2006

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Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1225

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EIGENVALUE CONDITION NUMBERS AND A FORMULA OF BURKE, LEWIS AND OVERTON∗

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Abstract. In a paper by Burke, Lewis and Overton, a first order expansion has been given for the minimum singular value of $A - zI$, $z \in \mathbb{C}$, about a nonderogatory eigenvalue $\lambda$ of $A \in \mathbb{C}^{n \times n}$. This note investigates the relationship of the expansion with the Jordan canonical form of $A$. Furthermore, formulas for the condition number of eigenvalues are derived from the expansion.

Key words. Eigenvalue condition numbers, Jordan canonical form, Singular values.

AMS subject classifications. 15A18, 65F35.

1. Introduction. By $\pi_\Sigma(A)$ we denote the product of the nonzero singular values of the matrix $A \in \mathbb{C}^{n \times m}$, counting multiplicities. For the zero matrix $0 \in \mathbb{C}^{n \times m}$ we set $\pi_\Sigma(0) = 1$. If $A$ is square then $\Lambda(A)$ denotes the spectrum and $\pi_\Lambda(A)$ stands for the product of the nonzero eigenvalues, counting multiplicities. If all eigenvalues of $A$ are zero then we set $\pi_\Lambda(A) = 1$. The subject of this note is the ratio

$$q(A, \lambda) := \frac{\pi_\Sigma(A - \lambda I_n)}{\pi_\Lambda(A - \lambda I_n)}$$

for $\lambda \in \Lambda(A)$.

In [1] the following first order expansion has been given for the function $z \mapsto \sigma_{\min}(A - zI_n)$, $z \in \mathbb{C}$, where $\sigma_{\min}(\cdot)$ denotes the minimum singular value and $I_n$ is the $n \times n$ identity matrix.

THEOREM 1.1. Let $\lambda \in \mathbb{C}$ be a nonderogatory eigenvalue of algebraic multiplicity $m$ of the matrix $A \in \mathbb{C}^{n \times n}$. Then

$$\sigma_{\min}(A - zI_n) = \frac{|z - \lambda|^m}{q(A, \lambda)} + O(|z - \lambda|^{m+1}), \quad z \in \mathbb{C}.$$

The relevance of this result for the perturbation theory of eigenvalues is as follows. The closed $\epsilon$− pseudospectrum of $A \in \mathbb{C}^{n \times n}$ with respect to the spectral norm, $\| \cdot \|$, is defined by

$$\Lambda_\epsilon(A) = \{ z \in \mathbb{C} \mid z \in \Lambda(A + \Delta), \Delta \in \mathbb{C}^{n \times n}, \| \Delta \| \leq \epsilon \}.$$

In words, $\Lambda_\epsilon(A)$ is the set of all eigenvalues of all matrices of the form $A + \Delta$ where the spectral norm of the perturbation $\Delta$ is bounded by $\epsilon > 0$. It is well known [10] that

$$\Lambda_\epsilon(A) = \{ z \in \mathbb{C} \mid \sigma_{\min}(A - zI) \leq \epsilon \}.$$
Theorem 1.1 yields an estimate for the size of pseudospectra for small $\epsilon$: Roughly speaking if $\epsilon$ is small enough then the connected component of $\Lambda_\epsilon(A)$ that contains the eigenvalue $\lambda$ is approximately a disk of radius $(q(A,\lambda)\epsilon)^{1/m}$ about $\lambda$. It follows that $q(A,\lambda)^{1/m}$ is the Hölder condition number of $\lambda$. We discuss this in detail in Section 4.

However, the main concern of this note is to establish the relationship of $q(A,\lambda)$ with the Jordan decomposition of $A$. For a simple eigenvalue the relationship is as follows. Let $x, y \in \mathbb{C}^n \setminus \{0\}$ be a right and a left eigenvector of $A$ to the eigenvalue $\lambda$ respectively, i.e. $Ax = \lambda x$, $y^*A = \lambda y^*$, where $y^*$ denotes the conjugate transpose of $y$. Then

$$P = (y^*x)^{-1}xy^* \in \mathbb{C}^{n \times n}$$

is a projection onto the one dimensional eigenspace $\mathbb{C}x$. The kernel of $P$ is the direct sum of all generalized eigenspaces belonging to the eigenvalues different from $\lambda$. As is well known [5, p.490],[3, p.202],[9, p.186], the condition number of $\lambda$ equals the norm of $P$. Combined with the considerations above this yields that

$$q(A,\lambda) = \|P\|. \quad (1.1)$$

In Section 3 we give an elementary proof of the identity (1.1) without using Theorem 1.1. Furthermore, we show that for a nondegenerate eigenvalue of algebraic multiplicity $m \geq 2$,

$$q(A,\lambda) = \|N^{m-1}\|, \quad (1.2)$$

where $N$ is the nilpotent operator associated with $\lambda$ in the Jordan decomposition of $A$. The formulas (1.1) and (1.2) are the main results of this note. The proofs also show that the assumption that $\lambda$ is nonderogatory is necessary.

The next section contains some preliminaries about the computation of the two products $\pi_\Sigma(A)$ and $\pi_\Lambda(A)$ and about the relationship of the Schur form of $A$ with the Jordan decomposition.

Throughout this note, $\|\cdot\|$ stands for the spectral norm.

2. Preliminaries. Below we list some easily verified properties of $\pi_\Lambda(A)$, the product of the nonzero eigenvalues of $A$, and of $\pi_\Sigma(A)$, the product of the nonzero singular values of $A$. In the sequel $A^T$ and $A^*$ denote the transpose and the conjugate transpose of $A$ respectively.

(a) If $A \in \mathbb{C}^{n \times n}$ is nonsingular then $\pi_\Lambda(A) = \det(A)$.

(b) For any $A \in \mathbb{C}^{n \times n}$ : $\pi_\Lambda(A^T) = \pi_\Lambda(A)$ and $\pi_\Lambda(A^*) = \overline{\pi_\Lambda(A)}$.

(c) Let $S \in \mathbb{C}^{n \times n}$ be nonsingular. Then for any $A \in \mathbb{C}^{n \times n}$, $\pi_\Lambda(SAS^{-1}) = \pi_\Lambda(A)$.

(d) Let $A_{11} \in \mathbb{C}^{n \times n}$, $A_{22} \in \mathbb{C}^{m \times m}$ and $A_{12} \in \mathbb{C}^{n \times m}$. Then

$$\pi_\Lambda \left( \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \right) = \pi_\Lambda(A_{11}) \pi_\Lambda(A_{22}).$$

(e) For any $A \in \mathbb{C}^{n \times m}$, $\pi_\Sigma(A)^2 = \pi_\Lambda(A^*A) = \pi_\Lambda(AA^*)$. 

In the next section we need the lemmas below.

Lemma 2.1. Let $M \in \mathbb{C}^{n \times n}$ be nonsingular, $X \in \mathbb{C}^{m \times n}$ and $Y = XM^{-1}$. Then

$$\pi_{\Sigma} \left( \begin{bmatrix} M \\ X \end{bmatrix} \right) = \pi_{\Sigma}(M) \sqrt{\det(I_n + Y^*Y)}.$$ 

Proof. We have

$$\pi_{\Sigma} \left( \begin{bmatrix} M \\ X \end{bmatrix} \right)^2 = \pi_{\Sigma} \left( \begin{bmatrix} M^* \ X^* \\ M \end{bmatrix} \right) = \det(M^*M + X^*X) = \det(M^*(I_n + YY^*)M) = \det(M^*) \det(M) \det(I_n + Y^*Y) = \pi_{\Sigma}(M)^2 \det(I_n + Y^*Y). \quad \square$$

Lemma 2.2. Let $Y \in \mathbb{C}^{m \times n}$. Then $\|I_n + Y^*Y\| = \|I_m + YY^*\|$ and $\det(I_n + Y^*Y) = \det(I_m + YY^*)$.

Proof. The case $Y = 0$ is trivial. Let $Y \neq 0$. The matrices $Y$ and $Y^*$ have the same nonzero singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p > 0$ say. The eigenvalues different from 1 of both $I_n + Y^*Y$ and $I_m + YY^*$ are $1 + \sigma_1^2 \geq 1 + \sigma_2^2 \ldots \geq 1 + \sigma_p^2$. Thus $\|I_n + Y^*Y\| = \|I_m + YY^*\| = 1 + \sigma_1^2$ and $\det(I_n + Y^*Y) = \det(I_m + YY^*) = \prod_{k=1}^{p}(1 + \sigma_k^2)$. \quad \square

We proceed with remarks on the Jordan decomposition. Let $\lambda_1, \ldots, \lambda_k$ be the pairwise different eigenvalues of $A \in \mathbb{C}^{n \times n}$. Let $X_j = \ker(A - \lambda_j I_n)$ be the generalized eigenspaces. By the Jordan decomposition theorem we have

$$A = \sum_{j=1}^{k} (\lambda_j P_j + N_j), \quad (2.1)$$

where $P_1, \ldots, P_k \in \mathbb{C}^{n \times n}$ are the projectors of direct decomposition $\mathbb{C}^n = \bigoplus_{j=1}^{k} X_j$, i.e.

$$P_j^2 = P_j, \quad \text{range}(P_j) = X_j, \quad \ker(P_j) = \bigoplus_{k=1, k \neq j}^{k} X_k,$$

and $N_1, \ldots, N_k \in \mathbb{C}^{n \times n}$ are the nilpotent matrices $N_j = (A - \lambda_j I_n)P_j$. The eigenvalue $\lambda_j$ is said to be

- semisimple (nondefective) if $X_j = \ker(A - \lambda_j I_n)$,
- simple if $\dim X_j = 1$,
- nonderogatory if $\dim \ker(A - \lambda_j I_n) = 1$. 
In the following $m$ denotes the algebraic multiplicity of $\lambda_j$. Note that if $m \geq 2$ then $\lambda_j$ is nonderogatory if and only if $N^m_{\lambda_j} \neq 0$. We now recall how to obtain the operators $P_j$ and $N_j$ from a Schur form of $A$. We only consider the nontrivial case that $A$ has at least two different eigenvalues. By the Schur decomposition theorem there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that

$$U^*AU = \begin{bmatrix} \lambda_j I_m + T & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{12} \in \mathbb{C}^{m \times (n-m)}$, $A_{22} \in \mathbb{C}^{(n-m) \times (n-m)}$, $\Lambda(A_{22}) = \Lambda(A) \setminus \{\lambda_j\}$ and $T \in \mathbb{C}^{n \times n}$ is strictly upper triangular,

$$T = \begin{bmatrix} 0 & t_{12} & \cdots & \cdots & t_{1m} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & t_{m-1,m} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

If $m = 1$ (i.e. $\lambda_j$ is simple) then $T$ is the $1 \times 1$ zero matrix. Since the spectra of $T$ and $A_{22} - \lambda_j I_{n-m}$ are disjoint the Sylvester equation

$$R(A_{22} - \lambda_j I_{n-m}) - TR = A_{12}. \quad (2.2)$$

has a unique solution $R \in \mathbb{C}^{m \times (n-m)}$.

**Proposition 2.3.** With the notation above the projector onto the generalized eigenspace and the nilpotent operator associated with $\lambda_j$ are given by

$$P_j = U \begin{bmatrix} I_m & -R \\ 0 & 0 \end{bmatrix} U^*, \quad \text{and} \quad N_j = U \begin{bmatrix} T & -TR \\ 0 & 0 \end{bmatrix} U^*.$$

For any integer $\ell \geq 1$ we have

$$N_j^\ell = U \begin{bmatrix} T^\ell & -T^\ell R \\ 0 & 0 \end{bmatrix} U^*. \quad (2.3)$$

The spectral norms of $P_j$ and of $N_j^\ell$ satisfy

$$\|P_j\| = \|I_m + RR^*\|^{1/2} \quad (2.4)$$

$$\|N_j^\ell\| = \|T^\ell(I_m + RR^*)(T^*)^\ell\|^{1/2}. \quad (2.5)$$

**Proof.** Let $X_1 := U \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in \mathbb{C}^{n \times m}$, $X_2 := U \begin{bmatrix} R \\ I_{n-m} \end{bmatrix} \in \mathbb{C}^{n \times (n-m)}$. Then obviously

$$\mathbb{C}^n = \text{range}(X_1) \oplus \text{range}(X_2)$$

and

$$AX_1 = X_1 (\lambda_j I_m + T). \quad (2.6)$$
Furthermore, (2.2) yields that
\[ AX_2 = X_2 A_{22}. \] (2.7)

Hence, range(\(X_1\)) and range(\(X_2\)) are complementary invariant subspaces of \(A\). The relations (2.6) and (2.7) imply that for any \(\lambda \in \mathbb{C}\) and any integer \(\ell \geq 1\),
\[
(A - \lambda I_n)^\ell X_1 = X_1 ((\lambda_j - \lambda)I_m + T)\ell,
\]
\[
(A - \lambda I_n)^\ell X_2 = X_2 (A_{22} - \lambda I_{n-m})\ell. \quad (2.8)
\]

Using this and the fact that \(\lambda_j \notin \Lambda(A_{22})\) it is easily verified that range(\(X_1\)) = ker \((A - \lambda_j I_n)^n\) and range(\(X_2\)) = \(\bigoplus_{k=1, k \neq j}^\infty\) ker \((A - \lambda_k I_n)^n\). The matrix
\[
P_j = U \begin{bmatrix} I_m & -R \\ 0 & 0 \end{bmatrix} U^*, \quad (2.9)
\]
satisfies \(P_j^2 = P_j\), \(P_j X_1 = X_1\) and \(P_j X_2 = 0\). Hence, \(P_j\) is the Jordan projector onto the generalized eigenspace ker \((A - \lambda_j I_n)^n\). For the associated nilpotent matrix \(N_j\) one obtains
\[
N_j = (A - \lambda_j I_n) P_j = U \begin{bmatrix} T & -T R \\ 0 & 0 \end{bmatrix} U^*. \quad (2.10)
\]

The formulas (2.3), (2.4) and (2.5) are immediate from (2.9) and (2.10). □

We give an expression for \(\|N_j^{m-1}\|\) which is a bit more explicit than formula (2.5). First note that if \(\lambda_j\) has algebraic multiplicity \(m \geq 2\) then
\[
T^{m-1} = \begin{bmatrix} 0 & \cdots & 0 & \tau \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \text{where} \quad \tau = \prod_{k=1}^{m-1} t_{k,k+1}.
\]

Let \(e_m^T = [0 \ldots 0 1]^T \in \mathbb{C}^m\) and \(r = e_m^T R\). Then \(r\) is the lower row of \(R\). Since the lower row of \(TR\) is zero it follows from the Sylvester equation (2.2) that
\[
r = e_m^T A_{12} (A_{22} - \lambda_j I_m)^{-1}. \quad (2.11)
\]

From (2.3) or (2.5) we obtain

**Proposition 2.4.** Suppose \(\lambda_j\) has algebraic multiplicity \(m \in \{2, \ldots, n - 1\}\). Then
\[
\|N_j^{m-1}\| = |\tau| \sqrt{1 + \|r\|^2}.
\]
3. Main result. We are now in a position to state and prove our main result on the ratio
\[
q(A, \lambda_j) = \frac{\pi_{\Sigma}(A - \lambda_j I_n)}{|\pi_{\Lambda}(A - \lambda_j I_n)|}, \quad \lambda_j \in \Lambda(A).
\] (3.1)

**Theorem 3.1.** Let \( \lambda_j \in \mathbb{C} \) be an eigenvalue of \( A \in \mathbb{C}^{n \times n} \). Let \( P_j \) and \( N_j \) be the eigenprojector and the nilpotent operator associated with \( \lambda_j \). Then the following holds.

(a) If \( \lambda_j \) is a semisimple eigenvalue then \( q(A, \lambda_j) = \pi_{\Sigma}(P_j) \).

(b) If \( \lambda_j \) is a simple eigenvalue then \( q(A, \lambda_j) = \|P_j\| \).

(c) If \( \lambda_j \) is a nonderogatory eigenvalue of algebraic multiplicity \( m \geq 2 \) then
\[
q(A, \lambda_j) = \|N_j^{m-1}\|.
\]

**Proof.** First, we treat the case that \( A \) has at least two different eigenvalues.

In view of Proposition 2.3 and since the products \( \pi_{\Sigma}(A - \lambda_j I_n) \), \( \pi_{\Lambda}(A - \lambda_j I_n) \) are invariant under unitary similarity transformations we may assume that
\[
A = \begin{bmatrix}
\lambda_j I_m + T & A_{12} \\
0 & A_{22}
\end{bmatrix}, \quad P_j = \begin{bmatrix}
I_m & -R \\
0 & 0
\end{bmatrix},
\]
where \( \Lambda(A_{22}) = \Lambda(A) \setminus \{\lambda_j\} \), \( T \in \mathbb{C}^{n \times n} \) is strictly upper triangular and \( R \in \mathbb{C}^{m \times (n-m)} \) is the solution of the Sylvester equation \( R(A_{22} - \lambda_j I_{n-m}) - TR = A_{12} \).

(a). Suppose \( \lambda_j \) is semisimple. Then \( T = 0 \) and \( R(A_{22} - \lambda_j I_{n-m}) = A_{12} \). Thus,
\[
(A - \lambda_j I_n)^*(A - \lambda_j I_n) = \begin{bmatrix}
0 & 0 \\
0 & (A_{22} - \lambda_j I_{n-m})^*(A_{22} - \lambda_j I_{n-m}) + A_{12}^* A_{12}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & (A_{22} - \lambda_j I_{n-m})^*(I_{n-m} + R^* R)(A_{22} - \lambda_j I_{n-m})
\end{bmatrix}.
\]
Thus
\[
\pi_{\Sigma}(A - \lambda_j I_n)^2 = \pi_{\Lambda}((A - \lambda_j I_n)^*(A - \lambda_j I_n))
= \det((A_{22} - \lambda_j I_{n-m})^*(I_{n-m} + R^* R)(A_{22} - \lambda_j I_{n-m}))
= \det((A_{22} - \lambda_j I_{n-m})^2 \det(I_{n-m} + R^* R)
= \|\pi_{\Lambda}(A - \lambda_j I_n)\|^2 \det(I_{n-m} + R^* R).
\] (3.2)

Furthermore we have \( P_j P_j^* = \begin{bmatrix}
I_m + RR^* & 0 \\
0 & 0
\end{bmatrix} \) and hence
\[
\pi_{\Sigma}(P_j)^2 = \det(I_m + RR^*) = \det(I_{n-m} + R^* R).
\] (3.4)

The latter equation holds by Lemma 2.2. By combining (3.3) and (3.4) we obtain (a).

(b). If \( m = 1 \) then \( P_j \) has rank 1 and hence, \( \pi_{\Sigma}(P_j) = \|P_j\| \). Thus (b) follows from (a).
(c) Suppose $m \geq 2$ and $\lambda_j$ is nonderogatory. Then $T = \begin{bmatrix} 0 & D \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \end{bmatrix}$, where $D \in \mathbb{C}^{(m-1) \times (m-1)}$ is upper triangular and nonsingular. In the following we write $A_{12} = \begin{bmatrix} \tilde{A} \\ a \end{bmatrix}$, where $a$ is the lower row of $A_{12}$. Let $r$ denote the lower row of $R$. By Formula (2.11) we have

$$r = a(A_{22} - \lambda_j I)^{-1}. \quad (3.5)$$

Let us determine $\pi_\Sigma(A)$. Since removing of a column of zeros and a permutation of rows does not change the nonzero singular values of a matrix we have

$$\pi_\Sigma(A - \lambda_j I_n) = \pi_\Sigma\left( \begin{bmatrix} 0 & D \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \tilde{A} \\ a \end{bmatrix} - \begin{bmatrix} D \\ 0 \\ A_{22} - \lambda_j I_{n-m} \end{bmatrix} \right) = \pi_\Sigma\left( \begin{bmatrix} D \\ 0 \\ A_{22} - \lambda_j I_{n-m} \end{bmatrix} \right).$$

Lemma 2.1 yields

$$\pi_\Sigma\left( \begin{bmatrix} D \\ 0 \\ A_{22} - \lambda_j I_{n-m} \end{bmatrix} \right) = \pi_\Sigma\left( \begin{bmatrix} D \\ 0 \\ A_{22} - \lambda_j I \end{bmatrix} \right) \sqrt{\det(1 + y^*y)}$$

$$= |\det(D)|\det(A_{22} - \lambda_j I)|\sqrt{1 + \|y\|^2}$$

$$= |\pi_\Sigma(A - \lambda_j I)| |\det(D)| \sqrt{1 + \|r\|^2},$$

where

$$y = \begin{bmatrix} \cdots & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} D \\ 0 \\ A_{22} - \lambda_j I \end{bmatrix}^{-1}.$$

From (3.5) it follows that $y = \begin{bmatrix} \cdots & 0 & \cdots & r \end{bmatrix}$ and hence, $\|y\| = \|r\|$. In summary,

$$\pi_\Sigma(A - \lambda_j I_n) = |\pi_\Sigma(A - \lambda_j I_n)| |\det(D)| \sqrt{1 + \|r\|^2}.$$

But $|\det(D)| \sqrt{1 + \|r\|^2} = \|N_j^{n-1}\|$ by Proposition 2.4. Hence, (c) holds.

Finally, we treat the case that $\lambda_1$ is the only eigenvalue of $A$. Let $U^*AU = \lambda_1 I_n + T$ be a Schur decomposition. The eigenprojection is $P_1 = I_n$ and the nilpotent operator is $N_1 = A - \lambda_1 I_n = U T U^*$. Since all eigenvalues of $A - \lambda_1 I_n$ are zero we have $\pi_\Sigma(A - \lambda_1 I_n) = 1$ by definition. If $\lambda_1$ is semisimple then also $\pi_\Sigma(A - \lambda_1 I_n) = \pi_\Sigma(0) = 1$. Hence, $q(A, \lambda_1) = 1 = \pi_\Sigma(P_1)$. Suppose $n \geq 2$ and $\lambda_1$ is nonderogatory. Then

$$q(A, \lambda_1) = \pi_\Sigma(A - \lambda_1 I_n) = \pi_\Sigma(T) = |\det(D)| = \|T^{n-1}\| = \|N_1^{n-1}\|,$$

where $T = \begin{bmatrix} 0 & D \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \end{bmatrix}.$}\]
4. Condition numbers. In this section we show that \( q(A, \lambda)^{1/m} \) equals the Hölder condition number of the nonderogatory eigenvalue \( \lambda \) of algebraic multiplicity \( m \). To this end we introduce some additional notation. By \( D_\lambda(r) \) we denote the closed disk of radius \( r > 0 \) about \( \lambda \in \mathbb{C} \). If \( \lambda \in \Lambda(A), \ A \in \mathbb{C}^{n \times n}, \) then \( C_\lambda(\epsilon) \) denotes the connected component of the \( \epsilon \)-pseudospectrum, \( \Lambda_\epsilon(A) \), that contains \( \lambda \). We define

\[
R_\lambda^-(\epsilon) := \inf\{ r > 0 \mid C_\lambda(\epsilon) \subseteq D_\lambda(r) \},
\]
\[
R_\lambda^+\epsilon) := \sup\{ r > 0 \mid D_\lambda(r) \subseteq C_\lambda(\epsilon) \}.
\]

Then

\[
D_\lambda(R_\lambda^-(\epsilon)) \subseteq C_\lambda(\epsilon) \subseteq D_\lambda(R_\lambda^+(\epsilon)).
\]

**Theorem 4.1.** Let \( \lambda \in \Lambda(A) \) be a nonderogatory eigenvalue of algebraic multiplicity \( m \). Then

\[
R_\lambda^\pm(\epsilon) = q(A, \lambda)^{1/m} \epsilon^{1/m} + o(\epsilon^{1/m}). \tag{4.1}
\]

The proof uses Theorem 1.1 and the lemma below.

**Lemma 4.2.** Let \( U \subseteq \mathbb{C}^n \) be an open neighborhood of \( z_0 \in \mathbb{C}^n \). Let \( f, g : U \to [0, \infty) \) be continuous functions. For \( \epsilon \geq 0 \) let \( S_f(\epsilon) \) and \( S_g(\epsilon) \) denote the connected component containing \( z_0 \) of the sublevel set \( \{ z \in U \mid f(z) \leq \epsilon \} \) and \( \{ z \in U \mid g(z) \leq \epsilon \} \) respectively. Assume that \( 0 = g(z_0) \) is an isolated zero of \( g \), and

\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = 1. \tag{4.2}
\]

Then there exists an \( \epsilon_0 > 0 \) and functions \( h_\pm : [0, \epsilon_0] \to [0, \infty) \) with \( \lim_{\epsilon \to 0} h_\pm(\epsilon) = 1 \) such that for all \( \epsilon \in [0, \epsilon_0] \),

\[
S_g(h_-(\epsilon) \epsilon) \subseteq S_f(\epsilon) \subseteq S_g(h_+(\epsilon) \epsilon). \tag{4.3}
\]

We postpone the proof of the lemma to the end of this section.

**Proof of Theorem 4.1:** Let in Lemma 4.2, \( z_0 = \lambda \) and

\[
f(z) = \sigma_{\min}(A - zI_n), \quad g(z) = \frac{|z - \lambda|^m}{q(A, \lambda)}, \quad z \in \mathbb{C}.
\]

Then \( S_f(\epsilon) = C_\lambda(\epsilon) \) and \( S_g(\epsilon) = D_\lambda((q(A, \lambda)\epsilon)^{1/m}) \). Theorem 1.1 yields \( \lim_{z \to \lambda} \frac{f(z)}{g(z)} = 1 \). Hence, by the lemma there are functions \( h_\pm \) with \( \lim_{\epsilon \to 0} h_\pm(\epsilon) = 1 \) and

\[
D_\lambda((q(A, \lambda) h_-(\epsilon) \epsilon)^{1/m}) \subseteq C_\lambda(\epsilon) \subseteq D_\lambda((q(A, \lambda) h_+(\epsilon) \epsilon)^{1/m}).
\]

This shows (4.1). \( \square \)

Now, we give the definition for the Hölder condition number of an eigenvalue of arbitrary multiplicity (see [2]). For \( \lambda \in \mathbb{C}, \ m \in \mathbb{N} \) and \( A \in \mathbb{C}^{n \times n} \) we set

\[
d_m(\tilde{A}, \lambda) := \min\{ r \geq 0 \mid D_\lambda(r) \text{ contains at least } m \text{ eigenvalues of } \tilde{A} \}.
\]
If \( \lambda \) is an eigenvalue of \( A \in \mathbb{C}^{n \times n} \) of algebraic multiplicity \( m \) then the Hölder condition number of \( \lambda \) to the order \( \alpha > 0 \) is defined by

\[
\text{cond}_\alpha(A, \lambda) = \lim_{\epsilon \to 0} \sup_{\|\Delta\| \leq \epsilon} \frac{d_m(A + \Delta, \lambda)}{\|\Delta\|^\alpha}.
\]

It is easily seen that \( 0 \neq \text{cond}_\alpha(A, \lambda) \neq \infty \) for at most one order \( \alpha > 0 \).

**Theorem 4.3.** Let \( \lambda \in \Lambda(A) \) be a nonderogatory eigenvalue of multiplicity \( m \). Then

\[
\text{cond}_{1/m}(A, \lambda) = q(A, \lambda)^{1/m} = \begin{cases} \|P\| & \text{if } m = 1, \\ \|\Lambda^{m-1}\|^{1/m} & \text{otherwise}, \end{cases}
\]

where \( P \in \mathbb{C}^{n \times n} \) is the eigenprojector onto the generalized eigenspace \( \ker(A - \lambda I_n)^m \), and \( N = (A - \lambda I_m)P \).

**Proof.** Let \( \Delta \in \mathbb{C}^{n \times n} \) with \( \|\Delta\| \leq \epsilon \). Then the continuity of eigenvalues yields, that for any \( t \in [0, 1] \) at least \( m \) eigenvalues of \( A + t\Delta \) are contained in \( C_\lambda(\epsilon) \) counting multiplicities. Hence

\[
d_m(A + \Delta, \lambda) \leq R^+_\lambda(\epsilon) = q(A, \lambda)^{1/m} \epsilon^{1/m} + o(\epsilon^{1/m}).
\]

By letting \( \epsilon = \|\Delta\| \) we obtain that for all \( \Delta \in \mathbb{C}^{n \times n} \),

\[
\frac{d_m(A + \Delta, \lambda)}{\|\Delta\|^{1/m}} \leq q(A, \lambda)^{1/m} + o(\|\Delta\|^{1/m})\|\Delta\|^{-(1/m)}.
\]

This yields

\[
\text{cond}_{1/m}(A, \lambda) \leq q(A, \lambda)^{1/m}.
\]

Let \( r > 0 \) be such that \( D_\lambda(r) \cap \Lambda(A) = \{\lambda\} \). Then by the continuity of eigenvalues there is an \( \epsilon_0 \) such that the following holds for all \( \epsilon < \epsilon_0 \),

(a) \( D_\lambda(r) \cap \Lambda(\epsilon) = C_\lambda(\epsilon) \).

(b) For any \( \Delta \in \mathbb{C}^{n \times n} \) with \( \|\Delta\| \leq \epsilon \), the set \( C_\lambda(\epsilon) \) contains precisely \( m \) eigenvalues of \( A + \Delta \) counting multiplicities.

Let \( \epsilon < \epsilon_0 \) and let \( z_\epsilon \in \mathbb{C} \) be a boundary point of \( C_\lambda(\epsilon) \). Then \( \sigma_{\min}(A - z_\epsilon I_n) = \epsilon \). Let \( \Delta_\epsilon = -\epsilon u v^* \), where \( u, v \in \mathbb{C}^n \) is a pair of normalized left and right singular vectors of \( A - z_\epsilon I_n \) belonging to the minimum singular value, i.e.

\[
(A - z_\epsilon I_n) v = \epsilon u, \quad u^*(A - z_\epsilon I_n) = \epsilon v^*, \quad \|u\| = \|v\| = 1.
\]

Then \( \|\Delta_\epsilon\| = \epsilon \) and \( z_\epsilon \in \Lambda(A + \Delta_\epsilon) \) since \( (A + \Delta_\epsilon)v = z_\epsilon v \). Thus, by (a) and (b),

\[
d_m(A + \Delta_\epsilon, \lambda) \geq |z_\epsilon - \lambda| \\
\geq R^+_\lambda(\epsilon) \\
= q(A, \lambda)^{1/m} \epsilon^{1/m} + o(\epsilon^{1/m}).
\]
and therefore
\[
\frac{d_m(A + \Delta, \lambda)}{\|\Delta\|^{1/m}} \geq q(A, \lambda)^{1/m} + o(\epsilon^{1/m})e^{-(1/m)}.
\]

Hence, \(\text{cond}_{1/m}(A, \lambda) \geq q(A, \lambda)^{1/m}\). \(\square\)

**Remark 4.4.** In [7] (see also [2, 4]) the following generalization of Theorem 4.3 has been shown. Let \(\lambda\) be an arbitrary eigenvalue of \(A\). If \(\lambda\) is semisimple then
\[
\text{cond}_1(A, \lambda) = \|P\|.
\]
If \(\lambda\) is not semisimple then
\[
\text{cond}_{1/m}(A, \lambda) = \|N^{m-1}\|^{1/m},
\]
where \(m\) denotes the index of nilpotency of \(N\), i.e. \(N^m = 0, N^{m-1} \neq 0\).

**Proof of Lemma 4.2.** By \(B_r\) we denote the closed ball of radius \(r > 0\) about \(z_0\). The condition that \(z_0\) is an isolated zero of \(g\) combined with (4.2) yields that \(z_0\) is also an isolated zero of \(f\). Hence, there is an \(r_0 > 0\) such that \(f(z) > 0\) for all \(z \in B_{r_0} \setminus \{z_0\}\). This implies that \(\epsilon_r := \min_{z \in \partial B_r} f(z) > 0\) for any \(r \in (0, r_0]\). If \(\epsilon < \epsilon_r\) then \(\partial B_r\) does not intersect the sublevel sets \(\{z \in U : f(z) \leq \epsilon\}\). Thus \(S_f(\epsilon)\) is contained in the interior of \(B_r\). Note that \(S_f(\epsilon)\) being a connected component of a closed set is closed. It follows that \(S_f(\epsilon)\) is compact if \(\epsilon < \epsilon_{r_0}\). Now, let
\[
\phi_{\pm}(z) := \begin{cases} 
(1 \pm \|z - z_0\|) \frac{f(z)}{f(z_0)} & z \in B_{r_0} \setminus \{z_0\}, \\
1 & z = z_0.
\end{cases}
\]
Condition (4.2) yields that the functions \(\phi_{\pm} : U \to \mathbb{R}\) are continuous. For \(\epsilon < \epsilon_{r_0}\) let
\[
h_-(\epsilon) := \min_{z \in S_f(\epsilon)} \phi_-(z), \quad h_+(\epsilon) := \max_{z \in S_f(\epsilon)} \phi_+(z).
\]
Then we have for all \(\epsilon < \epsilon_r\),
\[
\min_{z \in B_r} \phi_{\pm}(z) \leq h_-(\epsilon) \leq \max_{z \in B_r} \phi_{\pm}(z).
\]
As \(r\) tends to \(0\) the max and the min tend to \(\phi_{\pm}(z_0) = 1\). This yields \(\lim_{\epsilon \to 0} h_-(\epsilon) = 1\).
If \(z \in \partial S_f(\epsilon)\) then \(f(z) = \epsilon\) and \(g(z) > (1 - \|z - z_0\|) \frac{f(z)}{f(z_0)} \geq h_-(\epsilon)\). Thus \(\partial S_f(\epsilon)\) does not intersect \(E := \{z \in U : f(z) \leq h_-(\epsilon)\}\). Thus \(S_g(h_-(\epsilon))\) being a connected component of \(E\) is either contained in the interior of \(S_f(\epsilon)\) or in the complement of \(S_f(\epsilon)\). The latter is impossible since \(z_0 \in S_f(\epsilon) \cap S_g(h_-(\epsilon))\). Hence, \(S_g(h_-(\epsilon)) \subset S_f(\epsilon)\). This proves the first inclusion in (4.3). To prove the second suppose \(z_0 \neq z \in \partial S_g(h_+(\epsilon)) \cap S_f(\epsilon)\). Then \(g(z) = h_+(\epsilon)\) and \(0 < f(z) \leq \epsilon\). Hence \(g(z)/f(z) \geq h_+(\epsilon),\) a contradiction. Thus \(S_f(\epsilon)\) is contained in the interior of \(S_g(h_+(\epsilon))\). \(\square\)
REFERENCES


