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PRODUCTS OF M-MATRICES AND NESTED SEQUENCES OF
PRINCIPAL MINORS

D.A. GRUNDY†, C.R. JOHNSON‡, D.D. OLESKY§, AND P. VAN DEN DRIESSCHE†

Abstract. The question of whether or not the product of two nonsingular n-by-n M-matrices has
a nested sequence of positive principal minors (abbreviated to a nest) is considered. For
n = 2, 3, 4 such a product always has a nest, and this is conjectured for n = 5. For general n,
examples of products of two M-matrices with specified structure are identified as having a leading or trailing
nest. For n = 4, it is shown that the cube of an M-matrix need not have a nest.

Key words. M-matrix, Nested sequence of principal minors, P-matrix.

AMS subject classifications. 15A15.

1. Introduction. For C an n-by-n matrix and α, β ⊆ {1, 2, ..., n}, we denote by
C[α, β] the submatrix of C in rows α and columns β. The principal submatrix C[α, α] is abbreviated C[α]; det C[α] is a principal minor. The order of such a principal
submatrix or minor is |α|, the cardinality of α. By a nested sequence of principal
submatrices (minors) in C, we mean those corresponding to a nested sequence of
distinct subsets α1 ⊆ α2 ⊆ ... ⊆ αn = {1, ..., n}, in which |αk| = k. As each
subset in the nest brings in an additional index, we identify a nest via the sequence of
indices i1, i2, ..., in where αk = {i1, i2, ..., ik}. By a leading (trailing) nest, we mean
the sequence 1, 2, ..., n (n, n − 1, ..., 1). We say that a real n-by-n matrix C has
a nested sequence of positive principal minors (a nest, for short) if there is a nested
sequence isuch that det C[1, 1, ..., 1] > 0 for k = 1, ..., n. Note that
C[1, 1, ..., 1] is the principal submatrix of C with its rows and columns in the same
order as they occur in C. A nest clearly requires that det C be positive.

A matrix with nonpositive off-diagonal entries is called a Z-matrix. An M-matrix
is a square Z-matrix that has a nest. Furthermore, an M-matrix is a P-matrix, i.e.,
all of its principal minors are positive; see [2, 5] for this and many other equivalent
characterizations of M-matrices. We are mainly interested here in products of two
nonsingular M-matrices and whether such a product necessarily has a nest. For n = 4,
such a product need not be a P-matrix [8, Example 3], but a nest is not ruled out by
the example there. It has been shown that the product of a nonsingular M-matrix and
an inverse M-matrix does have a nest [9, Theorem 4.6], even though such a product
also need not be a P-matrix. (This is so for either order of the factors in such a
product and one order follows from the other by transposition, not by inversion as
stated in [9, proof of Theorem 4.6].) A similar question may be asked for a product
of two positive definite matrices, but the answer is negative as we show by example later (Example 6.1).

The question of the existence of a nest in the product of two $M$-matrices is quite different from that of an $M$-matrix and an inverse $M$-matrix, and it seems generally quite subtle. Here, our purpose is to popularize this question and to give quite a number of particular results about nests in general, principal minors in the product of two $M$-matrices and special situations in which a product of two $M$-matrices does have a nest. In the process, some interesting questions arise. A number of informative examples are also given.

Part of the motivation for our questions about a nest lies in the fact, due to Fisher and Fuller [3] and Ballantine [1], that if $C$ has a nest, then there is a positive diagonal matrix $D$ so that $DC$ is positive stable. For applications of this fact to negative stability, see [6].

2. Nest Preserving Transformations. If an $n$-by-$n$ matrix $C$ has the nest $i_1, i_2, \ldots, i_n$, then this nest is preserved under transposition and positive diagonal equivalence. In addition, by Jacobi's theorem, $C^{-1}$ has the nest $i_n, i_{n-1}, \ldots, i_1$, and the permutation similarity $P^T C P$ has a nest as determined by the permutation matrix $P$.

Note that if $C_1 C_2$ has a nest, then it is not in general true that $C_2 C_1$ has a nest. This is illustrated by the products

\[
C_1 C_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 1 \end{bmatrix},
\]

\[
C_2 C_1 = \begin{bmatrix} 3 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix},
\]

in which $C_1 C_2$ has the nest 2,1 but $C_2 C_1$ has no nest.

The following result shows that if a matrix has a nest, then so does its product with a triangular matrix having positive diagonal entries.

**Lemma 2.1.** Let $L$ and $U$ be lower and upper triangular matrices, respectively, with all diagonal entries positive. If $C$ has a leading nest, then $LC$ and $CU$ have leading nests, whereas $UC$ and $CL$ have trailing nests.

**Proof.** The first statement can be seen by partitioning the matrices so that

\[
LC = \begin{bmatrix} L_{11} & 0 \\ l_{21}^T & l_{22} \end{bmatrix} \begin{bmatrix} C_{11} & c_{12} \\ c_{21}^T & c_{22} \end{bmatrix} = \begin{bmatrix} L_{11} C_{11} & L_{11} c_{12} \\ l_{21}^T C_{11} + l_{22} c_{21}^T & l_{21}^T c_{12} + l_{22} c_{22} \end{bmatrix},
\]

and using induction. The other statements follow by transposition and/or permutation similarity with the backward identity permutation matrix. \(\square\)

3. Principal Minors in the Product of Two $M$-matrices. Let $A$ and $B$ be nonsingular $n$-by-$n$ $M$-matrices. Since the sign of every principal minor is preserved by positive diagonal equivalence, in considering the question of whether or not the product $AB$ of two $M$-matrices has a nest, without loss of generality it can be assumed
that all of the main diagonal entries of $A$ and $B$ are one. By the remark in Section 2 that inversion preserves a nest, this question is equivalent to the existence of a nest in the product of two inverse $M$-matrices.

If $A$ is an $M$-matrix, then there exist positive diagonal matrices $D_1, D_2$ such that $D_1 A$ is column diagonally dominant and $AD_2$ is row diagonally dominant. Thus, for positive diagonal matrices $D_i$, the product $(D_1 AD_2^{-1})(D_2 B D_3)$ shows that without loss of generality the $M$-matrix $A$ can be assumed to be both row and column diagonally dominant and the $M$-matrix $B$ row diagonally dominant. Since every $M$-matrix has an $LU$ factorization in which both the lower and upper triangular factors are $M$-matrices, it follows that $AB = L_A U_A L_B U_B$, where these four triangular factors are all $M$-matrices. Furthermore, if $A$ is both row and column diagonally dominant and $B$ is row diagonally dominant, then $L_A$ is column dominant and both of $U_A$ and $U_B$ are row dominant. Consequently, if $U_A L_B$ has a leading nest, then by Lemma 2.1 so does $AB$. It should be noted that although there is no certainty as to whether a nest exists in $AB = L_A U_A B$ or to the sequence of indices in such a nest, by Lemma 2.1 there does exist a trailing nest in $U_A B L_A$, which is triangularly similar to $AB$.

If the graphs associated with the matrices $A$ and $B$ are restricted, then some sufficient conditions for $AB$ to be a $P$-matrix are given in [7, 8]. The following result identifies some minors that are positive in the product of two arbitrary $M$-matrices.

**Proposition 3.1.** If $A, B$ are nonsingular $n$-by-$n$ $M$-matrices, then all principal minors of orders 1, $n - 1$ and $n$ of $AB$ are positive.

**Proof.** The $Z$-matrix sign pattern combined with the positivity of the diagonal entries of $A$ and $B$ shows that all order 1 principal minors of $AB$ are positive. As $\det A$ and $\det B$ are both positive, it follows that the order $n$ principal minor of $AB$ is positive. Lastly, by the positivity of the diagonal entries in $(AB)^{-1}$ and the positivity of $\det AB$, it follows that all order $n - 1$ principal minors in $AB$ are positive.

**Corollary 3.2.** If $n \leq 3$ then the product $AB$ of two nonsingular $n$-by-$n$ $M$-matrices is a $P$-matrix.

For $n = 2$, the product $AB$ is in fact an $M$-matrix. For $n = 4$, it is not necessarily true that $AB$ is a $P$-matrix; see [8, Example 3] in which one minor of order 2 is negative.

We now consider order 2 principal minors in the product for general $n$, and first prove two lemmas that are of independent interest. In the first lemma the inequality is entrywise.

**Lemma 3.3.** If $A, B$ are $M$-matrices, then $AB[\alpha] \geq A[\alpha]B[\alpha]$.

**Proof.** Without loss of generality let $\alpha = 1, 2, \ldots, k$. Let

$$
A = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & -B_{12} \\ -B_{21} & B_{22} \end{bmatrix}
$$

where $A_{11}, B_{11}$ are $M$-matrices of order $k$ and $A_{12}, A_{21}, B_{12}, B_{21} \geq 0$. Then the leading principal submatrix of order $k$ in $AB$ is $AB[\alpha] = A_{11}B_{11} + A_{12}B_{21} \geq A_{11}B_{11} = A[\alpha]B[\alpha]$.

**Lemma 3.4.** If $A, B$ are nonsingular $M$-matrices and $AB[\alpha]$ is a $Z$-matrix, then $AB[\alpha]$ is a nonsingular $M$-matrix.
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**Proof.** By Lemma 3.3, $AB[\alpha] \geq A[\alpha]B[\alpha]$. Thus if $AB[\alpha]$ is a $Z$-matrix, then the product $A[\alpha]B[\alpha]$ must also be a $Z$-matrix. However, since $A[\alpha]$ and $B[\alpha]$ are both nonsingular $M$-matrices, they are inverse nonnegative and thus $A[\alpha]B[\alpha]$ is a nonsingular $M$-matrix. By [5, Theorem 2.5.4(a)], it follows that $AB[\alpha]$ is a nonsingular $M$-matrix. \[\square\]

**Theorem 3.5.** Let $n \geq 2$ and $A, B$ be nonsingular $M$-matrices. Then the product $AB$ has at least $\lceil \frac{n}{2} \rceil$ positive principal minors of order 2.

**Proof.** Every principal submatrix of order 2 in $AB$ has one of the following forms:

$$\begin{bmatrix} + & 0 \\ * & + \end{bmatrix}, \begin{bmatrix} + & * \\ 0 & + \end{bmatrix}, \begin{bmatrix} + & - \\ * & + \end{bmatrix}, \begin{bmatrix} + & + \\ * & + \end{bmatrix}, \begin{bmatrix} + & + \\ - & + \end{bmatrix}, \begin{bmatrix} + & + \\ + & + \end{bmatrix}$$

where $*$ denotes an arbitrary entry. The determinant of a matrix of any of the first four forms is positive. Not all of the principal submatrices of order 2 can be of the fifth form since $AB$ cannot have all entries positive (as $(AB)^{-1}$ has every entry nonnegative). As the last form is a $Z$-matrix, by Lemma 3.4 it must be a nonsingular $M$-matrix and therefore has a positive determinant. Thus $AB$ has at least one positive minor of order 2. The lower bound on the number of positive principal minors of order 2 occurs when $A$ or $B$ is irreducible (in which case $(AB)^{-1}$ has every entry positive and thus every row and column of $AB$ has at least one negative entry) and all principal submatrices of order 2 that contain a negative entry in fact contain two negative entries. Thus $AB$ has at least $\lceil \frac{n}{2} \rceil$ positive principal minors of order 2. \[\square\]

From numerical evidence, the above lower bound of $\lceil \frac{n}{2} \rceil$ is very conservative. For example, if $n = 5$, we know of no such product $AB$ with fewer than eight positive principal minors of order 2.

**Corollary 3.6.** If $A, B$ are nonsingular $M$-matrices of order 4, then $AB$ has a nest.

**Proof.** Theorem 3.5 shows that there is at least one positive principal minor of order 2 in the product $AB$. By Proposition 3.1, all principal minors of orders 1, 3 and 4 are positive in $AB$. Therefore $AB$ has a nest. \[\square\]

In fact, by Theorem 3.5, since for $n = 4$ such a product $AB$ must have at least two positive principal minors of order 2, in this case $AB$ must have at least eight nests. Lemma 3.4 also gives the following result.

**Theorem 3.7.** If $A, B$ are nonsingular $M$-matrices and $AB[\alpha]$ is a $Z$-matrix with $|\alpha| = n - 2$, then $AB$ has a nest.

**Proof.** By Lemma 3.4, $AB[\alpha]$ is an $M$-matrix. The positivity of all principal minors of order $n - 1$ and the determinant of $AB$ complete a nest with the first $n - 2$ indices from $\alpha$. \[\square\]

**Example 3.8.** Let $A$ and $B$ be the 5-by-5 matrices

\[
A = \begin{bmatrix}
1 & -0.1 & -0.1 & -0.1 & -0.1 \\
-0.1 & 1 & -0.1 & -0.1 & -2 \\
-0.1 & -0.1 & 1 & -1 & -0.1 \\
-0.1 & -0.1 & -0.1 & 1 & -0.1 \\
-0.1 & -0.1 & -0.1 & -0.1 & 1
\end{bmatrix},
\]
Then $A$ and $B$ are both nonsingular $M$-matrices and

$$AB = \begin{bmatrix} 1.04 & -0.08 & 0.02 & -0.17 & -0.17 \\ 0.02 & 1.32 & 3.82 & 0.02 & -2.07 \\ -0.08 & 0.82 & 1.32 & -1.07 & -0.08 \\ -0.17 & -1.07 & 0.02 & 1.04 & -0.17 \\ -0.17 & -0.08 & -2.07 & -0.17 & 1.04 \end{bmatrix}.$$ 

Since $AB[145]$ is a $Z$-matrix of order 3, by Theorem 3.7 $AB$ has a nest. In particular $AB$ has the nest 1,4,5,2,3. Note that $det(AB)$ is a lower Hessenberg matrix.


We now prove that a nest in the product of two $M$-matrices is guaranteed if one of the factors is a Hessenberg matrix.

**Theorem 4.1.** Let $A, F, G$ be nonsingular $M$-matrices where $F$ is a lower Hessenberg matrix and $G$ is an upper Hessenberg matrix. Then $FA$ and $AG$ contain a leading nest and $GA$ and $AF$ contain a trailing nest.

**Proof.** Let

$$A = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}, \quad F = \begin{bmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{bmatrix}$$

where $A_{12}, A_{21}, F_{12}, F_{21}, G_{12}, G_{21} \geq 0$; $A_{11}, F_{11}, G_{11}$ are $M$-matrices of order $k$; $A_{22}, F_{22}, G_{22}$ are $M$-matrices of order $n - k$ and $2 \leq k \leq n - 2$. As well $F_{12}$ is $k$-by-$(n - k)$ and $G_{21}$ is $(n - k)$-by-$k$ with

$$F_{12} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & f \end{bmatrix}, \quad G_{21} = \begin{bmatrix} 0 & \cdots & 0 & g \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$$

where $f, g \geq 0$. Therefore, the leading principal submatrix of order $k$ in $FA$ is $F_{11}A_{11} + F_{12}A_{21}$. The structure of $F_{12}$ ensures that $F_{12}A_{21}$ is a rank one matrix that can be written as $xy^T$, where $x$ is the first column of $F_{12}$, $y^T$ is the first row of $A_{21}$ and $x, y \geq 0$. Therefore, by a well-known fact for a rank one perturbation of a matrix,

$$det(F_{11}A_{11} + F_{12}A_{21}) = det(F_{11}A_{11} + xy^T) = det(F_{11}A_{11}(1 + y^T[F_{11}A_{11}]^{-1}x)).$$

Since $F_{11}$, $A_{11}$ are nonsingular $M$-matrices, it follows that $det(F_{11}), det(A_{11}) > 0$ and therefore $det(F_{11}A_{11}) > 0$. As well $[F_{11}A_{11}]^{-1} \geq 0$. Thus $(1 + y^T[F_{11}A_{11}]^{-1}x) > 0$ and consequently $det(F_{11}A_{11} + F_{12}A_{21}) > 0$. As this is true for all $k$ and by Proposition
3.1 all principal minors of orders 1, n - 1 and n of FA are positive, it follows that FA contains a leading nest. The other statements follow by transposition and/or permutation similarity with the backward identity permutation matrix. \[\square\]

The following example shows that a product FA (as in Theorem 4.1) need not be a P-matrix.

**Example 4.2.**

\[ F = \begin{bmatrix} 1 & -0.1 & 0 & 0 \\ -0.1 & 1 & -0.1 & 0 \\ -0.1 & -2 & 1 & -0.1 \\ -2 & -0.1 & -0.1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -0.1 & -2 & -0.1 \\ -0.1 & 1 & -0.1 & -2 \\ -0.1 & -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & -0.1 & 1 \end{bmatrix} \]

are both M-matrices, but

\[ FA = \begin{bmatrix} 1.01 & -0.2 & -1.99 & 0.1 \\ -0.19 & 1.02 & 0 & -1.98 \\ 0.01 & -2.08 & 1.41 & 3.81 \\ -2.08 & 0.01 & 3.81 & 1.41 \end{bmatrix} \]

is not a P-matrix as \( \det FA[3, 4] < 0 \).

Corollary 3.2 and induction are now used to identify another M-matrix product FG that has a nest. The matrix F is lower Hessenberg with an extra diagonal immediately above the superdiagonal and G is upper Hessenberg with an extra diagonal immediately below the subdiagonal. Such a product FG includes the product of two pentadiagonal M-matrices.

**Theorem 4.3.** Let \( n \geq 3 \) and \( F = [f_{ij}] \), \( G = [g_{ij}] \) be nonsingular \( n \times n \) M-matrices such that \( f_{ij} = 0 \) if \( j - i \geq 3 \) and \( g_{ij} = 0 \) if \( i - j \geq 3 \). Then \( FG \) contains a leading nest and \( GF \) contains a trailing nest.

**Proof.** We use a proof by induction to show that \( FG \) contains a leading nest. By Corollary 3.2, when \( n = 3 \), \( FG \) is a P-matrix, and thus contains a leading nest.

Now suppose \( k \geq 4 \) and the statement is true for all \( m \in \mathbb{Z}^+ \) such that \( 3 \leq m < k \). For \( n = k \), we know that the order \( k - 1 \) principal minors of \( FG \) are positive and \( \det FG > 0 \). We can partition \( F \) and \( G \) as follows:

\[ F = \begin{bmatrix} F_{k-1} & -a \\ -b^T & f_{kk} \end{bmatrix}, \quad G = \begin{bmatrix} G_{k-1} & -c \\ -d^T & g_{kk} \end{bmatrix} \]

where \( a, d \geq 0 \), \( a^T = [0, 0, \ldots, a_{k-2}, a_{k-1}] \) and \( d^T = [0, 0, \ldots, d_{k-2}, d_{k-1}] \). Therefore the leading principal minor of order \( k - 1 \) in \( FG \) is \( F_{k-1}G_{k-1} + ad^T \) and

\[ ad^T = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{k-2}d_{k-2} & a_{k-2}d_{k-1} \\ \cdots & \cdots & 0 & a_{k-1}d_{k-2} & a_{k-1}d_{k-1} \end{bmatrix}. \]

Since \( F_{k-1}G_{k-1} \) contains a leading nest by the induction hypothesis, all leading principal minors of orders 1 to \( k - 3 \) in \( F_{k-1}G_{k-1} + ad^T \) and consequently in \( FG \) are
positive. As well, since $F_{k-1}G_{k-1} + ad^T$ is an order $k - 1$ principal submatrix of $FG$, its determinant is positive. It is thus only the leading principal minor of order $k - 2$ in $FG$ that must be considered, namely $det F[1, \ldots, k - 2]$. The only change from the order $k - 2$ leading principal submatrix of $F_{k-1}G_{k-1}$ to obtain the order $k - 2$ leading principal submatrix of $F_{k-1}G_{k-1} + ad^T$ is the nonnegative addition of $a_{k-2}d_{k-2}$ to the last main diagonal entry of the order $k - 2$ leading principal submatrix in $F_{k-1}G_{k-1}$. By the induction hypothesis it follows that the complementary minor of this diagonal entry in the leading principal submatrix of order $k - 2$ in $F_{k-1}G_{k-1}$ must be positive since it is a leading principal minor of order $k - 3$ in $FG$. Similarly, the induction hypothesis ensures that the leading principal minor of order $k - 2$ in $F_{k-1}G_{k-1}$ is positive because it is a leading principal minor of order less than $k$. By linearity of the determinant, the nonnegative addition to the diagonal entry leaves the sign of the order $k - 2$ minor in $F_{k-1}G_{k-1} + ad^T$ positive. Therefore, $FG$ contains a leading nest for $n = k$. Thus by induction, $FG$ contains a leading nest for $n \geq 3$. The other statement follows by permutation similarity with the backward identity permutation matrix. \[ \]

The next example shows that if the structure of $G$ is slightly more general than that of Theorem 4.3, then $FG$ need not have a leading nest.

**Example 4.4.** Consider the 4-by-4 $M$-matrices

\[
F = \begin{bmatrix}
1 & -0.1 & -2 & 0 \\
-0.1 & 1 & -0.1 & -2 \\
-0.1 & -0.1 & 1 & -0.1 \\
-0.1 & -0.1 & -0.1 & 1
\end{bmatrix}, \quad G = \begin{bmatrix}
1 & -0.1 & -0.1 & -0.1 \\
-0.1 & 1 & -0.1 & -0.1 \\
-0.1 & -2 & 1 & -0.1 \\
-2 & -0.1 & -0.1 & 1
\end{bmatrix}.
\]

Then

\[
FG = \begin{bmatrix}
1.21 & 3.8 & -2.09 & 0.11 \\
3.81 & 1.41 & 0.01 & -2.08 \\
0.01 & -2.08 & 1.03 & -0.18 \\
-2.08 & 0.01 & -0.18 & 1.03
\end{bmatrix}
\]

does not contain a leading nest as $FG[1, 2] < 0$. This example does, however, have a trailing nest.

**5. Products of More than Two $M$-Matrices.** The previous two sections discuss nests in the product of two $M$-matrices, and we now consider more general products. If $A_i$ are 2-by-2 nonsingular $M$-matrices, then $\prod_{i=1}^{k} A_i$ is a nonsingular $M$-matrix (and hence has a nest) for all positive integers $k$. We have a more restrictive result for 3-by-3 matrix products.

**Theorem 5.1.** If $A$ is a 3-by-3 nonsingular $M$-matrix, then $A^3$ has a nest.

**Proof.** Since $det A^3 > 0$ and $(A^3)^{-1}$ has positive diagonal entries, all order 2 and order 3 principal minors of $A^3$ are positive. It remains to show that $A^3$ has at least one positive diagonal entry, which is certainly true if trace $A^3$ is positive. This is now proved by considering eigenvalues. Since $A$ is a nonsingular $M$-matrix, it has eigenvalues $a$, $b$, $c$ or $a$, $b$, $c$ or $a$, $b$, $c$ where $a$, $b$, $c > 0$, and $0 < \theta < \frac{\pi}{2} - \frac{\theta}{2} = \frac{\theta}{2}$ [5, Theorem 2.5.9(b)]. In the first case, $A^3$ has eigenvalues $a^3$, $b^3$, $c^3$; thus trace
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$A^3 = a^3 + b^3 + c^3 > 0$. In the second case, $A^3$ has eigenvalues $a^3$, $b^3 e^{ \pm 3i\theta}$, giving trace $A^3 = a^3 + 2b^3 \cos 3\theta$. By the bounds on $\theta$, $0 < 3\theta < \frac{\pi}{2}$, thus $\cos 3\theta > 0$ and trace $A^3 > 0$. □

However, if $A$ is as in Theorem 5.1, then $A^3$ need not be a $P$-matrix [8, Example 2]. In fact, the following example shows that $A^3$ may have two negative diagonal entries.

Example 5.2.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -1 & 10 & -3 \\ -13 & 25 & 5 \\ 10 & -26 & -1 \end{bmatrix}.$$ 

The next example shows that the result of Theorem 5.1 need not be true if $A$ is a 4-by-4 $M$-matrix.

Example 5.3. Consider the 4-by-4 circulant $M$-matrix

$$A = \begin{bmatrix} 1 & -0.9 & 0 & 0 \\ 0 & 1 & -0.9 & 0 \\ 0 & 0 & 1 & -0.9 \\ -0.9 & 0 & 0 & 1 \end{bmatrix}.$$ 

Then

$$A^3 = \begin{bmatrix} 1 & -2.7 & 2.43 & -0.729 \\ -0.729 & 1 & -2.7 & 2.43 \\ 2.43 & -0.729 & 1 & -2.7 \\ -2.7 & 2.43 & -0.729 & 1 \end{bmatrix}$$

does not contain a nest as all the principal minors of order 2 in $A^3$ are negative. If $A$ is a symmetric or tridiagonal $M$-matrix, then $A^k$ is a $P$-matrix for all positive integers $k$ [8, Theorem 1 and Lemma 2].

6. Discussion. As stated in Corollary 3.2, the product of two nonsingular 3-by-3 $M$-matrices is a $P$-matrix (and thus has a nest). In contrast, the following example shows that the product of two 3-by-3 positive definite matrices may not even have a nest.

Example 6.1. The matrix

$$C = \begin{bmatrix} 6 & -1 & 0 \\ 0 & 0 & -1 \\ -60 & 11 & 0 \end{bmatrix}$$

has three distinct positive eigenvalues, namely $\{1, 2, 3\}$, and thus is the product of two positive definite matrices (see, e.g., [4, Problem 9, p. 468]). Note that $C$ does not have a nest.

The product of two nonsingular 4-by-4 $M$-matrices has a nest (Corollary 3.6) but is not necessarily a $P$-matrix. Extensive numerical calculations on the product of two 5-by-5 nonsingular $M$-matrices lead us to conjecture that such a product has a nest.
However, we do not know how to ensure the existence of a positive 3-by-3 minor, nor how to determine the positions of the positive 2-by-2 principal minors in the product of the two $M$-matrices.

As mentioned in the Introduction, if a matrix has a nest then it can be stabilized by premultiplication with a positive diagonal matrix. Our results thus give products of certain $M$-matrices that can be stabilized in this way. Example 4.2 illustrates this, as $FA$ has two eigenvalues with negative real parts. However, since $FA$ has a leading nest, there exists a positive diagonal matrix $D$ so that $DFA$ is positive stable.

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