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ON THE NULLITY OF GRAPHS

BO CHENG† AND BOLIAN LIU†

Abstract. The nullity of a graph $G$, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in its spectrum. It is known that $\eta(G) \leq n - 2$ if $G$ is a simple graph on $n$ vertices and $G$ is not isomorphic to $nK_1$. In this paper, we characterize the extremal graphs attaining the upper bound $n - 2$ and the second upper bound $n - 3$. The maximum nullity of simple graphs with $n$ vertices and $e$ edges, $M(n,e)$, is also discussed. We obtain an upper bound of $M(n,e)$, and characterize $n$ and $e$ for which the upper bound is achieved.

Key words. Graph eigenvalue, Nullity, Clique, Girth, Diameter.

AMS subject classifications. 05C50.

1. Introduction. Let $G$ be a simple graph. The vertex set of $G$ is referred to as $V(G)$, the edge set of $G$ as $E(G)$. If $W$ is a nonempty subset of $V(G)$, then the subgraph of $G$ obtained by taking the vertices in $W$ and joining those pairs of vertices in $W$ which are joined in $G$ is called the subgraph of $G$ induced by $W$ and is denoted by $G[W]$. We write $G - \{v_1, \ldots, v_k\}$ for the graph obtained from $G$ by removing the vertices $v_1, \ldots, v_k$ and all edges incident to them.

We define the union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, to be the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$. If $G_1$ and $G_2$ are disjoint we denote their union by $G_1 + G_2$. The disjoint union of $k$ copies of $G$ is often written $kG$. As usual, the complete graph and cycle of order $n$ are denoted by $K_n$ and $C_n$, respectively. An isolated vertex is sometimes denoted by $K_1$.

Let $r \geq 2$ be an integer. A graph $G$ is called $r$-partite if $V(G)$ admits a partition into $r$ classes $X_1, X_2, \ldots, X_r$ such that every edge has its ends in different classes; vertices in the same partition must not be adjacent. Such a partition $(X_1, X_2, \ldots, X_r)$ is called a $r$-partition of the graph. A complete $r$-partite graph is a simple $r$-partite graph with partition $(X_1, X_2, \ldots, X_r)$ in which each vertex of $X_i$ is joined to each vertex of $G - X_i$; if $|X_i| = n_i$, such a graph is denoted by $K_{n_1,n_2,\ldots,n_r}$. Instead of ‘2-partite’ (‘3-partite’) one usually says bipartite (tripartite).

Let $G$ and $G'$ be two graphs. Then $G$ and $G'$ are isomorphic if there exists a bijection $\varphi : V(G) \rightarrow V(G')$ with $xy \in E(G) \iff \varphi(x)\varphi(y) \in E(G')$ for all $x, y \in V(G)$.

The adjacency matrix $A(G)$ of graph $G$ of order $n$, having vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ symmetric matrix $[a_{ij}]$, such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and 0, otherwise. A graph is said to be singular (non-singular) if its
adjacency matrix is a singular (non-singular) matrix. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph $G$, and to form the spectrum of this graph. The number of zero eigenvalues in the spectrum of the graph $G$ is called its nullity and is denoted by $\eta(G)$. Let $r(A(G))$ be the rank of $A(G)$, clearly, $\eta(G) = n - r(A(G))$. The rank of a graph $G$ is the rank of its adjacency matrix $A(G)$, denoted by $r(G)$. Then $\eta(G) = n - r(G)$. Each of $\eta(G)$ and $r(G)$ determines the other.

It is known that $0 \leq \eta(G) \leq n - 2$ if $G$ is a simple graph on $n$ vertices and $G$ is not isomorphic to $nK_1$. In [3], L.Collatz and U.Sinogowitz first posed the problem of characterizing all graphs $G$ with $\eta(G) > 0$. This question is of great interest in chemistry, because, as has been shown in [4], for a bipartite graph (corresponding to an alternant hydrocarbon), if $\eta(G) > 0$, then it indicates the molecule which such a graph represents is unstable. The problem has not yet been solved completely; only for trees and bipartite graph some particular results are known (see [4] and [5]). In recent years, this problem has been investigated by many researchers([5], [7] and [8]).

A natural question is how to characterize the extremal matrices attaining the upper bound $n - 2$ and the second upper bound $n - 3$. The following theorems answer this question.

**Theorem 1.1.** Suppose that $G$ is a simple graph on $n$ vertices and $n \geq 2$. Then $\eta(G) = n - 2$ if and only if $G$ is isomorphic to $K_{n_1,n_2} + kK_1$, where $n_1 + n_2 + k = n$, $n_1, n_2 > 0$, and $k \geq 0$.

**Theorem 1.2.** Suppose that $G$ is a simple graph on $n$ vertices and $n \geq 3$. Then $\eta(G) = n - 3$ if and only if $G$ is isomorphic to $K_{n_1,n_2,n_3} + kK_1$, where $n_1 + n_2 + n_3 + k = n$, $n_1, n_2, n_3 > 0$, and $k \geq 0$.

We now introduce the definition of maximum nullity number, which is closely related to the upper bound of $\eta(G)$. Let $\Gamma(n, e)$ be the set of all simple graphs with $n$ vertices and $e$ edges. The maximum nullity number of simple graphs with $n$ vertices and $e$ edges, $M(n, e)$, is $\max \{\eta(A) : A \in \Gamma(n, e)\}$, where $n \geq 1$ and $0 \leq e \leq \binom{n}{2}$.

This paper is organized as follows. Theorem 1.1 and Theorem 1.2 are proved in section 2. In order to prove them, we obtain some inequalities concerning $\eta(G)$ in section 2. In section 4, we obtain an upper bound of $M(n, e)$, and characterize $n$ and $e$ for which the upper bound is achieved.

2. Some inequalities concerning $\eta(G)$. A path is a graph $P$ of the form $V(P) = \{v_1, v_2, \ldots, v_k\}$ and $E(P) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$, where the vertices $v_1, v_2, \ldots, v_k$ are all distinct. We say that $P$ is a path from $v_1$ to $v_k$, or a $(v_1, v_k)$-path. It can be denoted by $P_k$. The number of edges of the path is its length. The distance $d(x, y)$ in $G$ of two vertices $x, y$ is the length of a shortest $(x, y)$-path in $G$; if no such path exists, we define $d(x, y)$ to be infinite. The greatest distance between any two vertices in $G$ is the diameter of $G$, denoted by diam$(G)$. 
LEMMA 2.1. (see [6]) (i) The adjacency matrix of the complete graph $K_n$, $A(K_n)$, has 2 distinct eigenvalues $n - 1$, $-1$ with multiplicities $1$, $n - 1$ where $n > 1$.

(ii) The eigenvalues of $C_n$ are $\lambda_r = 2\cos\frac{2\pi r}{n}$, where $r = 0, \ldots, n - 1$.

(iii) The eigenvalues of $P_n$ are $\lambda_r = 2\cos\frac{2\pi r}{n+1}$, where $r = 1, 2, \ldots, n$.

LEMMA 2.2. (i) $r(K_n) = \begin{cases} 0 & \text{if } n = 1; \\ n & \text{if } n > 1. \end{cases}$

(ii) $r(C_n) = \begin{cases} n - 2, & \text{if } n \equiv 0(\text{mod}4); \\ n, & \text{otherwise}. \end{cases}$

(iii) $r(P_n) = \begin{cases} n - 1, & \text{if } n \text{ is odd}; \\ n, & \text{otherwise}. \end{cases}$

Proof. (i) and (iii) are direct consequences from Lemma 2.1.

(ii) We have $\lambda_r = 0$ if and only if $2\cos\frac{2\pi r}{n} = 0$ if and only if $\frac{2\pi r}{n} = \pi/2$ or $3\pi/2$. Therefore $\lambda_r = 0$ if and only if $r = n/4$ or $r = 3n/4$. Hence (ii) holds. □

The following result is straightforward.

LEMMA 2.3. (i) Let $H$ be an induced subgraph of $G$. Then $r(H) \leq r(G)$.

(ii) Let $G = G_1 + G_2$, then $r(G) = r(G_1) + r(G_2)$, i.e., $\eta(G) = \eta(G_1) + \eta(G_2)$.

In the remainder of this section, we give some inequalities concerning $\eta(G)$.

PROPOSITION 2.4. Let $G$ be a simple graph on $n$ vertices and $K_p$ be a subgraph of $G$, where $2 \leq p \leq n$. Then $\eta(G) \leq n - p$.

Proof. Immediate from Lemma 2.2(i) and Lemma 2.3(i). □

A clique of a simple graph $G$ is a subset $S$ of $V(G)$ such that $G[S]$ is complete. A clique $S$ is maximum if $G$ has no clique $S'$ with $|S'| > |S|$. The number of vertices in a maximum clique of $G$ is called the clique number of $G$ and is denoted by $\omega(G)$. The following inequality is clear from the above result.

COROLLARY 2.5. Let $G$ be a simple graph on $n$ vertices and $G$ is not isomorphic to $nK_1$. Then $\eta(G) + \omega(G) \leq n$.

PROPOSITION 2.6. Let $G$ be a simple graph on $n$ vertices and let $C_p$ be an induced subgraph of $G$, where $3 \leq p \leq n$. Then $\eta(G) \leq \begin{cases} n - p + 2, & \text{if } p \equiv 0(\text{mod}4); \\ n - p, & \text{otherwise}. \end{cases}$

Proof. This follows from Lemma 2.2(ii) and Lemma 2.3(i). □

The length of the shortest cycle in a graph $G$ is the girth of $G$, denoted by $gir(G)$. A relation between $gir(G)$ and $\eta(G)$ is given here.

COROLLARY 2.7. If $G$ is simple graph on $n$ vertices and $G$ has at least one cycle, then $\eta(G) \leq \begin{cases} n - gir(G) + 2, & \text{if } gir(G) \equiv 0(\text{mod}4); \\ n - gir(G), & \text{otherwise}. \end{cases}$
Proposition 2.8. Let \( G \) be a simple graph on \( n \) vertices and let \( P_k \) be an induced subgraph of \( G \), where \( 2 \leq k \leq n \). Then
\[
\eta(G) \leq \begin{cases} 
  n - k + 1, & \text{if } k \text{ is odd;} \\
  n - k, & \text{otherwise.} 
\end{cases}
\]

Proof. This is a direct consequence of Lemma 2.2(iii) and Lemma 2.3(i). \( \Box \)

Corollary 2.9. Suppose \( x \) and \( y \) are two vertices in \( G \) and there exists an \((x,y)\)-path in \( G \). Then
\[
\eta(G) \leq \begin{cases} 
  n - d(x,y), & \text{if } d(x,y) \text{ is even;} \\
  n - d(x,y) - 1, & \text{otherwise.} 
\end{cases}
\]

Proof. Let \( P_k \) be the shortest path between \( x \) and \( y \). Suppose \( v_1, v_2, \ldots, v_k \) are the vertices of \( P_k \). Then \( G[v_1, v_2, \ldots, v_k] \) is \( P_k \). From Proposition 2.8, we have
\[
\eta(G) \leq \begin{cases} 
  n - d(x,y), & \text{if } d(x,y) \text{ is even;} \\
  n - d(x,y) - 1, & \text{otherwise.} 
\end{cases}
\]

Corollary 2.10. Suppose \( G \) is a simple connected graph on \( n \) vertices. Then
\[
\eta(G) \leq \begin{cases} 
  n - \text{diam}(G), & \text{if } \text{diam}(G) \text{ is even;} \\
  n - \text{diam}(G) - 1, & \text{otherwise.} 
\end{cases}
\]

3. Extremal matrices and graphs. For any vertex \( x \in V(G) \), define \( \Gamma(x) = \{v : v \in V(G) \text{ and } v \text{ is adjacent to } x\} \). We first give the following lemma.

Lemma 3.1. Suppose that \( G \) is a simple graph on \( n \) vertices and \( G \) has no isolated vertex. Let \( x \) be an arbitrary vertex in \( G \). Let \( Y = \Gamma(x) \) and \( X = V(G) - Y \). If \( r(G) \leq 3 \), then
(i) No two vertices in \( X \) are adjacent.
(ii) Each vertex from \( X \) and each vertex from \( Y \) are adjacent.

Proof. (i) Suppose \( x_1 \in X \), \( x_2 \in X \), and \( x_1 \) and \( x_2 \) are adjacent. Since \( x_1 \in X \), \( x_1 \) and \( x \) are not adjacent. Similarly we have \( x_2 \) and \( x \) are not adjacent. Since \( G \) has no isolated vertex, \( x \) is not an isolated vertex. Then \( Y \) is not an empty set. Select any vertex \( y \) in \( Y \). Then \( G[x_1, x_2, y] \) is isomorphic to \( K_2 + K_1, K_{1,2} \) or \( K_3 \).

If \( G[x_1, x_2, y] \) is isomorphic to \( K_2 + K_1 \), then \( G[x, x_1, x_2, y] \) is isomorphic to \( P_2 + P_2 \). Since \( r(P_2 + P_2) = r(P_2) + r(P_2) = 2 + 2 = 4 \) by Lemma 2.3, we have \( r(G) \geq 4 \), a contradiction.

If \( G[x_1, x_2, y] \) is isomorphic to \( K_{1,2} \), then \( G[x, x_1, x_2, y] \) is isomorphic to \( P_3 \). Therefore \( r(G) \geq r(P_3) = 4 \), a contradiction.

If \( G[x_1, x_2, y] \) is isomorphic to \( K_3 \), then using the fact that neither \( x_1 \) nor \( x_2 \) is adjacent to \( x \), we can verify that \( r(G[x, x_1, y]) = 4 \), a contradiction.

Therefore no two vertices in \( X \) are adjacent.

(ii) Suppose not, then there exist \( x_1 \in X \) and \( y_1 \in Y \) such that \( x_1 \) and \( y_1 \) are not adjacent. Since \( x \) and \( y_1 \) are adjacent, we have \( x \) and \( x_1 \) are distinct. Due to the fact that \( G \) has no isolated vertex, we can choose a vertex \( z \) in \( G \) which is adjacent to \( x_1 \). By (i) we see \( z \in Y \). Then \( x \) and \( z \) are adjacent.
If $y_1$ and $z$ are not adjacent, then $G[x, x_1, y_1, z]$ is isomorphic to $P_4$. Hence $r(G[x, x_1, y_1, z]) > 3$, a contradiction.

If $y_1$ and $z$ are adjacent, then using the fact that neither $y_1$ nor $x$ is adjacent to $x_1$, we can verify that $r(G[x, x_1, y_1, z]) = 4$, a contradiction. Thus each vertex from $X$ and each vertex from $Y$ are adjacent. $\Box$

In order to prove Theorem 1.1, we prove the following lemma.

**Lemma 3.2.** Suppose that $G$ is a simple graph on $n$ vertices $(n \geq 2)$ and $G$ has no isolated vertex. Then $\eta(G) = n - 2$ if and only if $G$ is isomorphic to a complete bipartite graph $K_{n_1, n_2}$, where $n_1 + n_2 = n$, $n_1, n_2 > 0$.

**Proof.** The sufficiency is clear.

To prove the necessity, choose an arbitrary vertex $x$ in $G$. Let $Y = \Gamma(x)$ and $X = V(G) - Y$. Since $G$ has no isolated vertex, $x$ is not an isolated vertex. Then $Y$ is not an empty set. Since $x \in X$, $X$ is not empty.

We now prove any two vertices in $Y$ are not adjacent. Suppose that there exist $y_1 \in Y$ and $y_2 \in Y$ such that $y_1$ and $y_2$ are adjacent. Then $G[x, y_1, y_2]$ is a triangle. By Proposition 2.4, we have $\eta(G) \leq n - 3$, a contradiction.

From Lemma 3.1, we know that

(i) any two vertices in $X$ are not adjacent, and
(ii) any vertex from $X$ and any vertex from $Y$ are adjacent. Hence $G$ is isomorphic to a complete bipartite graph. $\Box$

Theorem 1.1 is immediate from the above lemma.

Two matrices $A_1$ and $A_2$ that are related by $B = P^{-1}AP$ where $P$ is a permutation matrix, are said to be permutation similar. Graphs $G_1$ and $G_2$ are isomorphic if and only if $A(G_1)$ and $A(G_2)$ are permutation similar.

We denote by $J_{p,q}$ the $p \times q$ matrix of all 1’s. Sometimes we simply use $J$ to denote an all 1’s matrix of appropriate or undetermined size. Similar conventions are used for zeros matrices with $O$ replacing $J$. Let $A_1$ and $A_2$ be two matrices. Define $A_1 \oplus A_2 = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}$ and $A_1 \boxplus A_2 = \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}$.

Then Theorem 1.1 can be written in the following equivalent form.

**Theorem 3.3.** Suppose that $G$ is a simple graph on $n$ vertices and $n \geq 2$. Then $\eta(G) = n - 2$ if and only if $A(G)$ is permutation similar to matrix $O_{n_1,n_2} \oplus O_{n_2,n_2} \oplus O_{k,k}$, where $n_1 + n_2 + k = n$, $n_1, n_2 > 0$, and $k \geq 0$.

Some lemmas are given before we prove Theorem 1.2.

**Lemma 3.4.** Let $A$ be a symmetric $n \times n$ matrix and let the rank of $A$ be $k$. Then there exists a nonsingular principal minor of order $k$.

**Lemma 3.5.** Suppose that $G$ is a simple graph on $n$ vertices $(n \geq 3)$ and $G$ has no isolated vertex. Then $\eta(G) = n - 3$ if and only if $G$ is isomorphic to a complete tripartite graph $K_{n_1,n_2,n_3}$, where $n_1, n_2, n_3 > 0$. 

Proof. If $G$ is isomorphic to a complete tripartite graph, then $A(G)$ is permutation similar to $O \oplus O \oplus O$. Thus we can verify that $r(G) = 3$, i.e., $\eta(G) = n - 3$. The sufficiency follows.

To prove the necessity, choose an arbitrary vertex $x$ in $G$. Let $Y = \Gamma(x)$ and $X = V(G) - Y$. Since $G$ has no isolated vertex, $x$ is not an isolated vertex. Then $Y$ is not an empty set. Since $x \in X$, $X$ is not empty.

By Lemma 3.1, we have the following results.

**Claim 3.6.** Any two vertices in $X$ are not adjacent.

**Claim 3.7.** Any vertex from $X$ and any vertex from $Y$ are adjacent.

We now consider $G - X$, and prove

**Claim 3.8.** $r(G - X) \leq 2$.

**Proof.** Suppose $r(G - X) > 2$. Due to the fact that $r(G - X) \leq r(G) = 3$, we see $r(G - X) = 3$. By Lemma 3.4, there exists an induced subgraph $H$ of $G - X$ such that $H$ is order 3 and $r(H) = 3$. Then $H$ is a triangle. Since $x$ is adjacent to each vertex of $H$, $K_4$ is a subgraph of $G$. Therefore $\eta(G) \leq n - 4$, a contradiction. \[ \boxdot \]

Furthermore, we can show

**Claim 3.9.** $r(G - X) = 2$.

**Proof.** Suppose $r(G - X) < 2$, then $r(G - X) = 0$. Hence $G - X = O$. Therefore $r(G) = 2$, which contradicts $\eta(G) = n - 3$. \[ \boxdot \]

By Theorem 1.1, $G - X$ is isomorphic to $K_{n_1,n_2} + kK_1$, where $n_1,n_2 > 0$, and $k \geq 0$.

If $k > 0$, then $A(G)$ is permutation similar to

$$
\begin{bmatrix}
O & J & J & J \\
J & O & J & O \\
J & J & O & O \\
J & O & O & O
\end{bmatrix}
$$

Then $r(G) = 4$, a contradiction. Thus $k = 0$. So $G - X$ is isomorphic to $K_{n_1,n_2}$.

By Claim 3.6 and 3.7, we see $G$ is isomorphic to a complete tripartite graph $K_{n_1,n_2,n_3}$, where $n_1,n_2,n_3 > 0$. \[ \boxdot \]

Theorem 1.2 is immediate from the above lemma. Theorem 1.2 also has the following equivalent form.

**Theorem 3.10.** Suppose that $G$ is a simple graph on $n$ vertices and $n \geq 3$. Then $\eta(G) = n - 3$ if and only if $A(G)$ is permutation similar to matrix

$$
O_{n_1,n_2} \oplus O_{n_2,n_3} \oplus O_{n_3,n_3} \oplus O_{k,k},
$$

where $n_1 + n_2 + n_3 + k = n$, $n_1,n_2,n_3 > 0$, and $k \geq 0$. 
4. Maximum nullity number of graphs. In the first section, we define

$$M(n, e) = \max \{ \eta(A) : A \in \Gamma(n, e) \}$$

where $\Gamma(n, e)$ is the set of all simple graphs with $n$ vertices and $e$ edges. In this section an upper bound of $M(n, e)$ is given. Let $g(m) = \max \{ k : k \mid m \text{ and } k \leq \sqrt{m} \}$, where $m$ is a positive integer, e.g., $g(1) = 1, g(2) = 1, g(4) = 2$.

**Theorem 4.1.** The following results hold:

(i) $M(n, 0) = n$. $M(n, \binom{n}{2}) = 0$.

(ii) $M(n, 1) = n - 2$ for $n \geq 2$.

(iii) $M(n, \binom{n}{2} - 1) = 1$ for $n > 2$.

(iv) $M(n, e) \leq n - 2$ for $e > 0$.

(v) $M(n, e) = n - 2$ if $e > 0$ and $g(e) + e/g(e) \leq n$.

(vi) $M(n, e) \leq n - 3$ if $e > 0$ and $g(e) + e/g(e) > n$.

**Proof.** (i) and (ii) are immediate from the definition.

(iii) Suppose $G \in \Gamma(n, \binom{n}{2} - 1)$. Then $G$ is isomorphic to $K_n$ with one edge deleted. Thus there exist two identical rows (columns) in $A(G)$. Therefore $A(G)$ is singular and $\eta(G) \geq 1$.

Since $G$ contains $K_{n-1}$, by Proposition 2.4, we have $\eta(G) \leq 1$. Hence $\eta(G) = 1$. Therefore $M(n, \binom{n}{2} - 1) = 1$.

(iv) From the fact that $\eta(G) \leq n - 2$ if $G$ is a simple graph on $n$ vertices and $G$ is not isomorphic to $nK_1$, we see that $M(n, e) \leq n - 2$ for $e > 0$.

(v) Let $n_1 = g(e), n_2 = e/g(e)$ and $k = n - n_1 - n_2$. Then $G = K_{n_1, n_2} + K_1 \in \Gamma(n, e)$ and $\eta(G) = n - 2$. Hence $M(n, e) = n - 2$.

(vi) Suppose $M(n, e) > n - 3$. Since $M(n, e) \leq n - 2$, we have $M(n, e) = n - 2$. Then there exists $G \in \Gamma(n, e)$ such that $\eta(G) = n - 2$. Hence $G = K_{n_1, n_2} + kK_1$. Therefore $n_1 \times n_2 = e$ and $n_1 + n_2 + k = n$. Without loss of generality, we may assume $n_1 \leq n_2$. Then $n_1 \leq \sqrt{e}$. Since $n_1 \mid n$, $n_1 \leq g(e)$.

Since $n_1 \leq \sqrt{e}$ and $g(e) \leq \sqrt{e}$, $g(e)n_1 \leq e$. Then $1 - \frac{e}{g(e)n_1} \leq 0$.

Since

$$g(e) + e/g(e) - n_1 - n_2 = g(e) - n_1 + e/g(e) - n_2 = g(e) - n_1 + e/g(e) - e/n_1$$

$$= g(e) - n_1 + e\frac{n_1 - g(e)}{g(e)n_1} = (g(e) - n_1)(1 - \frac{e}{g(e)n_1}) \leq 0,$$

$g(e) + e/g(e) \leq n_1 + n_2 \leq n$, which contradicts to $g(e) + e/g(e) > n$. \qed

The following immediate corollary gives an upper bound for $M(n, e)$ and characterizes when the upper bound is achieved.

**Corollary 4.2.** Suppose $e > 0$. Then $M(n, e) \leq n - 2$ and the equality holds if and only if $g(e) + e/g(e) \leq n$. 

Here we give a necessary condition for $M(n, e) = n - 3$.

**Theorem 4.3.** If $M(n, e) = n - 3$, then $e \leq n^2/3$.

**Proof.** Due to the fact that $M(n, e) = n - 3$, there exists $G \in \Gamma(n, e)$ such that $\eta(G) = n - 3$. Hence $G = K_{n_1, n_2, n_3} + kK_1$. Therefore $n_1 + n_2 + n_3 \leq n$ and $n_1n_2 + n_2n_3 + n_1n_3 = e$.

Since
\[(n_1 + n_2 + n_3)^2 = n_1^2 + n_2^2 + n_3^2 + 2(n_1n_2 + n_2n_3 + n_1n_3)\]
\[\geq n_1n_2 + n_2n_3 + n_1n_3 + 2(n_1n_2 + n_2n_3 + n_1n_3) = 3e,\]
then $n^2 \geq 3e$, i.e., $e \leq n^2/3$. \(\square\)

The following corollary is immediate.

**Corollary 4.4.** If $n^2/3 < e \leq \left(\frac{n}{3}\right)^2$, then $M(n, e) \leq n - 4$.

Finally we give a table for the exact values of $M(n, e)$, where $1 \leq n \leq 5$.

<table>
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<th>1</th>
<th>2</th>
<th>3</th>
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<th>5</th>
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<th>7</th>
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$M(5, 5) = 2$ is obtained by Theorem 4.1(vi) and the fact that $\eta(K_{1,1,2} + K_1) = 2$. $M(5, 7) = 2$ is from Theorem 4.1(vi) and the fact that $\eta(K_{1,1,3}) = 2$, and $M(5, 8) = 2$ is from Theorem 4.1(vi) and the fact that $\eta(K_{1,2,2}) = 2$.

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**REFERENCES**