2008

A note on the largest eigenvalue of non-regular graphs

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A NOTE ON THE LARGEST EIGENVALUE OF NON-REGULAR GRAPHS

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Abstract. The spectral radius of connected non-regular graphs is considered. Let $\lambda_1$ be the largest eigenvalue of the adjacency matrix of a graph $G$ on $n$ vertices with maximum degree $\Delta$. By studying the $\lambda_1$-extremal graphs, it is proved that if $G$ is non-regular and connected, then $\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}$. This improves the recent results by B.L. Liu et al.

AMS subject classifications. 05C50, 15A48.

Key words. Spectral radius, Non-regular graph, $\lambda_1$-extremal graph, Maximum degree.

1. Introduction. Let $G = (V, E)$ be a simple graph on vertex set $V$ and edge set $E$, where $|V| = n$. The eigenvalues of the adjacency matrix of $G$ are called the eigenvalues of $G$. The largest eigenvalue of $G$, denoted by $\lambda_1(G)$, is called the spectral radius of $G$. Let $D$ denote the diameter of $G$. We suppose throughout the paper that $G$ is a simple graph. For any vertex $u$, let $\Gamma(u)$ be the set of all neighbors of $u$ and $d(u) = |\Gamma(u)|$ be the degree of $u$. A nonincreasing sequence $\pi = (d_1, d_2, ..., d_n)$ of non-negative integers is called (connected) graphic if there exits a (connected) simple graph on $n$ vertices, for which $d_1, d_2, ..., d_n$ are the degrees of its vertices. Let $\Delta$ and $\delta$ be the maximum and minimum degree of vertices of $G$, respectively. A graph is called regular if $d(u) = \Delta$ for any $u \in V$. It is easy to see that the spectral radius of a regular graph is $\Delta$ with $(1, 1, ..., 1)^T$ as a corresponding eigenvector. We will use $G - e$ ($G + e$) to denote the graph obtained from $G$ by deleting (adding) the edge $e$. For other notations in graph theory, we follow from [2].

Stevanović [8] first found a lower bound of $\Delta - \lambda_1$ for the connected non-regular graphs. Then the results from [8] were improved in [9, 4, 7, 3]. In [4, 7], the authors showed that

$$\Delta - \lambda_1 \geq \frac{1}{n(D + 1)} \quad ([4, 7])$$

*Received by the editors November 21, 2007. Accepted for publication February 15, 2008. Handling Editor: Stephen J. Kirkland.
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and

\[ D \leq \frac{3n + \Delta - 5}{\Delta + 1} \]  \hspace{1cm} (1.2)

B.L. Liu et al. obtained

\[ \Delta - \lambda_1 \geq \frac{\Delta + 1}{n(3n + 2\Delta - 4)} \]  \hspace{1cm} (1.3)

Recently, S.M. Cioabă [3] improved (1.1) as follows:

\[ \Delta - \lambda_1 > \frac{1}{nD} \]  \hspace{1cm} (1.4)

Thus combining (1.2) and (1.4), the inequality (1.3) can be improved as follows:

\[ \Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 5)} \]  \hspace{1cm} (1.5)

In this note we improve the inequality (1.2) on \( \lambda_1 \)-extremal graphs. Furthermore, we obtain the following inequality which improves (1.5).

\[ \Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)} \]

2. Preparation. Firstly, we state a well-known result which is just Frobenius’s theorem applied to graphs.

Lemma 2.1. Let \( G \) be a connected graph and \( \lambda_1 (G) \) be its spectral radius. Then \( \lambda_1 (G+uv) > \lambda_1 (G) \) for any \( uv \notin E \).

Definition 2.2. [7] Let \( G \) be a connected non-regular graph. Then the graph \( G \) is called \( \lambda_1 \)-extremal if \( \lambda_1 (G) \geq \lambda_1 (G') \) for any other connected non-regular graph \( G' \) with the same number of vertices and maximum degree as \( G \).

Theorem 2.3. Let \( G \) be a \( \lambda_1 \)-extremal graph on \( n \) vertices with maximum degree \( \Delta \). Define

\[ V_{<\Delta} = \{ u : u \in V \text{ and } d(u) < \Delta \}. \]

Then \( G \) must have one of the following properties:

1. \( |V_{<\Delta}| \geq 2 \) and \( V_{<\Delta} \) induces a complete graph.
2. \( |V_{<\Delta}| = 1 \).
3. \( V_{<\Delta} = \{ u, v \}, uv \notin E(G) \) and \( d(u) = d(v) = \Delta - 1 \).

Proof. By contradiction, suppose that \( G \) is a \( \lambda_1 \)-extremal graph without properties (1), (2) and (3). It follows that \( |V_{<\Delta}| \geq 2 \). Then there are two cases.
**Case 1:** $V_{<\Delta} = \{u, v\}$, $uv \notin E(G)$, $d(u) < \Delta - 1$ and $d(v) \leq \Delta - 1$. Then the graph $G + uv$ has the same maximum degree as $G$. By Lemma 2.1, we obtain $\lambda_1(G + uv) > \lambda_1(G)$, contradicting the choice of $G$.

**Case 2:** $|V_{<\Delta}| > 2$ and $V_{<\Delta}$ does not induce a complete graph. Then there exist two vertices $u, v \in V_{<\Delta}$ and $uv \notin E(G)$. Similarly arguing to case 1, we obtain $\lambda_1(G + uv) > \lambda_1(G)$, contradicting the choice of $G$.

Combining the above two cases, the proof follows. □

Using the properties mentioned in Theorem 2.3, we give the following definition.

**Definition 2.4.** Let $G$ be a connected non-regular graph on $n$ vertices with maximum degree $\Delta$. Then
- the graph $G$ is called type-I if it has property (1),
- the graph $G$ is called type-II if it has property (2),
- the graph $G$ is called type-III if it has property (3).

**Lemma 2.5.** [6] Let $G$ be a simple connected graph with $n$ vertices, $m$ edges and spectral radius $\lambda_1(G)$. Then
\[
\lambda_1(G) \leq \frac{\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - \delta n)}}{2}
\]
and equality holds if and only if $G$ is either a regular graph or a graph in which each vertex has degree either $\delta$ or $n - 1$.

We first consider the $\lambda_1$-extremal graphs with $\Delta = 2$ or $\Delta = n - 1$. When $\Delta = 2$, the $\lambda_1$-extremal graph is the path with $\lambda_1(P_n) = 2\cos(\frac{\pi}{n+1})$. When $\Delta = n - 1$, similarly arguing to Theorem 2.3, we know that the $\lambda_1$-extremal graph is $K_n - e$. By Lemma 2.5, we obtain
\[
\lambda_1(K_n - e) = \frac{n - 3 + \sqrt{(n + 1)^2 - 8}}{2}. \tag{2.1}
\]
Theorem 2.3 shows that the $\lambda_1$-extremal graphs must be type-I, type-II or type-III, but in what follows, we will prove that when $2 < \Delta < n - 1$, any type-III graph is not $\lambda_1$-extremal.

**Lemma 2.6.** [5] Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of non-negative integers. Then $\pi$ is graphic if and only if
\[
\sum_{i=1}^{n} d_i \text{ is even and } \sum_{i=1}^{k} d_i \leq k(k - 1) + \sum_{i=k+1}^{n} \min\{k, d_i\}, \text{ for all } k = 1, 2, \ldots, n - 1. \tag{2.2}
\]

**Lemma 2.7.** Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of positive
Thus degree sequence \( u \) cases. Then we need only to prove that the inequality is true. We split our proof into four

**Lemma 2.8.** Let \( \pi = (d_1, d_2, \ldots, d_n) = (\Delta, \Delta, \ldots, \Delta, \Delta - 1, \Delta - 1) \) and \( \pi' = (d'_1, d'_2, \ldots, d'_n) = (\Delta, \Delta, \ldots, \Delta, \Delta - 2) \) with \( 2 < \Delta < n - 1 \). If \( \pi \) is connected graphic, then \( \pi' \) is connected graphic.

**Proof.** Since \( 2 < \Delta < n - 1 \), we obtain \( d'_{n-1} \geq 3 \) and \( d'_n \geq 1 \). Then by Lemma 2.7, we need only to prove that \( \pi' \) is graphic. Let \( G \) be a connected graph with the degree sequence \( \pi \). Since \( \pi \) is graphic, by Lemma 2.6, we obtain

\[
\sum_{i=1}^{n} d_i \text{ is even and } \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}, \text{ for all } k = 1, 2, \ldots, n-1.
\]

For \( \pi' \) we will prove that (2.2) is still true. Obviously, \( \sum_{i=1}^{n} d'_i = \sum_{i=1}^{n} d_i \) is even. Then we need only to prove that the inequality is true. We split our proof into four cases.

**Case 1:** \( 1 \leq k \leq \Delta - 2 \), then \( k \leq n - 4 \) and

\[
\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}
\]

\[
= k(k-1) + k(n-k)
\]

\[
= k(k-1) + \sum_{i=k+1}^{n} \min\{k, d'_i\}.
\]

**Case 2:** \( 1 < k = \Delta - 1 \), then \( k \leq n - 3 \) and

\[
k(k-1) + k(n-k) - \sum_{i=1}^{k} d_i = k(k-1) + k(n-k) - k\Delta = k(n-k-2) \geq k > 1.
\]

Thus

\[
\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i < k(k-1) + k(n-k) - 1 = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d'_i\}.
\]
Case 3: $\Delta \leq k \leq n - 2$, then
\[
\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} \\
= k(k-1) + (n-k-2)\Delta + 2\Delta - 2 \\
= k(k-1) + \sum_{i=k+1}^{n} \min\{k, d'_i\}.
\]

Case 4: $k = n-1$, then
\[
k(k-1) + \Delta - \sum_{i=1}^{k} d_i = (n-1)(n-2) + \Delta - [(n-1)\Delta - 1] \\
= (n-2)(n-1-\Delta) + 1 \geq 4,
\]
where the last inequality holds since $2 < \Delta < n - 1$. Hence
\[
\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i + 1 < k(k+1) + \Delta - 2 \\
= k(k-1) + \sum_{i=k+1}^{n} \min\{k, d'_i\}.
\]

Combining the above four cases, the inequality is true. Then by Lemma 2.6, the $\pi'$ is graphic. This completes the proof. \(\square\)

As we know, majorization on degree sequences is defined as follows: for two sequences $\pi = (d_1, d_2, \ldots, d_n)$, $\pi' = (d'_1, d'_2, \ldots, d'_n)$ we write $\pi \leq \pi'$ if and only if $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$ and $\sum_{i=1}^{j} d_i \leq \sum_{i=1}^{j} d'_i$ for all $j = 1, 2, \ldots, n$. We claim that $G$ has the greatest maximum eigenvalue if $\lambda_1(G) \geq \lambda_1(G')$ for any other graph $G'$ in the class $C_\pi$, where $C_\pi = \{G : G$ is a connected graph with the degree sequence $\pi\}$.

**Lemma 2.9.** [1] Let $\pi$ and $\pi'$ be two distinct degree sequences with $\pi \leq \pi'$. Let $G$ and $G'$ be graphs with the greatest maximum eigenvalues in classes $C_\pi$ and $C_{\pi'}$, respectively. Then $\lambda_1(G) < \lambda_1(G')$.

**Theorem 2.10.** Let $G$ be a connected graph with degree sequence
\[
\pi = (\Delta, \Delta, \ldots, \Delta, \Delta - 1, \Delta - 1)
\]
and $2 < \Delta < n - 1$. Then there exists a connected graph $G'$ with degree sequence $\pi' = (\Delta, \Delta, \ldots, \Delta, \Delta - 2)$ such that $\lambda_1(G) < \lambda_1(G')$.

**Proof.** By Lemma 2.8, there exists a connected graph $G'$ with degree sequence $\pi' = (\Delta, \Delta, \ldots, \Delta, \Delta - 2)$. We suppose that $G$ ($G'$) is the graph with the greatest
maximum eigenvalue in $C_\pi (C_{\pi'})$. It is obvious that $\pi \subseteq \pi'$. By Lemma 2.9, we obtain $\lambda_1(G) < \lambda_1(G')$. \[ \Box \]

**Theorem 2.11.** Let $G$ be a $\lambda_1$-extremal graph on $n$ vertices with the maximum degree $\Delta$ and $2 < \Delta < n - 1$. Then $G$ must be either type-I or type-II.

**Proof.** Suppose that $G$ is a type-III graph with the greatest maximum eigenvalue in class $C_\pi$, where $\pi = (\Delta, \Delta, ..., \Delta, \Delta - 1, \Delta - 1)$. By Theorem 2.10, there exists a graph $G'$ with degree sequence $\pi' = (\Delta, \Delta, ..., \Delta, \Delta - 2)$ and greatest maximum eigenvalue in class $C_{\pi'}$ such that $\lambda_1(G') > \lambda_1(G)$. It follows that $G$ is not $\lambda_1$-extremal. \[ \Box \]

**Remark.** Although Theorem 2.11 shows that the $\lambda_1$-extremal graph with $2 < \Delta < n - 1$ must be type-I or type-II, there exist some graphs with property (1) or (2) which are not $\lambda_1$-extremal. Let $G_1$, $G_2$ be connected graphs with degree sequences $(3, 3, 3, 3, 2, 2)$, $(5, 5, 5, 5, 5, 5, 2)$, respectively. Clearly $G_1$ ($G_2$) is a type-I (type-II) graph. However, by checking the Table 1 of [7], we know they are not the $\lambda_1$-extremal graphs. After some computer experiments, we give a conjecture about the $\lambda_1$-extremal graphs as follows:

**Conjecture 2.12.** Let $G$ be a connected non-regular graph on $n$ vertices and $2 < \Delta < n - 1$. Then $G$ is $\lambda_1$-extremal if and only if $G$ is a graph with the greatest maximum eigenvalue in classes $C_\pi$ and $\pi = (\Delta, \Delta, ..., \Delta, \delta)$, where

$$
\delta = \begin{cases} 
(\Delta - 1), & \text{when } n\Delta \text{ is odd}, \\
(\Delta - 2), & \text{when } n\Delta \text{ is even}.
\end{cases}
$$

3. Main Results.

**Theorem 3.1.** Let $G$ be a type-I or type-II graph on $n$ vertices with diameter $D$. Then

$$D \leq \frac{3n + \Delta - 8}{\Delta + 1}. \quad (3.1)$$

**Proof.** Since $G$ is a type-I or type-II graph, we have $\Delta \geq 3$. Let $u, v$ be two vertices at distance $D$ and $P : u = u_0 \leftrightarrow u_1 \leftrightarrow ... \leftrightarrow u_D = v$ be the shortest path connecting $u$ and $v$. We first claim that $|V_{\Delta} \cap V(P)| \leq 2$. Otherwise, $G$ must be a type-I graph and suppose $\{u_p, u_q, u_r\} \subseteq V(P) \cap V_{\Delta}$ with $p < q < r$. Then by definition of type-I graph, we obtain that $u_p u_q$, $u_q u_r$ and $u_p u_r \in E(G)$. Therefore, $P$ is not the shortest path connecting $u$ and $v$, a contradiction.

Then there are two cases.

**Case 1:** $V_{\Delta} \cap V(P) = \emptyset$. Define $T = \{i : i \equiv 0 \text{ mod } 3 \text{ and } i \leq (D - 3)\} \cup \{D\}$. Thus $|T| = \left\lceil \frac{D + 1}{3} \right\rceil$. Let $d(u_i, u_j)$ denote the distance between $u_i$ and $u_j$. Since $P$ is
the shortest path connecting \( u \) and \( v \), we have \( d(u_i, u_j) \geq 3 \) and \( \Gamma(u_i) \cap \Gamma(u_j) = \emptyset \) for any distinct \( i, j \in T \). Notice that \( u_i \in V(P) \) for any \( i \in T \). We obtain

\[
|\Gamma(u_i) - V(P)| = \begin{cases} 
\Delta - 1, & \text{if } i \in \{0, D\}, \\
\Delta - 2, & \text{otherwise}.
\end{cases}
\]

Then

\[
n \geq |V(P)| + \sum_{i \in T} |\Gamma(u_i) - V(P)| \\
\geq D + 1 + (|T| - 2)(\Delta - 2) + 2(\Delta - 1) \\
\geq D + 1 + \left(\frac{D + 1}{3} - 2\right)(\Delta - 2) + 2(\Delta - 1).
\]

Thus

\[
D \leq \frac{3n - \Delta - 7}{\Delta + 1}.
\]

**Case 2:** Either \( V_{<\Delta} \cap V(P) = \{u_p, u_q\} \) with \( q = p + 1 \) or \( V_{<\Delta} \cap V(P) = \{u_p\} \). The proof is similar to the proof of [7]. We obtain the same result

\[
D \leq \frac{3n + \Delta - 8}{\Delta + 1}.
\]

Combining the above two cases, the proof follows. \(\square\)

**Lemma 3.2.** [3] Let \( G \) be a connected non-regular graph on \( n \) vertices with maximum degree \( \Delta \) and diameter \( D \). Then

\[
\Delta - \lambda_1 > \frac{1}{nD}.
\]

**Theorem 3.3.** Let \( G \) be a connected non-regular graph on \( n \) vertices with maximum degree \( \Delta \). Then

\[
\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}.
\]

**Proof.** Without loss of generality, we suppose that \( G \) is a \( \lambda_1 \)-extremal graph. Since \( G \) is connected and non-regular, then \( n \geq 3 \) and \( \Delta \geq 2 \). When \( \Delta = n - 1 \) and \( n \geq 5 \), the \( \lambda_1 \)-extremal graph is \( K_n - e \) with \( D = 2 \). Then by Lemma 3.2, we obtain

\[
\lambda_1(K_n - e) < \Delta - \frac{1}{2n} < \Delta - \frac{\Delta + 1}{n(3n + \Delta - 8)}.
\]
When $\Delta = n - 1$ and $n = 3$, the $\lambda_1$-extremal graph is $P_3$ with $\lambda_1(P_3) = 1.4142$. When $\Delta = n - 1$ and $n = 4$, the $\lambda_1$-extremal graph is $K_4 - e$ with $\lambda_1(K_4 - e) = 2.5616$. By direct calculation, we know that the inequality is true. When $2 < \Delta < n - 1$, applying Theorem 3.1 and Lemma 3.2, we obtain the result. When $\Delta = 2$ and $n > 3$, the $\lambda_1$-extremal graph is $P_n$. By adding some edges to $P_n$, we can attain $K_n - e$. Then following the Lemma 2.1, we obtain $\lambda_1(P_n) < \lambda_1(K_n - e)$. This completes the proof. □

Acknowledgement. The authors would like to thank an anonymous referee for valuable comments and suggestions that improved our presentation.

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