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PRODUCTS OF $M$-MATRICES AND NESTED SEQUENCES OF PRINCIPAL MINORS

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Abstract. The question of whether or not the product of two nonsingular $n$-by-$n$ $M$-matrices has a nested sequence of positive principal minors (abbreviated to a nest) is considered. For $n = 2, 3, 4$ such a product always has a nest, and this is conjectured for $n = 5$. For general $n$, examples of products of two $M$-matrices with specified structure are identified as having a leading or trailing nest. For $n = 4$, it is shown that the cube of an $M$-matrix need not have a nest.

Key words. $M$-matrix, Nested sequence of principal minors, $P$-matrix.

AMS subject classifications. 15A15.

1. Introduction. For $C$ an $n$-by-$n$ matrix and $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, we denote by $C[\alpha, \beta]$ the submatrix of $C$ in rows $\alpha$ and columns $\beta$. The principal submatrix $C[\alpha]$ is abbreviated $C[\alpha]$; $\det C[\alpha]$ is a principal minor. The order of such a principal submatrix or minor is $|\alpha|$, the cardinality of $\alpha$. By a nested sequence of principal submatrices (minors) in $C$, we mean those corresponding to a nested sequence of distinct subsets $\alpha_1 \subseteq \alpha_2 \subseteq \ldots \subseteq \alpha_n = \{1, \ldots, n\}$, in which $|\alpha_k| = k$. As each subset in the nest brings in an additional index, we identify a nest via the sequence of indices $i_1, i_2, \ldots, i_n$ where $\alpha_k = \{i_1, i_2, \ldots, i_k\}$. By a leading (trailing) nest, we mean the sequence $1, 2, \ldots, n$ ($n, n - 1, \ldots, 1$). We say that a real $n$-by-$n$ matrix $C$ has a nested sequence of positive principal minors (a nest, for short) if there is a nested sequence $i_1, i_2, \ldots, i_n$ such that $\det C[\{i_1, i_2, \ldots, i_k\}] > 0$ for $k = 1, \ldots, n$. Note that $C[\{i_1, i_2, \ldots, i_k\}]$ is the principal submatrix of $C$ with its rows and columns in the same order as they occur in $C$. A nest clearly requires that $\det C$ be positive.

A matrix with nonpositive off-diagonal entries is called a $Z$-matrix. An $M$-matrix is a square $Z$-matrix that has a nest. Furthermore, an $M$-matrix is a $P$-matrix, i.e., all of its principal minors are positive; see [2, 5] for this and many other equivalent characterizations of $M$-matrices. We are mainly interested here in products of two nonsingular $M$-matrices and whether such a product necessarily has a nest. For $n = 4$, such a product need not be a $P$-matrix [8, Example 3], but a nest is not ruled out by the example there. It has been shown that the product of a nonsingular $M$-matrix and an inverse $M$-matrix does have a nest [9, Theorem 4.6], even though such a product also need not be a $P$-matrix. (This is so for either order of the factors in such a product and one order follows from the other by transposition, not by inversion as stated in [9, proof of Theorem 4.6].) A similar question may be asked for a product

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of two positive definite matrices, but the answer is negative as we show by example later (Example 6.1).

The question of the existence of a nest in the product of two $M$-matrices is quite different from that of an $M$-matrix and an inverse $M$-matrix, and it seems generally quite subtle. Here, our purpose is to popularize this question and to give quite a number of particular results about nests in general, principal minors in the product of two $M$-matrices and special situations in which a product of two $M$-matrices does have a nest. In the process, some interesting questions arise. A number of informative examples are also given.

Part of the motivation for our questions about a nest lies in the fact, due to Fisher and Fuller [3] and Ballantine [1], that if $C$ has a nest, then there is a positive diagonal matrix $D$ so that $DC$ is positive stable. For applications of this fact to negative stability, see [6].

2. Nest Preserving Transformations. If an $n$-by-$n$ matrix $C$ has the nest $i_1, i_2, \ldots, i_n$, then this nest is preserved under transposition and positive diagonal equivalence. In addition, by Jacobi’s theorem, $C^{-1}$ has the nest $i_n, i_{n-1}, \ldots, i_1$, and the permutation similarity $P^T CP$ has a nest as determined by the permutation matrix $P$.

Note that if $C_1C_2$ has a nest, then it is not in general true that $C_2C_1$ has a nest. This is illustrated by the products

$$C_1C_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 1 \end{bmatrix},$$

$$C_2C_1 = \begin{bmatrix} 3 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}$$

in which $C_1C_2$ has the nest 2,1 but $C_2C_1$ has no nest.

The following result shows that if a matrix has a nest, then so does its product with a triangular matrix having positive diagonal entries.

**Lemma 2.1.** Let $L$ and $U$ be lower and upper triangular matrices, respectively, with all diagonal entries positive. If $C$ has a leading nest, then $LC$ and $CU$ have leading nests, whereas $UC$ and $CL$ have trailing nests.

**Proof.** The first statement can be seen by partitioning the matrices so that

$$LC = \begin{bmatrix} L_{11} & 0 \\ l_{21}^T & l_{22} \\ \end{bmatrix} \begin{bmatrix} C_{11} & c_{12} \\ c_{21}^T & c_{22} \end{bmatrix} = \begin{bmatrix} L_{11}C_{11} \\ l_{21}^T C_{11} + l_{22} c_{21}^T \\ l_{21}^T c_{12} + l_{22} c_{22} \\ \end{bmatrix}$$

and using induction. The other statements follow by transposition and/or permutation similarity with the backward identity permutation matrix. $\square$

3. Principal Minors in the Product of Two $M$-matrices. Let $A$ and $B$ be nonsingular $n$-by-$n$ $M$-matrices. Since the sign of every principal minor is preserved by positive diagonal equivalence, in considering the question of whether or not the product $AB$ of two $M$-matrices has a nest, without loss of generality it can be assumed
that all of the main diagonal entries of $A$ and $B$ are one. By the remark in Section 2 that inversion preserves a nest, this question is equivalent to the existence of a nest in the product of two inverse $M$-matrices.

If $A$ is an $M$-matrix, then there exist positive diagonal matrices $D_1, D_2$ such that $D_1A$ is column diagonally dominant and $AD_2$ is row diagonally dominant. Thus, for positive diagonal matrices $D_1$, the product $(D_1AD_2^{-1})(D_2BD_3)$ shows that without loss of generality the $M$-matrix $A$ can be assumed to be both row and column diagonally dominant and the $M$-matrix $B$ row diagonally dominant. Since every $M$-matrix has an $LU$ factorization in which both the lower and upper triangular factors are $M$-matrices, it follows that $AB = L_AL_BL_B$, where these four triangular factors are all $M$-matrices. Furthermore, if $A$ is both row and column diagonally dominant and $B$ is row diagonally dominant, then $L_A$ is column dominant and both of $U_A$ and $U_B$ are row dominant. Consequently, if $U_AL_B$ has a leading nest, then by Lemma 2.1 so does $AB$. It should be noted that although there is no certainty as to whether a nest exists in $AB = L_AL_B$ or to the sequence of indices in such a nest, by Lemma 2.1 there does exist a trailing nest in $U_AL_B$, which is triangularly similar to $AB$.

If the graphs associated with the matrices $A$ and $B$ are restricted, then some sufficient conditions for $AB$ to be a $P$-matrix are given in [7, 8]. The following result identifies some minors that are positive in the product of two arbitrary $M$-matrices.

**Proposition 3.1.** If $A, B$ are nonsingular $n$-by-$n$ $M$-matrices, then all principal minors of orders 1, $n-1$ and $n$ of $AB$ are positive.

**Proof.** The $Z$-matrix sign pattern combined with the positivity of the diagonal entries of $A$ and $B$ shows that all order 1 principal minors of $AB$ are positive. As $det A$ and $det B$ are both positive, it follows that the order $n$ principal minor of $AB$ is positive. Lastly, by the positivity of the diagonal entries in $(AB)^{-1}$ and the positivity of $det AB$, it follows that all order $n-1$ principal minors in $AB$ are positive. □

**Corollary 3.2.** If $n \leq 3$ then the product $AB$ of two nonsingular $n$-by-$n$ $M$-matrices is a $P$-matrix.

For $n = 2$, the product $AB$ is in fact an $M$-matrix. For $n = 4$, it is not necessarily true that $AB$ is a $P$-matrix; see [8, Example 3] in which one minor of order 2 is negative. We now consider order 2 principal minors in the product for general $n$, and first prove two lemmas that are of independent interest. In the first lemma the inequality is entrywise.

**Lemma 3.3.** If $A, B$ are $M$-matrices, then $AB[\alpha] \geq A[\alpha]B[\alpha]$.

**Proof.** Without loss of generality let $\alpha = 1, 2, \ldots, k$. Let

$$A = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & -B_{12} \\ -B_{21} & B_{22} \end{bmatrix}$$

where $A_{11}$, $B_{11}$ are $M$-matrices of order $k$ and $A_{12}$, $A_{21}$, $B_{12}$, $B_{21} \geq 0$. Then the leading principal submatrix of order $k$ in $AB$ is $AB[\alpha] = A_{11}B_{11} + A_{12}B_{21} \geq A_{11}B_{11} = A[\alpha]B[\alpha]$. □

**Lemma 3.4.** If $A, B$ are nonsingular $M$-matrices and $AB[\alpha]$ is a $Z$-matrix, then $AB[\alpha]$ is a nonsingular $M$-matrix.
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Proof. By Lemma 3.3, $AB[\alpha] \geq A[\alpha]B[\alpha]$. Thus if $AB[\alpha]$ is a $Z$-matrix, then the product $A[\alpha]B[\alpha]$ must also be a $Z$-matrix. However, since $A[\alpha]$ and $B[\alpha]$ are both nonsingular $M$-matrices, they are inverse nonnegative and thus $A[\alpha]B[\alpha]$ is a nonsingular $M$-matrix. By [5, Theorem 2.5.4(a)], it follows that $AB[\alpha]$ is a nonsingular $M$-matrix. □

THEOREM 3.5. Let $n \geq 2$ and $A, B$ be nonsingular $M$-matrices. Then the product $AB$ has at least $\lceil \frac{2}{n} \rceil$ positive principal minors of order 2.

Proof. Every principal submatrix of order 2 in $AB$ has one of the following forms:

\[
\begin{bmatrix}
+ & 0 \\
* & +
\end{bmatrix},
\begin{bmatrix}
+ & * \\
0 & +
\end{bmatrix},
\begin{bmatrix}
+ & - \\
+ & +
\end{bmatrix},
\begin{bmatrix}
+ & + \\
- & +
\end{bmatrix},
\begin{bmatrix}
+ & + \\
+ & -
\end{bmatrix}
\]

where * denotes an arbitrary entry. The determinant of a matrix of any of the first four forms is positive. Not all of the principal submatrices of order 2 can be of the fifth form since $AB$ cannot have all entries positive (as $(AB)^{-1}$ has every entry nonnegative). As the last form is a $Z$-matrix, by Lemma 3.4 it must be a nonsingular $M$-matrix and therefore has a positive determinant. Thus $AB$ has at least one positive minor of order 2. The lower bound on the number of positive principal minors of order 2 occurs when $A$ or $B$ is irreducible (in which case $(AB)^{-1}$ has every entry positive and thus every row and column of $AB$ has at least one negative entry) and all principal submatrices of order 2 that contain a negative entry in fact contain two negative entries. Thus $AB$ has at least $\lceil \frac{2}{n} \rceil$ positive principal minors of order 2. □

From numerical evidence, the above lower bound of $\lceil \frac{2}{n} \rceil$ is very conservative. For example, if $n = 5$, we know of no such product $AB$ with fewer than eight positive principal minors of order 2.

COROLLARY 3.6. If $A, B$ are nonsingular $M$-matrices of order 4, then $AB$ has a nest.

Proof. Theorem 3.5 shows that there is at least one positive principal minor of order 2 in the product $AB$. By Proposition 3.1, all principal minors of orders 1, 3 and 4 are positive in $AB$. Therefore $AB$ has a nest. □

In fact, by Theorem 3.5, since for $n = 4$ such a product $AB$ must have at least two positive principal minors of order 2, in this case $AB$ must have at least eight nests. Lemma 3.4 also gives the following result.

THEOREM 3.7. If $A, B$ are nonsingular $M$-matrices and $AB[\alpha]$ is a $Z$-matrix with $|\alpha| = n - 2$, then $AB$ has a nest.

Proof. By Lemma 3.4, $AB[\alpha]$ is an $M$-matrix. The positivity of all principal minors of order $n - 1$ and the determinant of $AB$ complete a nest with the first $n - 2$ indices from $\alpha$. □

EXAMPLE 3.8. Let $A$ and $B$ be the 5-by-5 matrices

\[
A = \begin{bmatrix}
1 & -0.1 & -0.1 & -0.1 & -0.1 \\
-0.1 & 1 & -0.1 & -0.1 & -2 \\
-0.1 & -0.1 & 1 & -1 & -0.1 \\
-0.1 & -0.1 & -0.1 & 1 & -0.1 \\
-0.1 & -0.1 & -0.1 & -0.1 & 1
\end{bmatrix}
\]
Then $A$ and $B$ are both nonsingular $M$-matrices and

$$AB = \begin{bmatrix}
1.04 & -0.08 & 0.02 & -0.17 & -0.17 \\
0.02 & 1.32 & 3.82 & 0.02 & -2.07 \\
-0.08 & 0.82 & 1.32 & -1.07 & -0.08 \\
-0.17 & -1.07 & 0.02 & 1.04 & -0.17 \\
-0.17 & -0.08 & -2.07 & -0.17 & 1.04 \\
\end{bmatrix}.$$ 

Since $AB[145]$ is a $Z$-matrix of order $3$, by Theorem 3.7 $AB$ has a nest. In particular $AB$ has the nest $1, 4, 5, 2, 3$. Note that $\det AB$ is not a $P$-matrix.


We now prove that a nest in the product of two $M$-matrices is guaranteed if one of the factors is a Hessenberg matrix.

**Theorem 4.1.** Let $A, F, G$ be nonsingular $M$-matrices where $F$ is a lower Hessenberg matrix and $G$ is an upper Hessenberg matrix. Then $FA$ and $AG$ contain a leading nest and $GA$ and $AF$ contain a trailing nest.

**Proof.** Let

$$A = \begin{bmatrix}
A_{11} & -A_{12} \\
-A_{21} & A_{22} \\
\end{bmatrix}, \quad F = \begin{bmatrix}
F_{11} & -F_{12} \\
-F_{21} & F_{22} \\
\end{bmatrix}, \quad G = \begin{bmatrix}
G_{11} & -G_{12} \\
-G_{21} & G_{22} \\
\end{bmatrix}$$

where $A_{12}, A_{21}, F_{12}, F_{21}, G_{12}, G_{21} \geq 0$; $A_{11}, F_{11}, G_{11}$ are $M$-matrices of order $k$; $A_{22}, F_{22}, G_{22}$ are $M$-matrices of order $n-k$ and $2 \leq k \leq n-2$. As well $F_{12}$ is $k$-by-$(n-k)$ and $G_{21}$ is $(n-k)$-by-$k$ with

$$F_{12} = \begin{bmatrix}
0 & \cdots & 0 \\
0 & \ddots & 0 \\
f & \cdots & 0 \\
\end{bmatrix}, \quad G_{21} = \begin{bmatrix}
0 & \cdots & 0 & g \\
0 & \ddots & \cdots & \vdots \\
0 & \cdots & 0 \\
\end{bmatrix};$$

where $f, g \geq 0$. Therefore, the leading principal submatrix of order $k$ in $FA$ is $F_{11}A_{11} + F_{12}A_{21}$. The structure of $F_{12}$ ensures that $F_{12}A_{21}$ is a rank one matrix that can be written as $xy^T$, where $x$ is the first column of $F_{12}$, $y^T$ is the first row of $A_{21}$ and $x, y \geq 0$. Therefore, by a well known fact for a rank one perturbation of a matrix,

$$\det(F_{11}A_{11} + F_{12}A_{21}) = \det(F_{11}A_{11} + xy^T) = \det F_{11}A_{11}(1 + y^T[F_{11}A_{11}]^{-1}x).$$

Since $F_{11}, A_{11}$ are nonsingular $M$-matrices, it follows that $\det F_{11}, \det A_{11} > 0$ and therefore $\det F_{11}A_{11} > 0$. As well $[F_{11}A_{11}]^{-1} \geq 0$. Thus $(1 + y^T[F_{11}A_{11}]^{-1}x) > 0$ and consequently $\det(F_{11}A_{11} + F_{12}A_{21}) > 0$. As this is true for all $k$ and by Proposition
3.1 all principal minors of orders 1, \(n-1\) and \(n\) of \(FA\) are positive, it follows that \(FA\) contains a leading nest. The other statements follow by transposition and/or permutation similarity with the backward identity permutation matrix. \(\blacksquare\)

The following example shows that a product \(FA\) (as in Theorem 4.1) need not be a \(P\)-matrix.

**Example 4.2.**

\[
F = \begin{bmatrix}
1 & -0.1 & 0 & 0 \\
-0.1 & 1 & -0.1 & 0 \\
-0.1 & 2 & 1 & -0.1 \\
-2 & -0.1 & -0.1 & 1
\end{bmatrix}, \quad A = \begin{bmatrix}
1 & -0.1 & -2 & -0.1 \\
-0.1 & 1 & -0.1 & -2 \\
-0.1 & -0.1 & 1 & -0.1 \\
-0.1 & -0.1 & -0.1 & 1
\end{bmatrix}
\]

are both \(M\)-matrices, but

\[
FA = \begin{bmatrix}
0.01 & -0.2 & -1.99 & 0.1 \\
-0.19 & 1.02 & 0 & -1.98 \\
0.01 & -2.08 & 1.41 & 3.81 \\
-2.08 & 0.01 & 3.81 & 1.41
\end{bmatrix}
\]

is not a \(P\)-matrix as \(\det FA[3, 4] < 0\).

Corollary 3.2 and induction are now used to identify another \(M\)-matrix product \(FG\) that has a nest. The matrix \(F\) is lower Hessenberg with an extra diagonal immediately above the superdiagonal and \(G\) is upper Hessenberg with an extra diagonal immediately below the subdiagonal. Such a product \(FG\) includes the product of two pentadiagonal \(M\)-matrices.

**Theorem 4.3.** Let \(n \geq 3\) and \(F = [f_{ij}], \ G = [g_{ij}]\) be nonsingular \(n\times n\) \(M\)-matrices such that \(f_{ij} = 0\) if \(j - i \geq 3\) and \(g_{ij} = 0\) if \(i - j \geq 3\). Then \(FG\) contains a leading nest and \(GF\) contains a trailing nest.

**Proof.** We use a proof by induction to show that \(FG\) contains a leading nest. By Corollary 3.2, when \(n = 3\), \(FG\) is a \(P\)-matrix, and thus contains a leading nest.

Now suppose \(k \geq 4\) and the statement is true for all \(m \in \mathbb{Z}^+\) such that \(3 \leq m < k\). For \(n = k\), we know that the order \(k - 1\) principal minors of \(FG\) are positive and \(\det FG > 0\). We can partition \(F\) and \(G\) as follows:

\[
F = \begin{bmatrix}
F_{k-1} & -a \\
-b^T & f_{kk}
\end{bmatrix}, \quad G = \begin{bmatrix}
G_{k-1} & -c \\
-d^T & g_{kk}
\end{bmatrix}
\]

where \(a, d \geq 0\), \(a^T = [0, 0, \ldots, a_{k-2}, a_{k-1}]\) and \(d^T = [0, 0, \ldots, d_{k-2}, d_{k-1}]\). Therefore the leading principal minor of order \(k - 1\) in \(FG\) is \(F_{k-1}G_{k-1} + ad^T\) and

\[
ad^T = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & a_{k-2}d_{k-2} & a_{k-2}d_{k-1} \\
0 & \cdots & a_{k-1}d_{k-2} & a_{k-1}d_{k-1}
\end{bmatrix}
\]

Since \(F_{k-1}G_{k-1}\) contains a leading nest by the induction hypothesis, all leading principal minors of orders 1 to \(k - 3\) in \(F_{k-1}G_{k-1} + ad^T\) and consequently in \(FG\) are
positive. As well, since \( F_{k-1}G_{k-1} + ad^T \) is an order \( k-1 \) principal submatrix of \( FG \), its determinant is positive. It is thus only the leading principal minor of order \( k-2 \) in \( FG \) that must be considered, namely \( \det FG[1, \ldots, k-2] \). The only change from the order \( k-2 \) leading principal submatrix of \( F_{k-1}G_{k-1} \) to obtain the order \( k-2 \) leading principal submatrix of \( F_{k-1}G_{k-1} + ad^T \) is the nonnegative addition of \( a_{k-2}d_{k-2} \) to the last main diagonal entry of the order \( k-2 \) leading principal submatrix in \( F_{k-1}G_{k-1} \). By the induction hypothesis it follows that the complementary minor of this diagonal entry in the leading principal submatrix of order \( k-2 \) in \( F_{k-1}G_{k-1} \) must be positive since it is a leading principal minor of order \( k-3 \) in \( FG \). Similarly, the induction hypothesis ensures that the leading principal minor of order \( k-2 \) in \( F_{k-1}G_{k-1} \) is positive because it is a leading principal minor of order less than \( k \). By linearity of the determinant, the nonnegative addition to the diagonal entry leaves the sign of the order \( k-2 \) minor in \( F_{k-1}G_{k-1} + ad^T \) positive. Therefore, \( FG \) contains a leading nest for \( n \). Thus by induction, \( FG \) contains a leading nest for \( n \geq 3 \). The other statement follows by permutation similarity with the backward identity similarity permutation matrix. \( \square \)

The next example shows that if the structure of \( G \) is slightly more general than that of Theorem 4.3, then \( FG \) need not have a leading nest.

**Example 4.4.** Consider the 4-by-4 \( M \)-matrices

\[
F = \begin{bmatrix}
1 & -0.1 & -2 & 0 \\
-0.1 & 1 & -0.1 & -2 \\
-0.1 & -0.1 & 1 & -0.1 \\
-0.1 & -0.1 & -0.1 & 1
\end{bmatrix}, \quad
G = \begin{bmatrix}
1 & -0.1 & -0.1 & -0.1 \\
-0.1 & 1 & -0.1 & -0.1 \\
-0.1 & -2 & 1 & -0.1 \\
-2 & -0.1 & -0.1 & 1
\end{bmatrix}.
\]

Then

\[
FG = \begin{bmatrix}
1.21 & 3.8 & -2.09 & 0.11 \\
3.81 & 1.41 & 0.01 & -2.08 \\
0.01 & -2.08 & 1.03 & -0.18 \\
-2.08 & 0.01 & -0.18 & 1.03
\end{bmatrix}
\]

does not contain a leading nest as \( FG[1, 2] < 0 \). This example does, however, have a trailing nest.

**5. Products of More than Two \( M \)-Matrices.** The previous two sections discuss nests in the product of two \( M \)-matrices, and we now consider more general products. If \( A_i \) are 2-by-2 nonsingular \( M \)-matrices, then \( \prod_{i=1}^k A_i \) is a nonsingular \( M \)-matrix (and hence has a nest) for all positive integers \( k \). We have a more restrictive result for 3-by-3 \( M \)-matrix products.

**Theorem 5.1.** If \( A \) is a 3-by-3 nonsingular \( M \)-matrix, then \( A^3 \) has a nest.

**Proof.** Since \( \det A^3 > 0 \) and \( (A^3)^{-1} \) has positive diagonal entries, all order 2 and order 3 principal minorms of \( A^3 \) are positive. It remains to show that \( A^3 \) has at least one positive diagonal entry, which is certainly true if trace \( A^3 \) is positive. This is now proved by considering eigenvalues. Since \( A \) is a nonsingular \( M \)-matrix, it has eigenvalues \( a, b, c \) or \( a, be^{\pm i\theta} \) where \( a, b > 0 \), and \( 0 < \theta < \frac{\pi}{2}, \frac{\pi}{2} = \frac{\pi}{6} \) [5, Theorem 2.5.9(b)]. In the first case, \( A^3 \) has eigenvalues \( a^3, b^3, c^3 \); thus trace
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\[ A^3 = a^3 + b^3 + c^3 > 0. \] In the second case, \( A^3 \) has eigenvalues \( a^3, b^3 e^{\pm 3i\theta} \), giving trace \( A^3 = a^3 + 2b^3 \cos 3\theta \). By the bounds on \( \theta \), \( 0 < 3\theta < \frac{\pi}{2} \), thus \( \cos 3\theta > 0 \) and trace \( A^3 > 0. \)

However, if \( A \) is as in Theorem 5.1, then \( A^3 \) need not be a \( P \)-matrix [8, Example 2]. In fact, the following example shows that \( A^3 \) may have two negative diagonal entries.

**Example 5.2.**

\[ A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -1 & 10 & -3 \\ -13 & 25 & 5 \\ 10 & -26 & -1 \end{bmatrix}. \]

The next example shows that the result of Theorem 5.1 need not be true if \( A \) is a 4-by-4 \( M \)-matrix.

**Example 5.3.** Consider the 4-by-4 circulant \( M \)-matrix

\[ A = \begin{bmatrix} 1 & -0.9 & 0 & 0 \\ 0 & 1 & -0.9 & 0 \\ 0 & 0 & 1 & -0.9 \\ -0.9 & 0 & 0 & 1 \end{bmatrix}. \]

Then

\[ A^3 = \begin{bmatrix} 1 & -2.7 & 2.43 & -0.729 \\ -0.729 & 1 & -2.7 & 2.43 \\ 2.43 & -0.729 & 1 & -2.7 \\ -2.7 & 2.43 & -0.729 & 1 \end{bmatrix} \]

does not contain a nest as all the principal minors of order 2 in \( A^3 \) are negative. If \( A \) is a symmetric or tridiagonal \( M \)-matrix, then \( A^k \) is a \( P \)-matrix for all positive integers \( k \) [8, Theorem 1 and Lemma 2].

**6. Discussion.** As stated in Corollary 3.2, the product of two nonsingular 3-by-3 \( M \)-matrices is a \( P \)-matrix (and thus has a nest). In contrast, the following example shows that the product of two 3-by-3 positive definite matrices may not even have a nest.

**Example 6.1.** The matrix

\[ C = \begin{bmatrix} 6 & -1 & 0 \\ 0 & 0 & -1 \\ -60 & 11 & 0 \end{bmatrix} \]

has three distinct positive eigenvalues, namely \( \{1, 2, 3\} \), and thus is the product of two positive definite matrices (see, e.g., [4, Problem 9, p. 468]). Note that \( C \) does not have a nest.

The product of two nonsingular 4-by-4 \( M \)-matrices has a nest (Corollary 3.6) but is not necessarily a \( P \)-matrix. Extensive numerical calculations on the product of two 5-by-5 nonsingular \( M \)-matrices lead us to conjecture that such a product has a nest.
However, we do not know how to ensure the existence of a positive 3-by-3 minor, nor how to determine the positions of the positive 2-by-2 principal minors in the product of the two $M$-matrices.

As mentioned in the Introduction, if a matrix has a nest then it can be stabilized by premultiplication with a positive diagonal matrix. Our results thus give products of certain $M$-matrices that can be stabilized in this way. Example 4.2 illustrates this, as $FA$ has two eigenvalues with negative real parts. However, since $FA$ has a leading nest, there exists a positive diagonal matrix $D$ so that $DFA$ is positive stable.

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**REFERENCES**


