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A NEW EQUIVALENT CONDITION OF THE REVERSE ORDER LAW FOR G -INVERSES OF MULTIPLE MATRIX PRODUCTS*

BING ZHENG[†] AND ZHIPING XIONG[†]

Abstract. In 1999, Wei [M.We, Reverse order laws for generalized inverse of multiple matrix products, Linear Algebra Appl., 293 (1999), pp. 273-288] studied reverse order laws for generalized inverses of multiple matrix products and derived some necessary and sufficient conditions for

$$A_n\{1\}A_{n-1}\{1\}\cdots A_1\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}$$

by using P-SVD (Product Singular Value Decomposition). In this paper, using the maximal rank of the generalized Schur complement, a new simpler equivalent condition is obtained in terms of only the ranks of the known matrices for this inclusion.

Key words. Reverse order law, Generalized inverse, Matrix product, Maximal rank, Generalized Schur complement.

AMS subject classifications. 15A03, 15A09.

1. Introduction. Let A be an $m \times n$ matrix over the complex field. A^* and $r(A)$ denote the conjugate transpose and the rank of the matrix A , respectively. We recall that an $n \times m$ matrix G satisfying the equation $AGA = A$ is called a $\{1\}$ -inverse or a g -inverse of A and is denoted by $A^{(1)}$. The set of all $\{1\}$ -inverses of A is denoted by $A\{1\}$. We refer the reader to [1, 2] for basic results on the g -inverse.

The reverse order law for the generalized inverses of the multiple matrix products yields a class of interesting problems that are fundamental in the theory of generalized inverses of matrices and statistics. They have attracted considerable attention since the middle 1960s, and many interesting results have been obtained; see [3, 4, 5, 6, 7, 8].

In this paper, by applying the maximal rank of the generalized Schur complement [9], we derive a new and simple necessary and sufficient condition for the validity of the inclusion

$$(1.1) \quad A_n\{1\}A_{n-1}\{1\}\cdots A_1\{1\} \subseteq (A_1A_2\cdots A_n)\{1\}.$$

Compared with the conditions given in [6], our condition can be easily checked and the proof is very simple.

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The following two lemmas play a key role in this paper:

LEMMA 1.1. [9] *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times l}$, $C \in \mathbb{C}^{k \times n}$ and $D \in \mathbb{C}^{k \times l}$. Then*

$$(1.2) \quad \max_{A^{(1)} \in A\{1\}} r(D - CA^{(1)}B) = \min\{r(C, D), r\left(\begin{matrix} B \\ D \end{matrix}\right), r\left(\begin{matrix} A & B \\ C & D \end{matrix}\right) - r(A)\}.$$

Before presenting the next lemma, we first state the well-known Frobenius' inequality: If A, B, C are matrices such that ABC is defined, then

$$(1.3) \quad r(AB) + r(BC) \leq r(B) + r(ABC).$$

LEMMA 1.2. [8] *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \dots, n$. Then*

$$(1.4) \quad l_2 + l_3 + \dots + l_n + r(A_1 A_2 \dots A_n) \geq r(A_1) + r(A_2) + \dots + r(A_n).$$

Proof. Taking $A = A_1 A_2 \dots A_{i-1}$, $B = I_{l_i}$ and $C = A_i$ in (1.3), where $i = 2, 3, \dots, n$, we obtain

$$(1.5) \quad r(A_1 A_2 \dots A_{i-1} A_i) - r(A_1 A_2 \dots A_{i-1}) \geq r(A_i) - l_i.$$

So we have

$$(1.6) \quad r(A_1 A_2 \dots A_n) \geq \sum_{i=1}^n r(A_i) - \sum_{i=2}^n l_i. \quad \square$$

2. Main result. Define the following matrix function

$$(2.1) \quad \begin{aligned} & S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n) \\ &= A_1 A_2 \dots A_n - A_1 A_2 \dots A_n X_n X_{n-1} \dots X_1 A_1 A_2 \dots A_n, \end{aligned}$$

where X_1, X_2, \dots, X_n are any matrices of appropriate sizes. In order to present the new necessary and sufficient condition for the inclusion (1.1), we first give the maximum rank of matrix $S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)$ when each X_i ($i = 1, 2, \dots, n$) varies over the set $A_i\{1\}$ of all g -inverses of the matrix A_i .

THEOREM 2.1. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, \dots, n$, and $S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)$ be as in (2.1). Then*

$$(2.2) \quad \begin{aligned} & \max_{X_n, X_{n-1}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\ &= \min\{r(A_1 A_2 \dots A_n), r(A_1 A_2 \dots A_n) + \sum_{m=2}^n l_m - \sum_{m=1}^n r(A_m)\}, \end{aligned}$$

where X_i varies over $A_i\{1\}$ for $i = 1, 2, \dots, n$.

Proof. For $2 \leq i \leq n - 1$ and $X_j \in A_j\{1\}$, $j = i, i + 1, \dots, n$, we first prove

$$(2.3) \quad \begin{aligned} & \max_{X_i} r(A_1 A_2 \cdots A_{i-1} - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_i) \\ &= \min\{r(A_1 A_2 \cdots A_{i-1}), r(A_1 A_2 \cdots A_i - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}) \\ & \quad + l_i - r(A_i)\}. \end{aligned}$$

By Lemma 1.1 (with $A = A_i$, $B = I_{l_i}$, $D = A_1 \cdots A_{i-1}$, $C = A_1 \cdots A_n X_n \cdots X_{i+1}$), we have

$$\begin{aligned} & \max_{X_i} r(A_1 A_2 \cdots A_{i-1} - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_i) \\ &= \min\{r(A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}, A_1 A_2 \cdots A_{i-1}), r \left(\begin{array}{c} I_{l_i} \\ A_1 A_2 \cdots A_{i-1} \end{array} \right), \\ & \quad r \left(\begin{array}{cc} A_i & I_{l_i} \\ A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1} & A_1 A_2 \cdots A_{i-1} \end{array} \right) - r(A_i)\} \\ &= \min\{r(A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}, A_1 A_2 \cdots A_{i-1}), r \left(\begin{array}{c} I_{l_i} \\ A_1 A_2 \cdots A_{i-1} \end{array} \right), \\ & \quad r(A_1 A_2 \cdots A_i - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}) + l_i - r(A_i)\} \\ &= \min\{r(A_1 A_2 \cdots A_{i-1}), r(A_1 A_2 \cdots A_i - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}) \\ & \quad + l_i - r(A_i)\}, \end{aligned}$$

i.e., (2.3) holds, where the second equality holds as

$$\begin{aligned} & r \left(\begin{array}{cc} A_i & I_{l_i} \\ A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1} & A_1 A_2 \cdots A_{i-1} \end{array} \right) \\ &= r \left(\begin{array}{cc} O & I_{l_i} \\ A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1} - A_1 A_2 \cdots A_i & O \end{array} \right) \end{aligned}$$

and the last equality holds as

$$r(A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}, A_1 \cdots A_{i-1}) = r(A_1 \cdots A_{i-1})$$

and

$$r(A_1 A_2 \cdots A_{i-1}) \leq r(A_{i-1}) \leq l_i = r \left(\begin{array}{c} I_{l_i} \\ A_1 A_2 \cdots A_{i-1} \end{array} \right).$$

When $i = n$, again by Lemma 1.1 with $A = A_n$, $B = I_{l_n}$, $D = A_1 A_2 \cdots A_{n-1}$ and $C = A_1 A_2 \cdots A_n$, we have

$$(2.4) \quad \max_{X_n} r(A_1 A_2 \cdots A_{n-1} - A_1 A_2 \cdots A_n X_n)$$

$$\begin{aligned}
 &= \min\{r(A_1 A_2 \cdots A_n, A_1 A_2 \cdots A_{n-1}), r\left(\begin{array}{c} I_{l_n} \\ A_1 A_2 \cdots A_{n-1} \end{array}\right)\}, \\
 &\quad r\left(\begin{array}{cc} A_n & I_{l_n} \\ A_1 A_2 \cdots A_n & A_1 A_2 \cdots A_{n-1} \end{array}\right) - r(A_n)\} \\
 &= \min\{r(A_1 A_2 \cdots A_{n-1}), l_n - r(A_n)\}
 \end{aligned}$$

in which the last equality holds since

$$r(A_1 \cdots A_n, A_1 \cdots A_{n-1}) = r(A_1 \cdots A_{n-1}) \leq r(A_{n-1}) \leq l_n = r\left(\begin{array}{c} I_{l_n} \\ A_1 A_2 \cdots A_{n-1} \end{array}\right)$$

and

$$r\left(\begin{array}{cc} A_n & I_{l_n} \\ A_1 A_2 \cdots A_n & A_1 A_2 \cdots A_{n-1} \end{array}\right) = r\left(\begin{array}{c} I_{l_n} \\ A_1 A_2 \cdots A_{n-1} \end{array}\right).$$

We now prove (2.2). According to Lemma 1.1 with $A = A_1$, $B = A_1 A_2 \cdots A_n$, $C = A_1 A_2 \cdots A_n X_n X_{n-1} X_2$ and $D = A_1 A_2 \cdots A_n$, we have

$$\begin{aligned}
 &\max_{X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\
 &= \min\{r(A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2, A_1 A_2 \cdots A_n), r\left(\begin{array}{c} A_1 A_2 \cdots A_n \\ A_1 A_2 \cdots A_n \end{array}\right)\}, \\
 &\quad r\left(\begin{array}{cc} A_1 & A_1 A_2 \cdots A_n \\ A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2 & A_1 A_2 \cdots A_n \end{array}\right) - r(A_1)\} \\
 &= \min\{r(A_1 A_2 \cdots A_n), r(A_1 - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2) + r(A_1 A_2 \cdots A_n) - r(A_1)\},
 \end{aligned}$$

where the last equality holds as

$$r(A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2, A_1 A_2 \cdots A_n) = r(A_1 A_2 \cdots A_n) = r\left(\begin{array}{c} A_1 A_2 \cdots A_n \\ A_1 A_2 \cdots A_n \end{array}\right)$$

and

$$\begin{aligned}
 &r\left(\begin{array}{cc} A_1 & A_1 A_2 \cdots A_n \\ A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2 & A_1 A_2 \cdots A_n \end{array}\right) \\
 &= r(A_1 - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2) + r(A_1 A_2 \cdots A_n).
 \end{aligned}$$

Obviously $r(A_1 - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2) \leq r(A_1)$, so

$$\begin{aligned}
 (2.5) \quad &\max_{X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\
 &= r(A_1 - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_2) + r(A_1 A_2 \cdots A_n) - r(A_1).
 \end{aligned}$$

Combining (2.3) and (2.5), we have

$$\max_{X_2, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n))$$

$$\begin{aligned}
 &= \min\{r(A_1) + r(A_1 A_2 \cdots A_n) - r(A_1), \\
 &\quad r(A_1 A_2 \cdots A_n) + r(A_1 A_2 - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_3) + l_2 - r(A_2) - r(A_1)\} \\
 &= \min\{r(A_1 A_2 \cdots A_n), r(A_1 A_2 \cdots A_n) + r(A_1 A_2 - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_3) \\
 &\quad + l_2 - r(A_2) - r(A_1)\}.
 \end{aligned}$$

We contend that, for $2 \leq i \leq n-1$,

$$\begin{aligned}
 (2.6) \quad &\max_{X_i, X_{i-1}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\
 &= \min\{r(A_1 A_2 \cdots A_n), r(A_1 A_2 \cdots A_n) \\
 &\quad + r(A_1 A_2 \cdots A_i - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}) + \sum_{m=2}^i l_m - \sum_{m=1}^i r(A_m)\}.
 \end{aligned}$$

We proceed by induction on i . For $i = 2$, the equality relation (2.6) has been proved. Assume that (2.6) is true for $i-1$ ($i \geq 3$), that is,

$$\begin{aligned}
 (2.7) \quad &\max_{X_{i-1}, X_{i-2}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\
 &= \min\{r(A_1 A_2 \cdots A_n), r(A_1 A_2 \cdots A_n) \\
 &\quad + r(A_1 A_2 \cdots A_{i-1} - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_i) + \sum_{m=2}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m)\}.
 \end{aligned}$$

We now prove that (2.6) is also true for i . By (2.3) and (2.7), we have

$$\begin{aligned}
 &\max_{X_i, X_{i-1}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\
 &= \min\{r(A_1 A_2 \cdots A_n), \\
 &\quad r(A_1 A_2 \cdots A_n) + r(A_1 A_2 \cdots A_{i-1}) + \sum_{m=2}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m), \\
 &\quad r(A_1 A_2 \cdots A_n) + r(A_1 A_2 \cdots A_i - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}) \\
 &\quad + l_i - r(A_i) + \sum_{m=2}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m)\}.
 \end{aligned}$$

From Lemma 1.2 we know

$$l_2 + l_3 + \cdots + l_{i-1} + r(A_1 A_2 \cdots A_{i-1}) \geq r(A_1) + r(A_2) + \cdots + r(A_{i-1});$$

thus

$$\begin{aligned}
 &\max_{X_i, X_{i-1}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\
 &= \min\{r(A_1 A_2 \cdots A_n), \\
 &\quad r(A_1 A_2 \cdots A_n) + r(A_1 A_2 \cdots A_i - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_{i+1}) \\
 &\quad + \sum_{m=2}^i l_m - \sum_{m=1}^i r(A_m)\}.
 \end{aligned}$$

In particular, when $i = n - 1$, we have

$$(2.8) \quad \begin{aligned} & \max_{X_{n-1}, X_{n-2}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\ &= \min\{r(A_1 A_2 \cdots A_n), \quad r(A_1 A_2 \cdots A_n) \\ & \quad + r(A_1 A_2 \cdots A_{n-1} - A_1 A_2 \cdots A_n X_n) + \sum_{m=2}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m)\}. \end{aligned}$$

On account of (2.4) and (2.8), it is seen that

$$\begin{aligned} & \max_{X_n, X_{n-1}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\ &= \min\{r(A_1 A_2 \cdots A_n), \\ & \quad r(A_1 A_2 \cdots A_n) + r(A_1 A_2 \cdots A_{n-1}) + \sum_{m=2}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m), \\ & \quad r(A_1 A_2 \cdots A_n) + l_n - r(A_n) + \sum_{m=2}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m)\}. \end{aligned}$$

Noting that

$$r(A_1 A_2 \cdots A_{n-1}) + \sum_{m=2}^{n-1} l_m \geq \sum_{m=1}^{n-1} r(A_m),$$

we finally have

$$\begin{aligned} & \max_{X_n, X_{n-1}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) \\ &= \min\{r(A_1 A_2 \cdots A_n), r(A_1 A_2 \cdots A_n) + \sum_{m=2}^n l_m - \sum_{m=1}^n r(A_m)\}. \quad \square \end{aligned}$$

Since the inclusion (1.1) holds if and only if

$$\max_{X_n, X_{n-1}, \dots, X_1} r(S_{A_1, A_2, \dots, A_n}(X_1, X_2, \dots, X_n)) = 0,$$

by Theorem 2.1 we can immediately obtain the following result:

THEOREM 2.2. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \dots, n$. Then the inclusion (1.1) holds if and only if*

$$\min\{r(A_1 A_2 \cdots A_n), r(A_1 A_2 \cdots A_n) + \sum_{m=2}^n l_m - \sum_{m=1}^n r(A_m)\} = 0,$$

that is,

$$A_1 A_2 \cdots A_n = O$$

or

$$r(A_1 A_2 \cdots A_n) + \sum_{m=2}^n l_m - \sum_{m=1}^n r(A_m) = 0.$$

To end this paper, we now recover the equivalent condition established in [6] (by applying the multiple P-SVD [10], Product Singular Value Decomposition) for the inclusion (1.1).

THEOREM 2.3. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \dots, n$. Then the following statements are equivalent.*

- (1) $A_n \{1\} A_{n-1} \{1\} \cdots A_1 \{1\} \subseteq (A_1 A_2 \cdots A_n) \{1\}$;
- (2) [6] (2a') $A_1 A_2 \cdots A_n = O$ or (2b') $r(A_1 A_2 \cdots A_n) > 0$
 and for $i = 1, 2, \dots, n-1$,
 $r(A_1 A_2 \cdots A_i) + r(A_{i+1}) = l_{i+1} + r(A_1 A_2 \cdots A_{i+1})$;
- (3) (2a) $A_1 A_2 \cdots A_n = O$ or (2b) $r(A_1 A_2 \cdots A_n) + \sum_{m=2}^n l_m - \sum_{m=1}^n r(A_m) = 0$.

Proof. It suffices to prove that (2b') is equivalent to (2b). The implication (2b') \Rightarrow (2b) is easy. Now we show (2b) \Rightarrow (2b').

We first prove by induction (on i) the following identities:

$$(2.9) \quad r(A_1 A_2 \cdots A_{n-i}) = \sum_{m=1}^{n-i} r(A_m) - \sum_{m=2}^{n-i} l_m, \quad i = 0, 1, 2, \dots, n-2.$$

When $i = 0$, it reduces to identity (2b). Assume that the identity (2.9) holds for $i-1$, $1 \leq i \leq n-2$, i.e.,

$$(2.10) \quad r(A_1 A_2 \cdots A_{n-i+1}) = \sum_{m=1}^{n-i+1} r(A_m) - \sum_{m=2}^{n-i+1} l_m.$$

We now prove that the identity (2.9) is also true for i . Based on Lemma 1.2, we know

$$(2.11) \quad r(A_1 A_2 \cdots A_{n-i+1}) + l_{n-i+1} \geq r(A_1 A_2 \cdots A_{n-i}) + r(A_{n-i+1}).$$

Hence from (2.10) and (2.11), we have

$$l_{n-i+1} + \sum_{m=1}^{n-i+1} r(A_m) - \sum_{m=2}^{n-i+1} l_m \geq r(A_1 A_2 \cdots A_{n-i}) + r(A_{n-i+1}),$$

that is,

$$(2.12) \quad \sum_{m=1}^{n-i} r(A_m) \geq \sum_{m=2}^{n-i} l_m + r(A_1 A_2 \cdots A_{n-i}).$$

On the other hand, by Lemma 1.2 again, we know

$$(2.13) \quad \sum_{m=1}^{n-i} r(A_m) \leq \sum_{m=2}^{n-i} l_m + r(A_1 A_2 \cdots A_{n-i}).$$

Thus from (2.12) and (2.13), we get

$$\sum_{m=1}^{n-i} r(A_m) = \sum_{m=2}^{n-i} l_m + r(A_1 A_2 \cdots A_{n-i}).$$

This implies that the identity in (2.9) holds for $i = 0, 1, \dots, n - 2$. From (2.9) it is easy to check that the identity

$$r(A_1 A_2 \cdots A_i) + r(A_{i+1}) = l_{i+1} + r(A_1 A_2 \cdots A_{i+1})$$

holds for $i = 1, 2, \dots, n - 1$. \square

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