, Technion, Haifa 32000, Israel, (berman@technion.ac.il). Research supported by the New York Metropolitan Fund for Research at the Technion. Most of the work was done during a visit to TU Berlin; part of the work was done in FU Berlin, Charite - Benjamin Franklin.

<sup>‡</sup>Department of Mathematics, Technical University Berlin, Sekr. MA 6-4, Strasse des 17. Juni 136, 10623 Berlin, Germany (foerster@math.tu-berlin.de).

THEOREM 1.1. *The algebraic connectivity  $\mu(G_\omega^v)$  is a nondecreasing function of  $\omega$  and for every  $\omega$  and  $n > 1$*

$$\mu(G_\omega^v) \leq \mu(G).$$

*Proof.* Let  $L_\omega$  be the Laplacian matrix of  $G_\omega^v$  and let  $0 < \omega_1 \leq \omega_2$ . Then  $B = L_{\omega_2} - L_{\omega_1}$  is a singular rank one positive semidefinite matrix. By [7, Th. 4.3.1]

$$\lambda_k(L_{\omega_1}) \leq \lambda_k(L_{\omega_2}) \quad \text{for } k = 1 \dots n,$$

and for  $k = 2$ ,  $\mu(G_{\omega_1}^v) \leq \mu(G_{\omega_2}^v)$ .

To show that  $\mu(G_\omega^v)$  is bounded, write  $L_\omega$  as the sum of two block matrices

$$L_\omega = \begin{bmatrix} L(G) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \omega & -\omega \\ & -\omega & \omega \end{bmatrix},$$

where  $L(G)$  is  $n \times n$  and the left upper zero block in the second matrix is  $(n-1) \times (n-1)$ . By [7, Th. 4.3.4 (a), the case  $k = 2$ ],

$$\begin{aligned} \mu(G_\omega^v) &= \lambda_2(L_\omega) \leq \lambda_3(L(G) \oplus (0)) \\ &= \lambda_2(L(G)) = \mu(G). \quad \square \end{aligned}$$

REMARK 1.2. The theorem is essentially a consequence of Cor. 4.2 of [6]. It is proved for trees in [8].

EXAMPLE 1.3. For the complete graphs  $K_n, n > 1$ , with all weights equal to 1

$$\lim_{\omega \rightarrow \infty} \mu((K_n)_\omega^v) = \frac{n+1}{2} < n = \mu(K_n).$$

EXAMPLE 1.4. For the cycles  $C_n, n > 2$ , with weights equal to 1

$$\lim_{\omega \rightarrow \infty} \mu((C_n)_\omega^v) = \mu(C_{n+1}) < \mu(C_n).$$

EXAMPLE 1.5. Let  $G$  be the graph obtained from  $K_4$  by deleting an edge and let all the weights of  $G$  be equal to 1. If the degree of  $v$  is 3,

$$\lim_{\omega \rightarrow \infty} \mu(G_\omega^v) = 2 = \mu(G).$$

If the degree of  $v$  is 2

$$\lim_{\omega \rightarrow \infty} \mu(G_\omega^v) < \mu(G).$$

Since  $\mu(G_\omega^v)$  is bounded by  $\mu(G)$ , it is natural to ask when does

$$\lim_{\omega \rightarrow \infty} \mu(G_\omega^v) = \mu(G).$$

We answer this question in Section 3, in the case that  $G$  is a tree. The needed background on the algebraic connectivity of trees is described in Section 2.

**2. Results on trees.** Our paper relies heavily on the work of [12] so in this section we describe their main results and basic background on trees needed for these results and for the next section. In some cases we change the notation of [12].

**THEOREM 2.1.** [4, Th. 3.11] *Let  $T$  be a weighted tree with Laplacian matrix  $L$  and algebraic connectivity  $\mu$ . Let  $y$  be an eigenvector of  $L$  associated with  $\mu$ . Then exactly one of the following two cases occur:*

- (a) *Some entry of  $y$  is 0.*
- (b) *All entries of  $y$  are nonzero.*

*In the first case there exists a unique vertex  $c$  such that  $y_c = 0$  and  $c$  is adjacent to a vertex  $d$  with  $y_d \neq 0$ . In the second case there is a unique pair of vertices  $i$  and  $j$  adjacent in  $T$  such that  $y_i y_j < 0$ .*

**DEFINITION 2.2.** A weighted tree  $T$  is said to be of type I with a characteristic vertex  $c$  if case (a) of Theorem 2.1 holds, and of type II with characteristic vertices  $i$  and  $j$  in case (b). We use also the notation  $I_c$  in the first case and  $II_{i,j}$  in the second case.

The name characteristic vertices was coined in [11] by R. Merris who showed that if  $\mu$  is not a simple eigenvalue, then all the corresponding eigenvectors yield the same type of tree and the same characteristic vertices.

**DEFINITION 2.3.** Let  $v$  be a vertex of a tree  $T$ . Let  $L_v$  be the matrix obtained by deleting the row and column of the Laplacian matrix of  $T$  that correspond to  $v$ . The matrix  $M_{v,T} := L_v^{-1}$  is called the bottleneck matrix of  $T$  at  $v$ .

In [9] and [10], it is shown that the entry of  $M_{v,T}$  that corresponds to the vertices  $k$  and  $l$  is

$$m_{kl} = \sum \frac{1}{\omega(g)}$$

where the summation is on all edges  $g$  that lie on the intersection of the path between  $k$  and  $v$  and the path between  $l$  and  $v$ . The matrix  $M_{v,T}$  is permutationally similar to a block diagonal matrix, where the number of blocks is the degree of  $v$  and each block is a positive matrix which corresponds to a unique branch at  $v$ .

For vertices  $u, v$  of a tree  $T$  let  $v \rightarrow u$  denote the branch of  $T$  at  $v$ , that contains  $u$ . We denote by  $M_{v \rightarrow u, T}$  the block of  $M_{v,T}$  that corresponds to  $v \rightarrow u$ , and by  $M_{v \not\rightarrow u, T}$  the matrix obtained from  $M_{v,T}$  by deleting the rows and the columns corresponding to  $M_{v \rightarrow u, T}$ .

**DEFINITION 2.4.** A diagonal block of  $M_{v,T}$  whose spectral radius is equal to  $\rho(M_{v,T})$ , where  $\rho(A)$  denotes the spectral radius of the matrix  $A$ , is called a Perron block and the corresponding branch of  $T$  at  $v$  is called a Perron branch.

**THEOREM 2.5.** [9, Cor. 2.1] *Let  $T$  be a weighted tree. Then  $T$  is of type I with a characteristic vertex  $c$ , if and only if at  $c$ ,  $T$  has more than one Perron branch.*

*In this case,  $\mu(T)$ , the algebraic connectivity of  $T$  is equal to  $\frac{1}{\rho(M_{c,T})}$ .*

Let  $e$  be an edge of a graph  $G$ . Replace the weight at  $e$  by  $\omega$  and denote the resulting graph by  $G_\omega^e$ . Observe that since  $e = \{v, u\}$  is a pendant edge of  $G_\omega^e$ , then  $(G_\omega^e)_\omega^e = G_\omega^v$ . Let  $G_\infty^e$  denote the family of weighted graphs  $\{G_\omega^e, \omega > 0\}$ , and let  $G_\infty^v$  denote the family of weighted graphs  $\{G_\omega^v, \omega > 0\}$ .

THEOREM 2.6. [12, Corollary 1.1] *Let  $T$  be a weighted tree and let  $e$  be an edge of  $T$ . Then there exists a positive number  $\omega_0$  such that all the trees  $T_\omega^e$ ,  $\omega_0 < \omega < \infty$ , are of the same type and have the same characteristic vertices.*

The following definitions are used in [12].

DEFINITION 2.7. The family of trees  $T_\infty^e$  is a type I tree at infinity with characteristic vertex  $c$  if there exists an  $\omega_0 > 0$  such that for all  $\omega \in [\omega_0, \infty)$ ,  $T_\omega^e$  is of type  $I_c$ . Similarly,  $T_\infty^e$  is a type II tree at infinity with characteristic vertices  $i$  and  $j$  if there exists an  $\omega_0 > 0$  such that for all  $\omega \in [\omega_0, \infty)$ ,  $T_\omega^e$  is of type  $II_{i,j}$ .

We now can state the main result of [12].

THEOREM 2.8. [12, Th.1.8] *Let  $e = \{v, u\}$  be an edge that is not a pendant edge of a tree  $T$ . Let  $T_1$  and  $T_2$  be the resulting components arising from the deletion of  $e$ . Suppose  $v \in T_1$ ,  $u \in T_2$  and  $\mu(T_1) \leq \mu(T_2)$ . Then  $\lim_{\omega \rightarrow \infty} \mu(T_\omega^e) = \mu(T_1)$  iff  $T_1$  is a tree of type I with a characteristic vertex, say,  $c$ , and one of the following conditions holds:*

- (a)  $T_\infty^e$  is of type I with a characteristic vertex  $c$ .
- (b)  $c$  is incident to  $e$  and  $\rho(M_{u,T_2}) \leq \frac{1}{\mu(T_1)}$ .

We conclude the background results with the analogue of Theorem 2.5 for type II trees and two propositions that will be used in proving the main result.

THEOREM 2.9. [9, Th.1] *A weighted tree  $T$  is of type II iff at every vertex  $T$  has a unique Perron branch. If the characteristic vertices,  $i$  and  $j$ , of  $T$  are joined by an edge of weight  $\theta$ , then there exists a number  $0 < \gamma < 1$ , such that*

$$\rho(M_{i \rightarrow j, T} - \frac{\gamma}{\theta} J) = \rho(M_{j \rightarrow i, T} - \frac{1-\gamma}{\theta} J), \text{ and}$$

$$\mu(T) = \frac{1}{\rho(M_{i \rightarrow j, T} - \frac{\gamma}{\theta} J)} = \frac{1}{\rho(M_{j \rightarrow i, T} - \frac{1-\gamma}{\theta} J)},$$

where  $J$  denotes an all ones matrix.

PROPOSITION 2.10. [8, Cor. 1.1] *The characteristic vertices of  $T_\omega^v$  lie on the path between the characteristic vertices of  $T$  and  $u$ .*

PROPOSITION 2.11. [12, Claim 3.2] *Let  $T$  be a tree. Let  $\{i_k, j_k\}$  be edges in  $T$  with weights  $\alpha_k$ , for  $k = 1, 2$ , such that the path from  $i_1$  to  $j_2$  contains  $j_1$  and  $i_2$ , and let  $0 < \gamma_1, \gamma_2 < 1$ . Then*

$$\rho(M_{j_1 \rightarrow i_1, T} - \frac{\gamma_1}{\alpha_1} J) < \rho(M_{j_1 \rightarrow i_1, T}) < \rho(M_{j_2 \rightarrow i_2, T} - \frac{\gamma_2}{\alpha_2} J).$$

### 3. Assigning an arbitrarily large weight to a pendant edge of a tree.

In this section we consider the case where  $T$  is a tree and  $u$  is a pendant vertex of  $T \cup e$  where  $e = \{v, u\}$  and  $v \in T$ .

In some sense the question in this case may be considered as a special case of the discussion in [12]. To do this,  $\{u\}$  is to be considered as a "tree with algebraic connectivity  $\infty$ " and the spectral radius of an empty matrix has to be defined (for example as 0).

Our discussion is based on the analysis of the limits of the bottleneck matrices of  $T_\omega^v$  when  $\omega$  increases to  $\infty$ ; namely

$$M_{v,T_\omega^v} = M_{v,T} \oplus \left( \frac{1}{\omega} \right),$$

$$M_{u,T_\omega^v} = (M_{v,T} \oplus (0)) + \frac{1}{\omega} J$$

and if  $s \neq v$  is a vertex of  $T$ ,

$$M_{s,T_\omega^v} = \begin{pmatrix} M_{s,T} & M^{(v)} \\ M^{(v)t} & m_{vv} + \frac{1}{\omega} \end{pmatrix},$$

where  $M^{(v)}$  is the column of  $M_{s,T}$  corresponding to  $v$  and  $m_{vv}$  is the diagonal entry of  $M_{s,T}$ , corresponding to  $v$ . In particular, for all the branches of  $T$  at  $s$  that do not contain  $v$ , the diagonal blocks of  $M_{s,T_\omega^v}$  and of  $M_{s,T}$  are the same. Denoting  $\lim_{\omega \rightarrow \infty} M_{s,T_\omega^v}$  by  $M_{s,T_\infty^v}$  we see that for  $s \notin e$

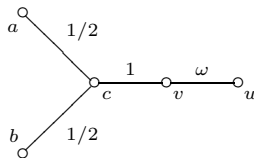
$$M_{s,T_\infty^v} = \begin{pmatrix} M_{s,T} & M^{(v)} \\ M^{(v)t} & m_{vv} \end{pmatrix} \tag{3.1}$$

and

$$M_{v,T_\infty^v} = M_{u,T_\infty^v} = M_{v,T} \oplus (0). \tag{3.2}$$

The reader should not be confused by the fact that  $T_\infty^v$  denotes a family of trees while  $M_{s,T_\infty^v}$  denotes a single matrix (up to permutation similarity).

EXAMPLE 3.1.



$$M_{v,T_\infty^v} = M_{u,T_\infty^v} = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{c,T_\infty^v} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$M_{a,T_\infty^v} = M_{b,T_\infty^v} = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{pmatrix}.$$

REMARK 3.2. The matrices  $M_{s,T_\infty^v}$  are of course singular, but they do contain information on  $\lim_{\omega \rightarrow \infty} \mu(T_\omega^v)$ .

As in the case of nonsingular bottleneck matrices we call the diagonal blocks of  $M_{s,T_\infty^v}$  whose spectral radius is maximal, Perron blocks. The corresponding branches of  $T_\omega^v$  do not depend on  $\omega$ ; they will be called the Perron branches of the family  $T_\infty^v$ .

LEMMA 3.3. *If  $M_{s,T_\infty^v}$  has more than one Perron block, then*

$$\lim_{\omega \rightarrow \infty} \mu(T_\omega^v) = \frac{1}{\rho(M_{s,T_\infty^v})}.$$

*Proof.* Consider the principal submatrix  $L_{s,T_\omega^v}$  obtained from the Laplacian matrix of  $T_\omega^v$  by deleting the row and column corresponding to  $s$ . Then

$$M_{s,T_\infty^v} = \lim_{\omega \rightarrow \infty} (L_{s,T_\omega^v})^{-1}.$$

By [7, Th. 4.3.15] for  $r = n - 1, k = 1$  and  $k = 2$ ,

$$\lambda_1(L_{s,T_\omega^v}) \leq \mu(T_\omega^v) \leq \lambda_2(L_{s,T_\omega^v}),$$

so

$$\lim_{\omega \rightarrow \infty} \lambda_1(L_{s,T_\omega^v}) \leq \lim_{\omega \rightarrow \infty} \mu(T_\omega^v) \leq \lim_{\omega \rightarrow \infty} \lambda_2(L_{s,T_\omega^v}),$$

Since  $M_{s,T_\infty^v}$  has at least two Perron blocks we obtain

$$\lim_{\omega \rightarrow \infty} \lambda_2(L_{s,T_\omega^v}) = \lim_{\omega \rightarrow \infty} \lambda_1(L_{s,T_\omega^v}) = \rho(M_{s,T_\infty^v}). \quad \square$$

REMARK 3.4. If there exists an  $\omega_0$  such that two of the Perron blocks of  $M_{s,T_\infty^v}$  are Perron blocks of  $M_{s,T_\omega^v}$  for  $\omega \geq \omega_0$  then

$$\lambda_2(L_{s,T_\omega^v}) = \lambda_1(L_{s,T_\omega^v}) \text{ for } \omega \geq \omega_0.$$

LEMMA 3.5. *Let  $s$  be a vertex of  $T$ . Suppose  $M_{s,T_\infty^v}$  has at least two Perron blocks and let  $t$  be another vertex of  $T$ . Then*

$$\rho(M_{t,T_\infty^v}) > \rho(M_{s,T_\infty^v})$$

*Proof.* By assumption, the family  $T_\infty^v$  has at least two Perron branches at  $s$ , so one of them, say  $s \rightarrow x$ , does not contain  $t$ . Let  $t \rightarrow s$  be the branch at  $t$  that contains  $s$ . Then it contains the branch  $s \rightarrow x$ , and we obtain

$$\rho(M_{t,T_\infty^v}) \geq \rho(M_{t \rightarrow s, T_\infty^v}) > \rho(M_{s \rightarrow x, T_\infty^v}) = \rho(M_{s, T_\infty^v}),$$

where the strict inequality follows from [1, Cor. 2.1.5] and the fact that  $M_{s \rightarrow x, T_\infty^v}$  is a submatrix of  $M_{t \rightarrow s, T_\infty^v}$ , which is positive.  $\square$

**COROLLARY 3.6.** *There is at most one vertex, say  $c$ , such that  $M_{c, T_\infty^v}$  has more than one Perron block.*

**DEFINITION 3.7.** In the case that there is a vertex  $c$  such that  $M_{c, T_\infty^v}$  has more than one Perron block, we will say that the family of trees  $T_\infty^v$  is a trii (tree in infinity) of type Ic. If no such  $c$  exists we say that  $T_\infty^v$  is a trii of type II.

**REMARK 3.8.**

(a) If the trees  $T_\omega^v$  are of type Ic for all sufficiently large  $\omega$ , then the family  $T_\infty^v$  is a trii of type Ic (and also a type I tree at infinity with characteristic vertex  $c$ ). In other words, if  $T_\infty^v$  is a trii of type II, then for all  $\omega$  large enough,  $T_\omega^v$  are trees of type II.

(b) Suppose the trees  $T_\omega^v$  are of type  $II_{p,q}$  for all sufficiently large  $\omega$ , then by the representation of  $L_\omega$  in the proof of Theorem 1.1 and by Theorem 2.1,  $\{p, q\}$  cannot be the pendant edge  $\{v, u\}$ .

(c) It is possible that  $T_\omega^v$  are of type II for all sufficiently large  $\omega$  (so  $T_\infty^v$  is a type II tree at infinity) but  $T_\omega^v$  is a trii of type I; see Lemma 3.10 and Subcase 4 of Example 3.13 in the following discussion.

**REMARK 3.9.** The proof of Lemma 3.5 shows that if  $T$  is a tree of type I with a characteristic vertex  $c$ , then for any other vertex  $s$  of  $T$

$$\rho(M_{s,T}) > \rho(M_{c,T}).$$

(This has already been established in Proposition 2 of [9].)

Consider Theorem 2.9 where  $T_\omega^v$  is of type  $II_{i,j}$  and the weight of the edge  $\{i, j\}$  is  $\theta$ . Then for every  $\omega$  (sufficiently large) there exist a number  $\gamma_\omega$ , between 0 and 1, such that

$$\mu(T_\omega^v) = \frac{1}{\rho(M_{i \rightarrow j, T_\omega^v} - \frac{\gamma_\omega}{\theta} J)} = \frac{1}{\rho(M_{j \rightarrow i, T_\omega^v} - \frac{1-\gamma_\omega}{\theta} J)}.$$

What happens to the the number  $\gamma_\omega$  when  $\omega$  goes to  $\infty$ ? We claim that  $\lim_{\omega \rightarrow \infty} \gamma_\omega$  exists. Indeed, one of the branches corresponding to  $M_{i \rightarrow j, T_\omega^v}$  and  $M_{j \rightarrow i, T_\omega^v}$  does not contain  $u$ . Suppose it is the second, so  $M_{j \rightarrow i, T_\omega^v} = M_{j \rightarrow i, T}$ . The numbers  $\mu(T_\omega^v)$  increase to a limit, see Theorem 1.1, so the numbers  $\rho(M_{j \rightarrow i, T_\omega^v} - \frac{1-\gamma_\omega}{\theta} J)$  decrease to a limit, which means that the numbers  $1-\gamma_\omega$  increase to a limit. This limit is at most 1 since  $0 < \gamma_\omega < 1$ .

**LEMMA 3.10.** *If the trees  $T_\omega^v$  are of type II, with characteristic vertices  $i, j$  for  $\omega_0 < \omega < \infty$ , and if  $\gamma = \lim_{\omega \rightarrow \infty} \gamma_\omega = 0$ , where  $\rho(M_{i \rightarrow j, T_\omega^v} - \frac{\gamma_\omega}{\theta} J) = \rho(M_{j \rightarrow i, T_\omega^v} - \frac{1-\gamma_\omega}{\theta} J)$*



and  $\theta$  is the weight of the edge  $\{i, j\}$ , then  $T_\infty^v$  is a trii of type  $I_i$ . Similarly, if  $\gamma = 1$  then  $T_\infty^v$  is a trii of type  $I_j$ .

*Proof.*  $M_{j \rightarrow i, T_\infty^v} = (M_{i \leftrightarrow j, T_\infty^v} \oplus (0)) + \frac{1}{\theta} J$  so if  $\gamma = 0$ , then  $\rho(M_{i \rightarrow j, T_\infty^v}) = \rho(M_{j \rightarrow i, T_\infty^v} - \frac{1}{\theta} J) = \rho(M_{i \leftrightarrow j, T_\infty^v})$  so  $T_\infty^e$  is a trii of type  $I_i$ .  $\square$

**COROLLARY 3.11.** *If the trees  $T_\omega^v$  are of type II and if  $T_\infty^v$  is a trii of type II, then  $0 < \gamma = \lim_{\omega \rightarrow \infty} \gamma_\omega < 1$ .*

**REMARK 3.12.** The tree  $T$  can be a tree of type I with a characteristic vertex, say  $c$ , or a tree of type II. In the first case there are 3 possibilities:

- 1  $T_\infty^v$  is a trii of type  $I_c$ ,
- 2  $T_\infty^v$  is a trii of type  $I_s$ , where  $s \neq c$ ,
- 3  $T_\infty^v$  is a trii of type II.

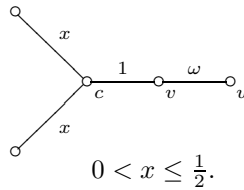
In the second case there are two possibilities:

- 4  $T_\infty^v$  is a trii of type  $I_s$  for some  $s$ ,
- 5  $T_\infty^v$  is a trii of type II.

The following example demonstrates that all five subcases are possible.

**EXAMPLE 3.13.**

Subcase 1

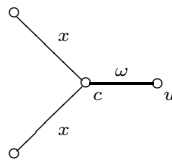


Here

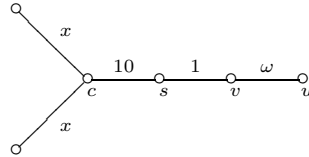
$$M_{c, T_\omega^v} = \begin{pmatrix} 1/x & 0 & 0 & 0 \\ 0 & 1/x & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 + \frac{1}{\omega} \end{pmatrix},$$

so for  $x < \frac{1}{2}$ ,  $T_\infty^v$  is a tree of type I with a characteristic vertex  $c$  and for  $x = \frac{1}{2}$  it is only a trii of type  $I_c$ .

Another example is when  $c = v$



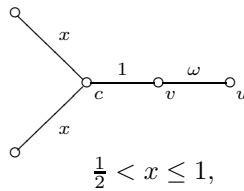
Subcase 2



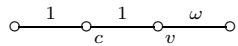
where

$$\rho \left[ \begin{pmatrix} 1/x + 0.1 & 0.1 & 0.1 \\ 0.1 & 1/x + 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix} \right] = 2.$$

Subcase 3



or

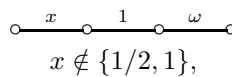
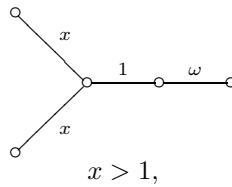


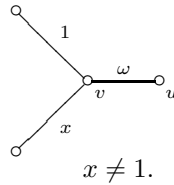
Subcase 4

$$\text{---} \overset{1}{\circ} \text{---} \overset{x}{\circ} \text{---} \overset{1}{\circ} \text{---} \overset{\omega}{\circ} \text{---}, \quad \rho \left[ \begin{pmatrix} 1 + 1/x & 1/x \\ 1/x & 1/x \end{pmatrix} \right] = 2.$$

Subcase 5

Here we suggest 3 examples:





We are now ready to state and prove the main result.

**THEOREM 3.14.** *Let  $T$  be a tree. Then*

$$\lim_{\omega \rightarrow \infty} \mu(T_\omega^v) = \mu(T) \tag{3.3}$$

*if and only if*

(a)  $T$  is a tree of type I with characteristic vertex say  $c$ ,  
 and

(b)  $\rho(M_{c \rightarrow u, T_\infty^v}) \leq \rho(M_{c, T}) = \frac{1}{\mu(T)}$ .

*Proof.* We prove the theorem by considering the five subcases of Remark 3.12, and showing that (3), (a) and (b) hold in Subcase 1 and only in this case, i.e. if and only if  $T$  and  $T_\infty^v$  are of type  $I_c$  for some vertex  $c$ .

Subcase 1: Obviously (a) holds. From (1) and (2) follows that  $\rho(M_{c, T_\omega^v}) \geq \rho(M_{c, T})$ . But if  $T$  and  $T_\omega^v$  are of type  $I_c$ , then equality holds. Thus

$$\rho(M_{c \rightarrow u, T_\infty^v}) \leq \rho(M_{c, T}),$$

proving (b), and

$$\mu(T) = \frac{1}{\rho(M_{c, T})} = \frac{1}{\rho(M_{c, T_\omega^v})} = \lim_{\omega \rightarrow \infty} \mu(T_\omega^v),$$

proving (3). This completes the proof in Subcase 1.

If  $T$  is a tree of type  $I_c$  and (b) holds, then it follows easily that  $T_\omega^v$  is a tree of type  $I_c$ . Therefore (b) does not hold in Subcases 2 and 3, while (a) obviously does not hold in Subcases 4 and 5. Now we will prove that (3) does not hold in the last four subcases.

Subcase 2: We have to show that (3) does not hold. Indeed

$$\begin{aligned} \mu(T) &= \frac{1}{\rho(M_{c, T})} > \frac{1}{\rho(M_{s, T})} \\ &= \frac{1}{\rho(M_{s, T_\omega^v})}, \text{ by Lemma 3.5} \\ &= \lim_{\omega \rightarrow \infty} \mu(T_\omega^v), \text{ by Lemma 3.3.} \end{aligned}$$

Subcase 3: By Remark 3.8(a) the trees  $T_\omega^v$  are for sufficiently large  $\omega$  of type II, say of type  $II_{p, q}$ , see Theorem 2.6, and the edge  $\{p, q\}$  of  $T$  has weight  $\theta$ , by Remark

3.8(b) it does not depend on  $\omega$ . Without loss of generality,  $p$  lies on the path between  $q$  and  $c$ .

By Proposition 2.10 the vertices  $p$  and  $q$  lie on the path between  $c$  and  $u$ . Let  $i$  be a neighbor of  $c$  such that  $c, p$  and  $q$  lie on the path between  $i$  and  $u$  and  $c \rightarrow i$  is a Perron branch of  $T$ . Then we obtain

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \mu(T_\omega^v) &= \lim_{\omega \rightarrow \infty} \frac{1}{\rho(M_{q \rightarrow p, T_\omega^v} - \frac{\gamma_\omega}{\theta} J)}, \text{ by Theorem 2.9,} \\ &= \lim_{\omega \rightarrow \infty} \frac{1}{\rho(M_{q \rightarrow p, T} - \frac{\gamma_\omega}{\theta} J)}, \text{ since } q \rightarrow p \text{ is in } T, \\ &= \frac{1}{\rho(M_{q \rightarrow p, T} - \frac{\gamma}{\theta} J)}, \text{ where } 0 < \gamma < 1, \text{ by Corollary 3.11,} \\ &< \frac{1}{\rho(M_{c \rightarrow i, T})}, \text{ by Proposition 2.11,} \\ &= \frac{1}{\rho(M_{c, T})}, \text{ since } c \rightarrow i \text{ is a Perron branch of } T, \\ &= \mu(T), \text{ since } T \text{ is of type } I_c, \end{aligned}$$

so (3) does not hold.

Subcase 4: Here again we have to show that (3) does not hold. Suppose  $T$  is of type  $II_{ij}$ , where  $j$  lies on the path from  $i$  to  $u$ . Let  $\theta$  and  $\gamma$  be as in Theorem 2.9. Since  $T_\omega^v$  is a trii of type  $I_s$ , the by Proposition 2.10,  $s$  lies on the path from  $i$  to  $u$ . Therefore

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \mu(T_\omega^v) &= \lim_{\omega \rightarrow \infty} \frac{1}{\rho(M_{s \rightarrow i, T_\omega^v})} = \lim_{\omega \rightarrow \infty} \frac{1}{\rho(M_{s \rightarrow i, T})} \\ &\leq \frac{1}{\rho(M_{j \rightarrow i, T})} < \frac{1}{\rho(M_{j \rightarrow i, T} - \frac{\gamma}{\theta} J)} = \mu(T). \end{aligned}$$

Subcase 5: Here  $T$  is of, say, type  $II_{ij}$  and for  $\omega$  large enough,  $T_\omega^v$  are of, say, type  $II_{pq}$ , where by Proposition 2.10, we may take, without loss of generality,  $p$  and  $q$  to lie between  $i$  and  $j$ . Let  $\theta$  and  $\gamma$  be as in Theorem 2.9 for the edge  $\{i, j\}$  in  $T$  and let  $\hat{\theta}$  and  $\gamma_\omega$  be the corresponding pair for the edge  $\{p, q\}$  in  $T_\omega^v$ . Observe that  $\hat{\theta}$  does not depend on  $\omega$  by Remark 3.8(b). Let  $\hat{\gamma} = \lim_{\omega \rightarrow \infty} \gamma_\omega$ . By Corollary 3.13 we have  $0 < \hat{\gamma} < 1$ . Now

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \mu(T_\omega^v) &= \lim_{\omega \rightarrow \infty} \frac{1}{\rho(M_{q \rightarrow p, T_\omega^v} - \frac{\gamma_\omega}{\hat{\theta}} J)} = \lim_{\omega \rightarrow \infty} \frac{1}{\rho(M_{q \rightarrow p, T} - \frac{\gamma_\omega}{\hat{\theta}} J)} \\ &= \frac{1}{\rho(M_{q \rightarrow p, T} - \frac{\hat{\gamma}}{\hat{\theta}} J)} < \frac{1}{\rho(M_{j \rightarrow i, T} - \frac{\gamma}{\theta} J)} = \mu(T), \end{aligned}$$

where the inequality follows from Proposition 2.11.  $\square$

**Acknowledgments.** We are indebted to Mr. Felix Goldberg [5] for suggesting Example 1.5 which shows that  $G$  does not have to be a tree for  $\lim_{\omega \rightarrow \infty} \mu(G_\omega^v) = \mu(G)$

to hold. The question of when does  $\lim_{\omega \rightarrow \infty} \mu(G_\omega^v) = \mu(G)$ , when  $G$  is a general graph, seems to be much more difficult than the one in the case that  $G$  is a tree. We are grateful to the referee for his or her important remarks and for suggesting that Propositions 1.3 and 1.4, as well as Lemma 2.2 of [2], may be useful in dealing with the general case.

#### REFERENCES

- [1] A. Berman and R.J. Plemmons. Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.
- [2] S. Fallat and S. Kirkland. Extremizing algebraic connectivity subject to graph theoretic constraints. *Electronic Journal of Linear Algebra*, 3:48–74, 1998.
- [3] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23: 298–305, 1973.
- [4] M. Fiedler. A property of eigenvectors of nonnegative symmetric matrices and its applications to graph theory. *Czechoslovak Mathematical Journal*, 25: 619–633, 1975.
- [5] F. Goldberg. Private communication.
- [6] R. Grone, R. Merris, and V. Sunder. The Laplacian spectrum of a graph. *SIAM Journal on Matrix Analysis and Applications*, 11: 218–238, 1990.
- [7] R.J. Horn and C.R. Johnson. Matrix Analysis. Cambridge University Press, Cambridge, 1985.
- [8] S. Kirkland and M. Neumann. Algebraic connectivity of weighted trees under perturbation. *Linear and Multilinear Algebra*, 42: 187–203, 1997.
- [9] S. Kirkland, M. Neumann, and B. Shader. Characteristic vertices of weighted trees via Perron values. *Linear and Multilinear Algebra*, 40: 311–325, 1996.
- [10] S. Kirkland, M. Neumann, and B. Shader. Distances in weighted trees and group inverse of Laplacian Matrices. *SIAM Journal on Matrix Analysis and Applications*, 18: 827–841, 1997.
- [11] R. Merris. Characteristic vertices of trees. *Linear and Multilinear Algebra*, 22: 115–131, 1987.
- [12] J.J. Moliterno and M. Neumann. The algebraic connectivity of two trees connected by an edge of infinite weight. *Electronic Journal of Linear Algebra*, 8: 1–13, 2001.