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THE MINIMUM SPECTRAL RADIUS OF GRAPHS WITH A GIVEN CLIQUE NUMBER*

DRAGAN STEVANOVIĆ† AND PIERRE HANSEN‡

Abstract. In this paper, it is shown that among connected graphs with maximum clique size ω, the minimum value of the spectral radius of adjacency matrix is attained for a kite graph $PK_{n-\omega,\omega}$, which consists of a complete graph K_ω to a vertex of which a path $P_{n-\omega}$ is attached. For any fixed ω, a small interval to which the spectral radii of kites $PK_{m,\omega}$, $m \geq 1$, belong is exhibited.

Key words. Adjacency matrix, Largest eigenvalue, Spectral radius, Clique number, Kite graph.

AMS subject classifications. 05C35, 05C50, 05C69.

1. Introduction. All graphs in this note are simple and undirected. For a graph G, let $A(G)$ denote its adjacency matrix, $\rho(G)$ the spectral radius of $A(G)$, and $P_G \equiv P_G(\lambda)$ the characteristic polynomial of $A(G)$. For other undefined notions, the reader is referred to [2].

Brualdi and Solheid [3] proposed the following general problem, which became one of the classic problems of spectral graph theory:

> *Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radius in this set and characterize the graphs in which the maximal spectral radius is attained.*

Such extremal graphs, for example, asymptotically have more closed walks of any given length than the other graphs in the set. However, let us point out that it may be of (practical) interest to characterize the graphs having the minimum spectral radius as well: Wang et al. [15] recently proposed a new analytic model that accurately models the propagation of computer viruses on arbitrary network graph G, which, under reasonable approximations, has an epidemic threshold of $1/\rho(G)$, below which the number of infected nodes in the network decays exponentially. Thus, the computer networks with smaller spectral radii are more resistant to virus propagation.

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As expected, there are far less results on the minimum spectral radius than on the maximum spectral radius of graphs: some recent results are given in [1, 8], while the older results are presented in Section 3 of [6].

We study here the set $\mathcal{G}_{n,\omega}$, $n \geq \omega \geq 2$, of connected graphs of order n with a maximum clique size ω. Recently, Nikiforov [12] showed that the Turán graphs attain the maximum spectral radius in $\mathcal{G}_{n,\omega}$. Experimenting with the computer system *AutoGraphiX* [4], which is well suited to find extremal graphs and support conjecture making, we observed that kite graphs attain the minimum spectral radius in $\mathcal{G}_{n,\omega}$. Recall that the *kite* graph $PK_{m,w}$ is a graph on $m+w$ vertices obtained from the path P_m and the complete graph K_w by adding an edge between an end vertex of P_m and a vertex of K_w. This observation turned out to be correct. Specifically,

THEOREM 1.1. *If $G \in \mathcal{G}_{n,\omega}$, $n \geq \omega \geq 2$, then*

$$\rho(G) \geq \rho(PK_{n-\omega,\omega}).$$

The equality holds if and only if G is isomorphic to $PK_{n-\omega,\omega}$.

We are also interested to find a good estimate of $\rho(PK_{m,\omega})$. It is obvious that

$$\rho(PK_{0,\omega}) = \rho(K_\omega) = \omega - 1.$$

However, the other values of $\rho(PK_{m,\omega})$ are not that straightforward to obtain. For any $m \geq 0$, $PK_{m,\omega}$ is a proper subgraph of $PK_{m+1,\omega}$, so the sequence $\big(\rho(PK_{m,\omega})\big)_{m\geq 0}$ is strictly increasing. Further, with $\rho(PK_{m,\omega})$ being bounded from above by ω (recall that the spectral radius is at most the maximum vertex degree, see, e.g., [5, p. 85]), we see that $\lim_{m\to\infty} \rho(PK_{m,\omega})$ exists with a value between $\omega - 1$ and ω. Thus, the values $\rho(PK_{1,\omega})$ and $\lim_{m\to\infty} \rho(PK_{m,\omega})$ represent the sharp lower and upper bound on $\rho(PK_{m,\infty})$, $m \geq 1$. These values turn out to be very close to each other.

THEOREM 1.2. *For any integers $\omega \geq 3$ and $m \geq 1$,*

$$\omega - 1 + \frac{1}{\omega^2} + \frac{1}{\omega^3} < \rho(PK_{m,\omega}) < \omega - 1 + \frac{1}{4\omega} + \frac{1}{\omega^2 - 2\omega}.$$

The proofs of these theorems are given in the next two sections.

2. Proof of Theorem 1.1. We will use the following auxiliary results in the proof. The first lemma is by Li and Feng from [9], and is also found as Theorem 6.2.2 in [6].

LEMMA 2.1. *Let u and v be two adjacent vertices of the connected graph G and for positive integers k and l, let $G(k,l)$ denote the graph obtained from G by adding*

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pendent paths of length k at u and length l at v. If $k \geq l \geq 1$, then $\rho(G(k,l)) > \rho(G(k+1,l-1))$.

The next lemma is by Zhang, Zhang and Zhang [16], and is also found as Theorem 6.2.3 of [6]. Let $SvwT$ denote the graph obtained from disjoint graphs S, T by adding an edge joining the vertex v of S to the vertex w of T. Further, let S_v denote the graph obtained from S by adding a pendent edge at vertex v.

LEMMA 2.2. *If $P_{S_u}(\lambda) < P_{S_v}(\lambda)$ for all $\lambda > \rho(S_u)$, then $\rho(SvwT) < \rho(SuwT)$ for any vertex w of T.*

REMARK 2.3. *Note that the formulation of previous lemma in [6] contains a typo: instead of $x > \mu_1(H_v)$ it should contain the condition $x > \mu_1(H_u)$.*

We may prove the following lemma on the basis of the previous two.

LEMMA 2.4. *Let vw be a bridge of a connected graph G and suppose that there is a path of length k, $k \geq 1$, attached at v, with u being the other end vertex of this path. Then*

$$\rho(G - vw + uw) \leq \rho(G),$$

with equality if and only if the vertex v has degree two.

Proof of Lemma 2.4. Let S and T be the subgraphs of G as indicated in Figure 2.1. If v has degree two, then $G - vw + uw \cong G$ and the statement follows.

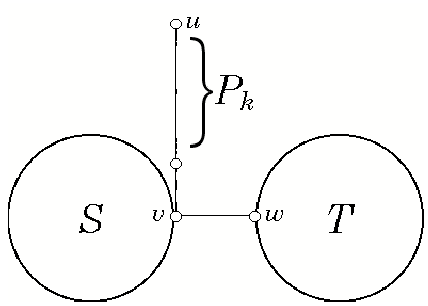

FIG. 2.1. *The structure of G.*

Suppose now that the degree of v is at least three, i.e., that S is a non-trivial graph. Let S' denote the subgraph formed by S and the path from v to u. In the terms of Lemmas 2.1 and 2.2, we have that

$$S'_u = S(k+1, 0), \quad S'_v = S(k, 1).$$

From the proof of Lemma 2.1, it is evident that $P_{S'_u}(\lambda) > P_{S'_v}(\lambda)$ for all $\lambda > \rho(S'_v)$.

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Thus, from Lemma 2.2 it follows that

$$\rho(G - vw + uw) = \rho(S'uwT) < \rho(S'vwT) = \rho(G). \qquad \square$$

Proof of Theorem 1.1. Suppose that n and ω, $n \geq \omega \geq 2$, are given.

If $n = \omega$, then $\mathcal{G}_{n,\omega}$ consists of a single graph K_n, which is also a (degenerate) kite graph $PK_{0,n}$.

If $\omega = 2$, then the path P_n has the minimum spectral radius among all connected graphs of order n [5, p. 78], and thus also in $\mathcal{G}_{n,2}$. Note that the path P_n is also a (degenerate) kite graph $PK_{n,0} = PK_{n-1,1} = PK_{n-2,2}$.

Thus, suppose that $n > \omega \geq 3$. We will transform an arbitrary graph $G \in \mathcal{G}_{n,\omega}$, containing a clique K of size ω, into a kite graph $PK_{n-\omega,\omega}$, in such a way that the spectral radius of transformed graph decreases at each step.

i) It is known that deletion of an edge from a connected graph strictly decreases its spectral radius [5]. In order to keep a graph within $\mathcal{G}_{n,\omega}$, we may delete from G any edge not in K which belongs to a cycle. Let G_1 be a subgraph of G obtained by deleting such edges in an arbitrary order as long as they exist. G_1 will then consist of a clique K with a number of rooted trees attached to clique vertices.

ii) At this step, we will "flatten out" the trees attached to clique vertices. Let T be a rooted tree of G_1, attached to a clique vertex. Let u be the leaf of T farthest from the clique K, and let v be the vertex of T of degree at least three, that is closest to u. If T is not a path, then there is a neighbor w of v in T, which is not on the path from v to u. Then in the tree $T - vw + uw$ the distance from the farthest leaf to K increases, while from Lemma 2.4 its spectral radius decreases. Processing further with edge deletions and additions by always choosing farthest leaves in rooted trees, we decrease the spectral radius at each iteration until we reach a graph G_2 in which every rooted tree, attached to a vertex of K, becomes a path.

iii) We may suppose that the graph G_2 consists of clique K and the paths P_{k_1}, P_{k_2}, $\ldots$, P_{k_m} attached to m distinct vertices of K. With the repeated use of Lemma 2.1 to paths P_{k_1} and P_{k_i}, $2 \leq i \leq m$, we may decrease the spectral radius of G_2 until the attached paths P_{k_2}, $\ldots$, P_{k_m} disappear, and we finally arrive to the kite graph $PK_{n-\omega,\omega}$.

Since we have (strictly) decreased the spectral radius at each step, we may conclude that the kite graph $PK_{n-\omega,\omega}$ has smaller spectral radius than any other graph in $\mathcal{G}_{n,\omega}$. This finishes the proof of Theorem 1.1. $\square$

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3. Proof of Theorem 1.2. To prove this theorem, we estimate the values of $\rho(PK_{1,\omega})$ and $\lim_{m\to\infty}\rho(PK_{m,\omega})$ in the next two subsections.

3.1. Estimating $\rho(PK_{1,\omega})$. Let x be a principal eigenvector of $PK_{1,\omega}$, let u be a leaf of $PK_{1,\omega}$ and let v be the neighbor of u. All other vertices of $PK_{1,\omega}$ are similar, and since $\rho(PK_{1,\omega})$ is a simple eigenvalue, we have that x has the same value, name it y, at these vertices. Now the eigenvalue equation at u, v and arbitrary other vertex of $PK_{1,\omega}$ reads (where we write just ρ for $\rho(PK_{1,m})$):

$$\begin{aligned}\rho x_u &= x_v,\\ \rho x_v &= x_u + (\omega-1)y,\\ \rho y &= (\omega-2)y + x_v.\end{aligned}$$

These equations, taken together, imply that

$$\rho^3 - (\omega-2)\rho^2 - \omega\rho + (\omega-2) = 0.$$

Introducing the variable change $\rho = \omega - 1 + \sigma$, the above cubic equation transforms into

$$\sigma^3 + (2\omega-1)\sigma^2 + (\omega^2-\omega-1)\sigma - 1 = 0.$$

While solving this cubic equation explicitly is possible, the obtained solution is cumbersome and sports at least the square root of a sixth degree polynomial in ω (which itself is not a square). However, if we denote

$$f(\sigma) = \sigma^3 + (2\omega-1)\sigma^2 + (\omega^2-\omega-1)\sigma - 1,$$

we can see that, for $\omega \geq 3$,

$$\begin{aligned}&f\left(\frac{1}{\omega^2}\left(1+\frac{1}{\omega}\right)\right)\\ &= \frac{1}{\omega^6}\left(1+\frac{1}{\omega}\right)^3 + (2\omega-1)\frac{1}{\omega^4}\left(1+\frac{1}{\omega}\right)^2 + (\omega^2-\omega-1)\frac{1}{\omega^2}\left(1+\frac{1}{\omega}\right) - 1\\ &= -\frac{2}{\omega^2} + \frac{1}{\omega^3} + \Theta\left(\frac{1}{\omega^4}\right) < 0,\end{aligned}$$

while

$$\begin{aligned}&f\left(\frac{1}{\omega^2}\left(1+\frac{2}{\omega}\right)\right)\\ &= \frac{1}{\omega^6}\left(1+\frac{2}{\omega}\right)^3 + (2\omega-1)\frac{1}{\omega^4}\left(1+\frac{2}{\omega}\right)^2 + (\omega^2-\omega-1)\frac{1}{\omega^2}\left(1+\frac{2}{\omega}\right) - 1\\ &= \frac{1}{\omega} - \frac{3}{\omega^2} + \Theta\left(\frac{1}{\omega^4}\right) > 0.\end{aligned}$$

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Thus, a root of $f(\sigma)$ belongs to the interval $\left(\frac{1}{\omega^2}(1+\frac{1}{\omega}), \frac{1}{\omega^2}(1+\frac{2}{\omega})\right)$ of length $\frac{1}{\omega^3}$, and we conclude that, for $m \geq 1$,

$$\rho(PK_{m,\omega}) \geq \rho(PK_{1,\omega}) > \omega - 1 + \frac{1}{\omega^2}\left(1+\frac{1}{\omega}\right)$$

is a simple, yet rather good lower bound.

3.2. Estimating $\lim_{m\to\infty}\rho(PK_{m,\omega})$**.** Let $PK_{\infty,\omega}$ denote the *infinite* kite graph which consists of a clique K_ω, to one vertex of which an infinite path is attached. A textbook introduction to infinite graphs may be found in Chapter 8 of [7], while the more specific results on the spectra of infinite graphs appear in [10, 11].

In particular, for an infinite graph G with bounded vertex degrees, let $\rho(G)$ denote the spectral radius of the adjacency matrix of G. We say that a sequence of subgraphs G_n converges to G, if each edge of G is contained in all but finitely many of the G_n. From Theorem 4.13 of [11] we have that the sequence $\rho(G_n)$ converges to $\rho(G)$ from below. In particular,

$$\rho(PK_{\infty,\omega}) = \lim_{m\to\infty}\rho(PK_{m,\omega}).$$

We are set to determine the value of $\rho(PK_{\infty,\omega})$ by finding a value μ and a positive vector x such that

$$A(PK_{\infty,\omega})x = \mu x \tag{3.1}$$

holds. In general, we may not say that μ is an eigenvalue of $PK_{\infty,\omega}$ as it may happen that the vector x does not belong to the space on which $A(PK_{\infty,\omega})$ acts as a linear operator. Nevertheless, the equality $\mu = \rho(PK_{\infty,\omega})$ is ensured from Theorem 6.2 of [11] (see [13, 14] for further details), which states that, for real μ and an infinite, connected and locally finite graph G, the inequality $A(G)x \leq \mu x$ has a nonnegative solution $x \neq 0$ if and only if $\mu \geq \rho(G)$, and if the equality $A(G)x = \mu x$ holds, then $\mu = \rho(G)$. Moreover, the vector x is unique in such case (up to constant multiples). The rest of this subsection is devoted to finding a pair (μ, x) which satisfies (3.1).

Let us denote the common component of x at the clique vertices by x_{-1}, the component at a clique vertex to which the path is attached by x_0, and the components at path vertices by x_1, x_2, x_3, ..., respectively. The equation (3.1) yields the following:

$$\mu x_{-1} = (\omega-2)x_{-1} + x_0 \tag{3.2}$$

$$\mu x_0 = (\omega-1)x_{-1} + x_1 \tag{3.3}$$

$$\mu x_i = x_{i-1} + x_{i+1}, \quad i \geq 1. \tag{3.4}$$

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The equation (3.4) represents a familiar linear recurrence equation. For $\mu = 2$, the positivity of x_i implies that $\omega = 2$ and, in such case, the solution is given by

$$x_i = i + 2, \quad i \geq -1.$$

For $\mu > 2$, (3.4) has the solution in the form

$$x_i = at_1^i + bt_2^i, \quad i \geq 0,$$

where a and b are constants, and

$$t_1 = \frac{\mu - \sqrt{\mu^2 - 4}}{2}, \quad t_2 = \frac{\mu + \sqrt{\mu^2 - 4}}{2}$$

are the distinct solutions of the characteristic equation $\mu t = 1 + t^2$.

When $\omega \geq 3$, the components of the positive unit eigenvector of any finite $PK_{m,\omega}$ decrease along the path from the clique towards its leaf. Besides the fact that $\rho(PK_{\infty,\omega}) = \lim_{m\to\infty} \rho(PK_{m,\omega})$ and as we would also like x to be a limit of these eigenvectors, we may impose additional condition that $\lim_{i\to\infty} x_i = 0$. This translates into $b = 0$ and, with appropriate scaling, we may suppose that

$$x_0 = 1, \quad x_1 = t_1.$$

The equations (3.2, 3.3) then lead to

$$\frac{1}{\mu - \omega + 2} = \frac{\mu - t_1}{\omega - 1}.$$

After replacing the value of t_1 and simplifying yields the quadratic equation

$$\mu^2 - (\omega - 3)\mu - \left(2\omega - 2 + \frac{1}{\omega - 2}\right) = 0,$$

whose positive solution is

$$\mu = \frac{\omega - 3 + \sqrt{(\omega + 1)^2 + \frac{4}{\omega - 2}}}{2}.$$

It is straightforward to see that μ and x satisfy (3.1), and therefore, for every finite m, it follows that

$$\rho(PK_{m,\omega}) < \rho(PK_{\infty,\omega}) = \frac{\omega - 3 + \sqrt{(\omega + 1)^2 + \frac{4}{\omega - 2}}}{2}.$$

Finally, from the simple inequality $\sqrt{1 + x} \leq 1 + \frac{x}{2}$, we get

$$\begin{aligned} \rho(PK_{\infty,\omega}) &= \frac{\omega}{2}\left(1 - \frac{3}{\omega} + \sqrt{1 + \frac{2}{\omega} + \frac{1}{\omega^2} + \frac{4}{\omega^2(\omega - 2)}}\right) \\ &\leq \omega - 1 + \frac{1}{4\omega} + \frac{1}{\omega^2 - 2\omega}. \qquad \square \end{aligned}$$

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REFERENCES

[1] F. Belardo, E.M. Li Marzi and S. Simić. Some results on the index of unicyclic graphs. *Linear Algebra Appl.*, 416:1048–1059, 2006.

[2] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications.* Macmillan Press, New York, 1976.

[3] R.A. Brualdi and E.S. Solheid. On the spectral radius of complementary acyclic matrices of zeros and ones. *SIAM J. Algebra. Discrete Method* 7:265–272, 1986.

[4] G. Caporossi and P. Hansen. Variable Neighborhood Search for Extremal Graphs, I. The AutoGraphiX System. *Discrete Math.*, 212:29–44, 2000.

[5] D. Cvetković, M. Doob and H. Sachs. *Spectra of Graphs—Theory and Application.* 3rd edition, Johann Ambrosius Barth Verlag, 1995.

[6] D. Cvetković, P. Rowlinson and S. Simić. *Eigenspaces of Graphs.* Encyclopedia of Mathematics and Its Applications, Vol.66, Cambridge University Press, Cambridge, 1997.

[7] R. Diestel. *Graph Theory.* Third edition, Graduate Texts in Mathematics, Vol. 173, Springer-Verlag, Heidelberg-New York, 2005.

[8] E. van Dam and R. Kooij. The minimal spectral radius of graphs with a given diameter. *Linear Algebra Appl.*, 423:408–419, 2007.

[9] Q. Li and K. Feng. On the largest eigenvalue of a graph. *Acta Math. Appl. Sinica*, 2:167–175, 1979 (in Chinese).

[10] B. Mohar. The Spectrum of an Infinite Graph. *Linear Algebra Appl.*, 48:245–256, 1982.

[11] B. Mohar and W. Woess. A Survey on Spectra of Infinite Graphs. *Bull. London Math. Soc*, 21:209–234, 1989.

[12] V. Nikiforov. Bounds on Graph Eigenvalues, II. Preprint. Available at http://arxiv.org/pdf/math/0612461.

[13] W.E. Pruitt. Eigenvalues of Non-negative Matrices. *Ann. Math. Statist.*, 35:1797–1800, 1964.

[14] D. Vere-Jones. Ergodic Properties of Non-negative Matrices I. *Pacific J. Math.*, 22:361–386, 1967.

[15] Y. Wang, D. Chakrabarti, C. Wang, and C. Faloutsos. Epidemic spreading in real networks: an eigenvalue viewpoint. in 22nd Symposium in Reliable Distributed Computing, Florence, Italy, October 6–8, 2003.

[16] F.J. Zhang, Z.N. Zhang, and Y.H. Zhang. Some theorems about the largest eigenvalue of graphs. *J. Xinjiang Univ. Nat. Sci*, 3:84–90, 1984 (in Chinese).