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NOTE ON DELETING A VERTEX AND WEAK INTERLACING OF THE LAPLACIAN SPECTRUM

ZVI LOTKER

Abstract. The question of what happens to the eigenvalues of the Laplacian of a graph when we delete a vertex is addressed. It is shown that
\[ \lambda_i - 1 \leq \lambda_v^i \leq \lambda_i + 1, \]
where \( \lambda_i \) is the \( i \)th smallest eigenvalues of the Laplacian of the original graph and \( \lambda_v^i \) is the \( i \)th smallest eigenvalues of the Laplacian of the graph \( G[V - v] \); i.e., the graph obtained after removing the vertex \( v \). It is shown that the average number of leaves in a random spanning tree \( F(G) > \frac{2|E|e^{-\alpha}}{\lambda_n} \), if \( \lambda_2 > \alpha n \).

Key words. Spectrum, Random spanning trees, Cayley formula, Laplacian, Number of leaves.

AMS subject classifications. 05C30, 34L15, 34L40.

1. Introduction. Given a graph \( G = (V, E) \) with \( n \) vertices \( V = \{1, ..., n\} \) and \( E \) edges, let \( A \) be the adjacency matrix of \( G \), i.e. \( a_{i,j} = 1 \) if vertex \( i \) is adjacent to vertex \( j \) in \( V \) and \( a_{i,j} = 0 \) otherwise. The Laplacian matrix of graph \( G \) is \( L = D - A \), where \( D \) is a diagonal matrix where \( d_{i,i} \) is equal to the degree \( d_i \) of vertex \( i \) in \( G \). The Laplacian of a graph is one of the basic matrices associated with a graph. The spectrum of the Laplacian fully characterizes the Laplacian (for more detail see [1]). Since \( L \) is symmetric and positive semidefinite, its eigenvalues are all nonnegative. We denote them by \( \lambda_1 \leq ... \leq \lambda_n \). One of the elementary operations on a graph is deleting a vertex \( v \in V \), we denote the graph obtained from deleting the node \( v \) by \( G[V - v] \), and the Laplacian Matrix of \( G[V - v] \) by \( L' \). Finally let \( \lambda_1^v \leq ... \leq \lambda_{n-1}^v \) be the eigenvalues of \( L_v' \). A well known theorem in Algebraic Graph theory is the interlacing of Laplacian spectrum under addition/deletion of an edge; see for example [1, Thm. 13.6.2]) quoted next.

Theorem 1.1. Let \( X \) be a graph with \( n \) vertices and let \( Y \) be obtained from \( X \) by adding an edge joining distinct vertices of \( X \) then
\[ \lambda_{i-1}(L(Y)) \leq \lambda_i(L(X)) \leq \lambda_i(L(Y)), \]
for all \( i = 1, ..., n \), (we assume that \( \lambda_0 = -\infty \)).

We remark that the eigenvalues of adjacency matrices \( A(G) \) and \( A(G[V - v]) \) also interlace; see, for example, [1, Thm. 9.1.1]. A natural question is whether we get a similar behavior for the Laplacian when we add/delete a vertex. In this note we study this question.
Related Work. This work uses two theorems from Matrix Analysis. The first is Cauchy’s Interlacing theorem which states that the eigenvalues of a Hermitian matrix \( A \) of order \( n \) interlace the eigenvalues of the principal submatrix of order \( n - 1 \), obtained by removing the \( i \)th row and the \( i \)th column for each \( i \in \{1, \ldots, n\} \).

**Theorem 1.2.** Let \( A \) be a Hermitian matrix of order \( n \) and let \( B \) be a principal submatrix of \( A \) of order \( n - 1 \). Then the eigenvalues of \( A \) and \( B \) are interlacing i.e. \( \lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_{n-1}(B) \leq \lambda_n(A) \).

Proof of this theorem can be found in [2].

The second theorem we use is the Courant-Fischer Theorem. This theorem is an extremely useful characterization of the eigenvalues of symmetric matrices.

**Theorem 1.3.** Let \( L \) be a symmetric matrix. Then

1. the \( i \)th eigenvalue \( \lambda_i \) of \( L \) is given by

\[
\lambda_i = \min_{U} \max_{x \in U} \frac{x^t L x}{x^t x},
\]

2. the \((n - i + 1)\)st eigenvalue \( \lambda_{n-i+1} \) of \( L \) is given by

\[
\lambda_{n-i+1} = \max_{U} \min_{x \in U} \frac{x^t L x}{x^t x},
\]

where \( U \) ranges over all \( i \) dimensional subspaces.

Proof of this theorem can be found in [3, p. 186]. Let \( v \in V \) be a vertex. Let \( P \) be the principal submatrix after we delete the row and column that correspond to the vertex \( v \) of the Laplacian. Denote the eigenvalues of \( P \) by \( \rho_1 \leq \cdots \leq \rho_{n-1} \).

**2. Weak Interlace for the \( L, L^v \).** In this section we show a weak interlacing connection between the \( L \) and \( L^v \). Since \( L \) is a symmetric matrix we can use Cauchy’s interlacing theorem. The next corollary simply applies this theorem for \( L \) and \( P \).

**Corollary 2.1.** \( \lambda_1 \leq \rho_1 \leq \cdots \leq \rho_{n-1} \leq \lambda_n \).

The next lemma uses the Courant-Fischer Theorem in order to prove weak interlacing for \( L, P \).

**Lemma 2.2.** For all \( i = 1, \ldots, n-1 \), \( \rho_i \leq \lambda_i^v + 1 \)

Proof. Let \( I_v = P - L^v \). Note that \( I_v \) is a \((0,1)\) diagonal matrix whose \( j \)th diagonal entry is 1 if and only if \( j \) is connected to \( v \) in \( G \). Fix \( i \in \{1, \ldots, n-1\} \). Using the Courant-Fischer Theorem it follows that

\[
\rho_{n-i+1} = \max_{U} \min_{x \in U} \left\{ \frac{x^t P x}{x^t x} : U \subseteq \mathbb{R}^n, \dim(U) = i, x \in U = \text{span}(U) \right\},
\]

where \( x^t \) is the transpose of \( x \). Substituting \( L^v + I_v \) in \( P \) it follows that

\[
\rho_{n-i+1} = \max_{U} \min_{x \in U} \left\{ \frac{x^t (L^v + I_v) x}{x^t x} : U \subseteq \mathbb{R}^n, \dim(U) = i, x \in U = \text{span}(U) \right\}.
\]

Using standard calculus we get

\[
\rho_{n-i+1} \leq \max_{U} \min_{x \in U} \left\{ \frac{x^t L^v x}{x^t x} : U \subseteq \mathbb{R}^n, \dim(U) = i, x \in U = \text{span}(U) \right\}
\]
We now use the previous lemma to get a lower bound on $\lambda_i^v$.

**Lemma 2.3.** For all $v = 1, \ldots, n$ and for all $i = 1, \ldots, n-1$,

$$\lambda_i - 1 \leq \lambda_i^v.$$  

**Proof.** Fix $i \in \{1, \ldots, n-1\}$. From Lemma 2.2 it follows that $\rho_i \leq \lambda_i^v + 1$. Now this lemma follows from substituting the conclusion of Corollary 2.1 into the previous inequality $\lambda_i \leq \rho_i \leq \lambda_i^v + 1$. 

The next lemma provides an upper bound on $\lambda_i^v$.

**Lemma 2.4.** For all $v = 1, \ldots, n$ and for all $i = 1, \ldots, n-1$,

$$\lambda_i^v \leq \lambda_{i+1}.$$  

**Proof.** We prove this lemma by induction on $d_v$, the degree of the node $v$. If the degree is $d_v = 0$, then by removing the node $v$ we reduce the multiplicity of the small eigenvalues, which is 0. Formally $\lambda_i^v = \lambda_{i+1}$ for $i = 1, \ldots, n-1$. Therefore the lemma holds in this case. For the induction step, suppose that the statement holds for $d_v = k$ and consider the case $d_v = k+1$. Since $d_v > 0$ it follows that there exists an edge $e$ connecting the vertex $v$ to some other node $u$. Denote the graph obtained by removing the edge $e$ from the graph $G$ by $X$. Let $\sigma_1 \leq \ldots \leq \sigma_{n-1}$ be the eigenvalues of the Laplacian of the graph $X$. From Theorem 1.1 it follows that $\sigma_i \leq \lambda_i$ for all $i = 1, \ldots, n$. Using induction we obtain that $\lambda_{i+1} \leq \sigma_i \leq \lambda_i$, for all $i = 2, \ldots, n$.

Now we present our main theorem.

**Theorem 2.5.** For all $v = 1, \ldots, n$ and for all $i = 1, \ldots, n-1$,

$$\lambda_i - 1 \leq \lambda_i^v \leq \lambda_{i+1}.$$  

**Proof.** The proof is a direct consequence of Lemmas 2.3 and 2.4.

We remark that both inequalities above are tight. To see that, we show there exist graphs such that $\lambda_i - 1 = \lambda_i^v$. Consider the graph $K_n$. It is well known that the eigenvalues of $K_n$ are $0, n, \ldots, n$, where the multiplicity of the eigenvalue $n$ is $n-1$ and 0 is a simple eigenvalue. Now removing a vertex from $K_n$ produces the graph $K_{n-1}$. Again the eigenvalues of $K_{n-1}$ are $0, n-1, \ldots, n-1$, where the multiplicity of the eigenvalue $n-1$ is $n-2$ and 0 is a simple eigenvalue. To see that there are graphs that satisfy $\lambda_i^v = \lambda_{i+1}$, consider the graph without any edges.

3. **Application to average leafy trees.** In this section we use the weak interlacing Theorem 2.5 to obtain a bound on the average number of leaves in a random spanning tree $F(G)$. Our bound is useful when $\lambda_2 > \alpha n$, for fixed $\alpha > 0$ and $|E| = O(n^2)$. We call such a graph a *dense expander*; in this case we show that the bound is linear in the number of vertices.
It is well known that the smallest eigenvalue of $L$ is 0 and that its corresponding eigenvector is $(1, 1, \ldots, 1)$. If $G$ is connected, all other eigenvalues are greater than 0. Let $P^v$ denote the submatrix of $L$ obtained by deleting the $v$th row and $v$th column. Then, by the Matrix Tree Theorem, for each vertex $v \in V$ we have $t(G) = |\det(P^v)|$, where $t(G)$ is the number of spanning trees of $G$. One can rephrase the Matrix Tree Theorem in terms of the spectrum of the Laplacian matrix. The next theorem appears in [1, p. 284]; it connects the eigenvalues of the Laplacian of $G$ and $t(G)$.

**Theorem 3.1.** Let $G$ be a graph on $n$ vertices and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the Laplacian of $G$. Then the number of labeled spanning trees in $G$ is

$$
\frac{1}{n} \prod_{i=2}^{n} \lambda_i.
$$

Let $G$ be a graph. Using the previous theorem it is possible to define the following probability space: $\Omega(G) = \{T : T$ is a spanning tree in $G\}$. On this set we take a spanning tree in a uniform probability. We are interested in finding the average number of leaves in a random spanning tree. Let $T$ be a random spanning tree taken from $\Omega(G)$ with the uniform distribution. Denote by $F(G)$ the expected number of leaves in $T$. Using the matrix theorem we can get a formula to compute the average number of leaves in a random spanning tree.

**Lemma 3.2.**

$$
F(G) = \sum_{v \in V} nd_v \prod_{i=2}^{n-1} \frac{\lambda_i}{\lambda_i}.
$$

**Proof.** The number of trees that have vertex $v$ as a leaf is $\frac{nd_v \prod_{i=2}^{n-1} \lambda_i}{(n-1) \prod_{i=2}^{n} \lambda_i}$. The lemma follows by summing over all vertices and dividing by the total number of trees. $\square$

The weak interlacing theorem enables us to bound the average number of leaves in a dense expander graph. More precisely, we show that $F(G) = O(n)$.

**Theorem 3.3.** Let $G$ be a graph. If $\lambda_2 > \alpha n$, then the average number of leaves in $T$ is bigger than $\frac{2|E|e^{-\frac{1}{\lambda_n}}}{\lambda_n}$.

**Proof.**

$$
F(G) = \sum_{v \in V} nd_v \prod_{i=2}^{n-1} \frac{\lambda_i}{\lambda_i}
\geq \sum_{v \in V} nd_v \prod_{i=2}^{n-1} \frac{(\lambda_i - 1)}{(n-1) \prod_{i=2}^{n} \lambda_i}
= \sum_{v \in V} nd_v \prod_{i=2}^{n-1} \frac{\lambda_i - 1}{(n-1) \lambda_i}
= \sum_{v \in V} nd_v \prod_{i=2}^{n-1} \frac{(1 - \frac{1}{\lambda_i})}{(n-1) \lambda_i}
\geq \sum_{k \in V} nd_k \left(1 - \frac{1}{\lambda_2}\right)^n
\geq \sum_{k \in V} nd_k (1 - \frac{1}{\lambda_2})^n
\geq \frac{2|E|e^{-\frac{1}{\lambda_n}}}{\lambda_n}.
$$
\[
\begin{align*}
\geq & \frac{2|E|e^{-\frac{n}{2\lambda}}}{\lambda_n} \\
\geq & \frac{2|E|e^{-\frac{1}{\alpha}}}{\lambda_n}.
\end{align*}
\]

**Corollary 3.4.** For any constant \( \alpha > 0 \), if \( \lambda_2 > \alpha n \), and \( |E| = O(n^2) \), then the average number of leaves in \( T \) is \( O(n) \).

**Conclusion.** In this paper we proved a weak interlacing theorem for the Laplacian. Using this theorem we showed that in a dense expander the average number of leaves is \( O(n) \). A natural open question is to show that the average number of leaves in a random tree is an approximation to the maximal spanning leafy tree.

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