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ON POLYNOMIALS IN TWO PROJECTIONS

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Abstract. Conditions are established on a polynomial $f$ in two variables under which the equality $f(P_1, P_2) = 0$ for two orthogonal projections $P_1, P_2$ is possible only if $P_1$ and $P_2$ commute.

Key words. Orthogonal projections, Polynomial equations.

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1. Introduction. Denote by $\mathbb{C}^{n\times n}$ the set of all $n \times n$ matrices with complex entries. Let $P_1, P_2 \in \mathbb{C}^{n\times n}$ be two orthogonal projections, that is,

$$P_j = P_j^* = P_j^2, \quad j = 1, 2.$$ 

Then an arbitrary polynomial $f$ in two variables $P_1, P_2$ has the form

$$f(P_1, P_2) = \sum c_{(m,i)} P_{(m,i)}$$

with $m$ assuming natural values and $i \in \{1, 2\}$. Here $c_{(m,i)} \in \mathbb{C}$, and $P_{(m,i)}$ is the notation for an alternating product of $m$ multiples $P_1, P_2$ starting with $P_i$.

We are interested in the question for which polynomials $f$ it is true that

$$f(P_1, P_2) = 0$$

implies the commutativity of $P_1$ and $P_2$.

The particular case of a binomial $f$ was considered earlier in [3], and the situation of three or four terms was dealt with in [1]. As a matter of fact, the problem of describing all polynomials $f$ for which condition (1.2) does not imply that $P_1$ and $P_2$ commute was also stated in [1], though not explicitly and in slightly different terms.

Our approach is different from that proposed in [3, 1], and is based on a (known) canonical form in which a pair of orthogonal projections can be put by a unitary equivalence. In Section 2 we recall this canonical form and prove the general result. Section 3 is devoted to its particular cases. Finally, the infinite dimensional variations are discussed in Section 4.

2. Main result. The following result is well known; relevant references will be given in Section 4.

**Lemma 2.1.** For any two orthogonal projections $P_1, P_2$ there exists a unitary transformation $U$ such that $U^*P_1U$ is the orthogonal sum of one dimensional blocks $0, 1$ and two dimensional blocks $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ while $U^*P_2U$ is the sum of (the same
number of) one dimensional blocks 0, 1 and two dimensional blocks

\[ p_t = \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix}. \]

The number of two dimensional blocks coincides with the number of the eigenvalues of \( P_1P_2P_1 \) different from 0 and 1 (and therefore lying strictly in between), and the parameter \( t \) runs through the set of all such eigenvalues (counting their multiplicities).

To formulate our main result, we need to introduce certain notation. Namely, let us partition the set of indices in the right hand side of (1.1) into four disjoint classes, corresponding to \( m \) being odd or even and \( i \) equal 1 or 2. Accordingly, introduce four scalar polynomials:

\[ \phi_1(t) = \sum c_{(2k+1,1)}t^k, \quad \phi_2(t) = \sum c_{(2k,1)}t^{k-1}, \]
\[ \phi_3(t) = \sum c_{(2k+1,2)}t^k, \quad \phi_4(t) = \sum c_{(2k,2)}t^{k-1} \quad (2.1) \]

(as usual, we agree that a sum with the void set of indices is equal to zero).

**Theorem 2.2.** Let \( P_1, P_2 \) be two orthogonal projections, and let \( f \) be a polynomial in \( P_1, P_2 \) given by (1.1). Then statement (1.2) holds if and only if all four polynomials \( \phi_j \) defined by (2.1) do not vanish simultaneously at any point in \((0, 1)\).

**Proof.** Let us start by computing \( f(p, p_t) \), where \( p \) and \( p_t (t \in (0, 1)) \) are \( 2 \times 2 \) Hermitian idempotent matrices introduced in Lemma 2.1. This is a technical part, needed both for the proof of necessity and sufficiency.

Direct computations show that

\[ pp_t = \begin{bmatrix} t & \sqrt{t(1-t)} \\ 0 & 0 \end{bmatrix}, \quad pp_t = tp_t, \quad pp_t = \begin{bmatrix} t & 0 \\ \sqrt{t(1-t)} & 0 \end{bmatrix}, \quad \text{and} \quad pp_t = tp_t. \]

Consequently,

\[ f(p, p_t) = \phi_1(t)p + \phi_2(t)pp_t + \phi_3(t)p_t + \phi_4(t)p_t p. \quad (2.2) \]

**Necessity.** Suppose that all four polynomials \( \phi_j \) have a common zero (say, \( t_0 \)) in \((0, 1)\). Then, according to (2.2), \( f(p, p_{t_0}) = 0 \). On the other hand, the orthogonal projections \( p, p_{t_0} \) do not commute, since

\[ pp_t - p_t p = \begin{bmatrix} 0 & \sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 0 \end{bmatrix}. \]

** Sufficiency.** Suppose that the polynomials \( \phi_j, j = 1, \ldots, 4 \), do not have joint zeros on \((0, 1)\) and that the orthogonal projections \( P_1, P_2 \) do not commute. According to Lemma 2.1, \( f(P_1, P_2) \) is then unitarily equivalent to the orthogonal sum in which at least one of the summands is a two dimensional block of the form \( f(p, p_t) \) with \( t \in (0, 1) \). Combining formula (2.2) with the fact that the matrices \( p, p_t, pp_t \) and \( p_t p \) are linearly independent, we conclude that \( f(p, p_t) \neq 0 \). Hence, \( f(P_1, P_2) \neq 0. \]
3. Particular cases. Apparently, zero is the only root of a monomial. This simple observation, when combined with Theorem 2.2, leads to the following statement.

**Corollary 3.1.** Suppose that expression (1.1) contains exactly one term with odd (or even) number of multiples and starting with $P_1$ (or $P_2$). Then (1.2) holds.

The situation is only slightly more complicated when one of the polynomials $\phi_j$ is a binomial, since the latter has exactly one non-zero root.

**Corollary 3.2.** Suppose that one of the polynomials (2.1) is a binomial, say $d_1 t^{k_1} + d_2 t^{k_2}$. Then either of conditions

(i) $d_1 d_2 > 0$,

(ii) $(k_1 - k_2)(|d_1| - |d_2|) \leq 0$,

(iii) at least one of the remaining polynomials $\phi_j$ assumes a non-zero value at $x_0 = (-d_2/d_1)^{1/(k_1-k_2)}$,

is sufficient for (1.2) to hold.

**Proof.** Under condition (i) or (ii) the roots of $d_1 t^{k_1} + d_2 t^{k_2}$ lie outside of $(0, 1)$, and Theorem 2.2 applies in a trivial way. If conditions (i) and (ii) do not hold, then $x_0$ is the only root of $d_1 t^{k_1} + d_2 t^{k_2}$ lying in $(0, 1)$, so that the applicability of Theorem 2.2 is guaranteed by condition (iii). □

We can now give an alternative proof for the following result from [3].

**Theorem 3.3.** Let

$$P_{(m,i)} = P_{(l,k)} \text{ with } (m,i) \neq (l,k).$$

(3.1)

Then $P_1$ and $P_2$ commute.

**Proof.** Consider $f(P_1, P_2) = P_{(m,i)} - P_{(l,k)}$. If $m-l$ is even (but non-zero) and $i = k$, then exactly one of the polynomials $\phi_j$ associated with $f$ is different from zero, and this polynomial is in fact a binomial $t^{s_1} - t^{s_2}$ with $s_1 \neq s_2$. Commutativity of $P_1$ and $P_2$ follows then from Corollary 3.2, since condition (ii) of this statement is met. In all other cases there are two non-trivial polynomials $\phi_j$, both being monomials, so that Corollary 3.1 applies. □

It is straightforward (and was also observed in [3]) that the converse of Theorem 3.3 holds provided that $m, l \geq 2$. The necessary and sufficient conditions for (3.1) to hold can be easily established when $\min\{m, l\} = 1$ but they are not worth the further discussion.

**Theorem 3.4.** Let the polynomial (1.1) contain at most five summands. Suppose that the set of the indices $(m,i)$ is such that the following possibilities are excluded:

(i) all indices are of the same type, that is, all $m$ are of the same evenness and all $i$ are the same;

(ii) there are two different types of the indices, one of them corresponding to exactly two coefficients of the opposite sign.

Then statement (1.2) holds.

**Proof.** Suppose first that the number of types of the indices present (that is, the number of non-zero polynomials (2.1) associated with $f$) is three or higher. Then at least one of $\phi_j$ is a monomial, and Corollary 3.1 applies. This corollary applies also when there are two non-zero polynomials $\phi_j$ but one of them is still a monomial. It
remains to consider the case when there are exactly two non-zero polynomials \( \phi_j \), neither of which being a monomial. At least one of them must then be a binomial, and its coefficients are of the same sign due to condition (ii). This is the situation (i) of Corollary 3.2.

A particular case of Theorem 3.4, dealing with four summands and not utilizing the properties of the coefficients, was proved in [1].

4. Infinite dimensional setting. Lemma 2.1 is a finite dimensional adaptation of the following result.

**Theorem 4.1.** Let \( P_1 \) and \( P_2 \) be two orthogonal projections acting on a Hilbert space \( \mathcal{H} \). Then there exist an orthogonal decomposition

\[
\mathcal{H} = \mathcal{M}_{00} \oplus \mathcal{M}_{01} \oplus \mathcal{M}_{10} \oplus \mathcal{M}_{11} \oplus \mathcal{M}_0 \oplus \mathcal{M}_1,
\]

(4.1)

a Hermitian operator \( H \) on \( \mathcal{M}_0 \) with the spectrum in \([0, 1]\) not having 0, 1 as its eigenvalues, and a unitary operator \( W : \mathcal{M}_1 \to \mathcal{M}_0 \) such that with respect to decomposition (4.1):

\[
P_1 = I \oplus I \oplus 0 \oplus 0 \oplus \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
P_2 = I \oplus 0 \oplus 0 \oplus I \oplus \begin{bmatrix} I & 0 \\ 0 & W^* \end{bmatrix} \begin{bmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}.
\]

(4.2)

As a matter of fact, the summands in (4.1) and the operator \( H \) are defined uniquely. Namely,

\[
\mathcal{M}_{00} = \text{Im} \, P_1 \cap \text{Im} \, P_2, \quad \mathcal{M}_{01} = \text{Im} \, P_1 \cap \text{Ker} \, P_2,
\]

\[
\mathcal{M}_{10} = \text{Ker} \, P_1 \cap \text{Ker} \, P_2, \quad \mathcal{M}_{11} = \text{Ker} \, P_1 \cap \text{Im} \, P_2,
\]

\( \mathcal{M}_0 \) is the orthogonal complement of \( \mathcal{M}_{00} \oplus \mathcal{M}_{01} \) in \( \text{Im} \, P_1 \), and \( H \) is the restriction of \( P_1 P_2 P_1 \) onto \( \mathcal{M}_0 \). The non-trivial part of Theorem 4.1 is the statement that \( H \) is unitarily equivalent to the restriction of \((I - P_1)(I - P_2)(I - P_1)\) onto the orthogonal complement \( \mathcal{M}_1 \) of \( \mathcal{M}_{10} \oplus \mathcal{M}_{11} \) in \( \text{Ker} \, P_1 \); the operator \( W \) arises from this unitary equivalence. The operators \( P_1 \) and \( P_2 \) commute if and only if the subspace \( \mathcal{M}_0 \) (and therefore \( \mathcal{M}_1 \)) has zero dimension, that is, the orthogonal sums in (4.1), (4.2) degenerate to the first four summands.

Theorem 4.1 in various disguises can be found, for example, in [7, 4, 5], and some of its applications are in [9, 8, 6].

The question on the validity of (1.2) can be asked in the setting of Hilbert spaces. Using the notation (2.1) and formula (4.2), it is easy to see that \( f(P_1, P_2) \) is the sum of a diagonal operator acting on \( \mathcal{M}_{00} \oplus \mathcal{M}_{01} \oplus \mathcal{M}_{10} \oplus \mathcal{M}_{11} \) with the operator unitarily equivalent to

\[
\begin{bmatrix}
\phi_1(H) + H(\phi_2 + \phi_3 + \phi_4)(H) & (\phi_2 + \phi_3)(H)\sqrt{H(I-H)} \\
(\phi_2 + \phi_3)(H)\sqrt{H(I-H)} & \phi_3(H)(I-H)
\end{bmatrix}.
\]

\(^1\) Added in proof: The case of four summands was also considered in [2, Theorem 2.3].
Due to the injectivity of $H$ and $I - H$, the latter operator is zero if and only if $\phi_j(H) = 0$ for $j = 1, \ldots, 4$. The spectral mapping theorem implies then that the polynomials $\phi_j$ have a common root on $(0, 1)$, unless the summands $M_0$ and $M_1$ are missing in the decomposition (4.1). The latter happens if and only if the operators $P_1$ and $P_2$ commute, which proves the operator version of Theorem 2.2. From here it follows that the results of Section 3 also remain valid verbatim in the infinite dimensional setting.

In fact, our results hold in a (formally) more general situation when $P_1$ and $P_2$ are selfadjoint idempotent elements of any $C^*$-algebra $\mathcal{A}$, because any such $\mathcal{A}$ is isomorphic to a subalgebra of the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. On the other hand, the selfadjointness condition is essential: even Theorem 3.3 is not valid if $P_1$, $P_2$ are idempotent non-selfadjoint matrices.

REFERENCES