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BOUNDS FOR THE SPECTRAL RADIUS OF BLOCK H-MATRICES

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1. Introduction. H-matrices have important applications in many fields, such as numerical analysis, control theory, and mathematical physics. Recently, Huang and Ran [5] have presented a simple upper bound for the spectral radius of (block) H-matrices. In this paper, we give some new upper bounds.

A square complex or real matrix $A$ is called an H-matrix if there exists a square positive diagonal matrix $X$ such that $AX$ is strictly diagonally dominant (SDD) [5]. Let $\mathbb{C}^{n,n}(\mathbb{R}^{n,n})$ denote the set of $n \times n$ complex (real) matrices. If $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we write $|A| = |a_{ij}|$, where $|a_{ij}|$ is the modulus of $a_{ij}$. We denote by $\rho(A)$ the spectral radius of $A$, which is just the radius of the smallest disc centered at the origin in the complex plane that includes all the eigenvalues of $A$ (see [4, Def. 1.1.4]).

Throughout the paper, we let $\|\cdot\|$ denote a consistent family of norms on matrices of all sizes, which satisfies the following four axioms:

1. $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = 0$;
2. $\|cA\| = |c| \|A\|$ for all complex scalars $c$;
3. $\|A + B\| \leq \|A\| + \|B\|$, where $A$ and $B$ are in the same size; and
4. $\|AB\| \leq \|A\| \|B\|$ provided that $AB$ is defined.

Axioms (1) and (4) ensure that $\|I\| \geq 1$, where $I$ is the identity matrix. For example, the Frobenius norm $\|\cdot\|_F$, 1-norm $\|\cdot\|_1$, and $\infty$-norm $\|\cdot\|_{\infty}$ (see e.g., [4, Chap. 5]) are all consistent families of norms.

Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be partitioned in the following form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix}$$

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in which \( A_{ij} \in \mathbb{C}^{n_i,n_j} \) and \( \sum_{i=1}^{k} n_i = n \). If each diagonal block \( A_{ii} \) is nonsingular and
\[
\left\| A_{ii}^{-1} \right\|^{-1} > \sum_{j \neq i} \left\| A_{ij} \right\| \quad \text{for all} \quad i = 1, 2, \ldots, k,
\]
then \( A \) is said to be \textit{block strictly diagonally dominant with respect to} \( \| \cdot \| \) (BSDD) [3]; if there exist positive numbers \( x_1, x_2, \ldots, x_k \) such that
\[
x_i \left\| A_{ii}^{-1} \right\|^{-1} > \sum_{j \neq i} x_j \left\| A_{ij} \right\| \quad \text{for all} \quad i = 1, 2, \ldots, k,
\]
then \( A \) is said to be \textit{block H-matrix with respect to} \( \| \cdot \| \) [6].

**Theorem 1.1.** ([5]) Let \( A = [a_{ij}] \in \mathbb{C}^{n,n} \). If \( A \) is an H-matrix, then
\[
\rho(A) < 2 \max_i |a_{ii}|.
\]

**Theorem 1.2.** ([5]) Let \( A \in \mathbb{C}^{n,n} \) be partitioned as in (1.1). Let \( \| \cdot \| \) be a consistent family of norms. If \( A \) is a block H-matrix with respect to \( \| \cdot \| \), then
\[
\rho(A) < \max_i \left\{ \| A_{ii} \| + \left\| A_{ii}^{-1} \right\|^{-1} \right\}.
\]

2. **Main results.** In this section, we present some new bounds for the spectral radius of an H-matrix and a block H-matrix, respectively. We need the following two lemmas.

**Lemma 2.1.** ([4, Thm 8.1.18]) Let \( A = [a_{ij}] \in \mathbb{C}^{n,n} \). Then \( \rho(A) \leq \rho(|A|) \).

**Lemma 2.2.** ([1]) Let \( A = [a_{ij}] \in \mathbb{R}^{n,n} \) be a nonnegative matrix. Then
\[
\rho(A) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii} + a_{jj} + \left[ (a_{ii} - a_{jj})^2 + 4 \sum_{k \neq i} a_{ik} \sum_{k \neq j} a_{jk} \right]^{1/2} \right\}.
\]

The following is one of the main results of this paper.

**Theorem 2.3.** Let \( A = [a_{ij}] \in \mathbb{C}^{n,n} \) be an H-matrix. Then
\[
\rho(A) < \max_{i \neq j} (|a_{ii}| + |a_{jj}|) \leq 2 \max_i |a_{ii}|.
\]

**Proof.** Let \( X = \text{diag}(x_1, x_2, \ldots, x_n) \) be a square positive diagonal matrix such that \( AX \) is SDD. Then \( X^{-1}AX \) is also SDD, i.e.,
\[
|a_{ii}| = |X^{-1}AX|_{ii} > \sum_{j \neq i} |X^{-1}AX|_{ij} = \sum_{j \neq i} \frac{|a_{ij}|x_j}{x_i} \quad \text{for all} \quad i = 1, 2, \ldots, n.
\]
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The spectral radii of $A$ and $X^{-1}AX$ are equal since the two matrices are similar. Lemma 2.1 and Lemma 2.2 ensure that

$$\rho(A) = \rho(X^{-1}AX) \leq \rho(|X^{-1}AX|)$$

$$\leq \max_{i \neq j} \frac{1}{2} \left\{ |a_{ii}| + |a_{jj}| + \left( |a_{ii}| - |a_{jj}| \right)^2 + 4 \sum_{k \neq i} |a_{ik}|x_k \sum_{k \neq j} |a_{jk}|x_k \right\}^{\frac{1}{2}}$$

$$< \max_{i \neq j} \frac{1}{2} \left\{ |a_{ii}| + |a_{jj}| + \left( |a_{ii}| - |a_{jj}| \right)^2 + 4|a_{i1}||a_{j1}| \right\}^{\frac{1}{2}}$$

$$= \max_{i} (|a_{ii}| + |a_{jj}|) \leq 2 \max_{i} |a_{ii}|. \quad \square$$

We now consider block H-matrices. The following lemma was stated in [2] but the proof offered there is not correct.

**LEMMA 2.4.** ([2]) Let $A = [A_{ij}] \in \mathbb{R}^{n \times n}$ be a nonnegative block matrix of the form (1.1). Let $B = \|A_{ij}\|$, where $\|\|$ is a consistent family of norms. Then

$$\rho(A) \leq \rho(B).$$

**Proof.** First we assume that $A$ is a positive matrix. By Perron’s Theorem [4, Thm 8.2.11], $\rho(A)$ is an eigenvalue of $A$ corresponding to a positive eigenvector $x$, i.e.,

$$Ax = \rho(A)x, \quad x > 0.$$  

Partition $x^T = (x_1^T, \ldots, x_k^T)$, where each $x_i \in \mathbb{R}^{n_i}$, $i = 1, \ldots, k$. Let $z_i = \|x_i\|$. Define $z := (z_1, \ldots, z_k)^T \in \mathbb{R}^k$, so $z > 0$ and for all $1 \leq i \leq k$,

$$\sum_{j=1}^{k} A_{ij}x_j = \rho(A)x_i,$$

which implies

$$\rho(A)z_i = \rho(A)\|x_i\| = \left\| \sum_{j=1}^{k} A_{ij}x_j \right\| \leq \sum_{j=1}^{k} \|A_{ij}\| \|x_j\| = \sum_{j=1}^{k} \|A_{ij}\| z_j.$$  

Since the inequality $\rho(A)z_i \leq \sum_{j=1}^{k} \|A_{ij}\| z_j$ holds for all $i = 1, \ldots, k$, we have

$$\rho(A)z \leq Bz.$$  

Since $B$ is nonnegative and $z > 0$, we obtain $\rho(A) \leq \rho(B)$ [4, Cor. 8.1.29].

Next we show the inequality (2.2) holds for all nonnegative matrices $A$. For any given $\varepsilon > 0$, define $A(\varepsilon) := [a_{ij} + \varepsilon]$ and let $B(\varepsilon) := \|A_{ij}(\varepsilon)\|$. Since every $A_{ij}(\varepsilon)$ is positive, therefore $\rho(A(\varepsilon)) \leq \rho(B(\varepsilon))$. By the continuity of $\rho(\cdot)$, we have

$$\rho(A) = \lim_{\varepsilon \to 0} \rho(A(\varepsilon)) \leq \lim_{\varepsilon \to 0} \rho(B(\varepsilon)) = \rho(B). \quad \square$$
THEOREM 2.5. Let $A \in \mathbb{C}^{n \times n}$ be partitioned as in (1.1). Suppose $A$ is a block H-matrix with respect to a consistent family of norms $\|\cdot\|$. Then

\[\rho(A) \leq \max_{i \neq j} \frac{1}{2} \left\{ \left\| A_{ii} \right\| + \left\| A_{jj} \right\| + \left[ \left\| A_{ii} \right\| - \left\| A_{jj} \right\| \right]^2 + 4 \left\| A_{ii}^{-1} \right\|^{-1} \left\| A_{jj}^{-1} \right\|^{-1} \right\}^{\frac{1}{2}} \right\}

\[\leq \max_{i \neq j} \left( \left\| A_{ii} \right\| + \left\| A_{jj} \right\| \right) + \frac{1}{2} \left\{ \left\| A_{ii} \right\| + \left\| A_{jj} \right\| + \left[ \left\| A_{ii} \right\| - \left\| A_{jj} \right\| \right]^2 + 4 \left\| A_{ii}^{-1} \right\|^{-1} \left\| A_{jj}^{-1} \right\|^{-1} \right\}^{\frac{1}{2}} \right\}

Proof. Let $x_1, x_2, \ldots, x_k$ be positive numbers such that

\[x_i \left\| A_{ii}^{-1} \right\|^{-1} > \sum_{j \neq i} x_j \left\| A_{ij} \right\| \quad \text{for all} \quad i = 1, 2, \ldots, k.

Let $X = \text{diag}(x_1 I_{n_1}, x_2 I_{n_2}, \ldots, x_k I_{n_k})$. Then $AX$ is BSDD. Let

\[B = X^{-1}AX = \begin{pmatrix}
A_{11} & \ldots & \frac{x_k}{x_i} A_{1k} \\
\frac{x_2}{x_1} A_{21} & \ldots & \frac{x_k}{x_1} A_{2k} \\
\vdots & \ddots & \vdots \\
\frac{x_k}{x_n} A_{k1} & \ldots & A_{kk}
\end{pmatrix},
\]

Then $B = [B_{ij}]$ is also BSDD. Let $C = [\|B_{ij}\|] \in \mathbb{R}^{k \times k}$. Then Lemma 2.2 and Lemma 2.4 ensure that

\[\rho(A) = \rho(B) \leq \rho(C)
\]

\[\leq \max_{i \neq j} \frac{1}{2} \left\{ \left\| A_{ii} \right\| + \left\| A_{jj} \right\| + \left[ \left\| A_{ii} \right\| - \left\| A_{jj} \right\| \right]^2 + 4 \left\| A_{ii}^{-1} \right\|^{-1} \left\| A_{jj}^{-1} \right\|^{-1} \right\}^{\frac{1}{2}} \right\}

Moreover, we have $1 \leq \|I\| = \|A_{ii}A_{ii}^{-1}\| \leq \|A_{ii}\| \|A_{ii}^{-1}\|$, so $\|A_{ii}^{-1}\|^{-1} \leq \|A_{ii}\|$ and

\[\|A_{ii}\| + \|A_{jj}\| + \left[ \|A_{ii}\| - \|A_{jj}\| \right]^2 + 4 \|A_{ii}^{-1}\|^{-1} \|A_{jj}^{-1}\|^{-1} \right\}^{\frac{1}{2}} \right\}

\[\leq \|A_{ii}\| + \|A_{jj}\| + \left[ \|A_{ii}\| - \|A_{jj}\| \right]^2 + 4 \|A_{ii}\| \|A_{jj}\| \right\}^{\frac{1}{2}} \quad \square
\]

REMARK 2.6. Without loss of generality, for given $i \neq j$, assume that

\[\|A_{ii}\| + \|A_{ii}^{-1}\|^{-1} \geq \|A_{jj}\| + \|A_{jj}^{-1}\|^{-1}.
\]
Then
\[
\|A_{ii}\| + \|A_{jj}\| + \left[\left(\|A_{ii}\| - \|A_{jj}\|\right)^2 + 4\|A_{ii}^{-1}\|^{-1}\|A_{jj}^{-1}\|^{-1}\right]^{\frac{1}{2}}
\]
\[
\leq \|A_{ii}\| + \|A_{jj}\| + \left[\left(\|A_{ii}\| - \|A_{jj}\|\right)^2 + 4\|A_{ii}^{-1}\|^{-1}\left(\|A_{ii}\| + \|A_{jj}^{-1}\|^{-1} - \|A_{jj}\|\right)\right]^{\frac{1}{2}}
\]
\[
= \|A_{ii}\| + \|A_{jj}\| + \left(\|A_{ii}\| - \|A_{jj}\| + 2\|A_{ii}^{-1}\|^{-1}\right)^{\frac{1}{2}}
\]
\[
= \|A_{ii}\| + \|A_{jj}\| + \left(\|A_{ii}\| + \|A_{ii}^{-1}\|^{-1} - \|A_{jj}\|\right) + \|A_{ii}^{-1}\|^{-1}
\]
\[
= 2\left(\|A_{ii}\| + \|A_{ii}^{-1}\|^{-1}\right) \leq 2\max_i \left(\|A_{ii}\| + \|A_{ii}^{-1}\|^{-1}\right).
\]

Hence, the first bound in (2.3) is at least as good as the bound (1.2).

Example. Consider the block matrix
\[
A = \begin{bmatrix}
4 & -2 & 1.5 & 0.5 \\
-2 & 6 & 1 & -0.5 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 1 & 0 \\
0.5 & 0.5 & 0 & 1
\end{bmatrix}
\]
and the norms \(\|\cdot\|_\infty\). Then \(A\) is a block H-matrix with spectral radius 7.2152. The bound in Theorem 1.2 is \(\rho(A) \leq 10.5\). The bound in Theorem 2.5 is \(\rho(A) \leq 8.34\).

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