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ON THE BRUALDI-LIU CONJECTURE FOR THE EVEN PERMANENT

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Abstract. Counterexamples are given to Brualdi and Liu’s conjectured even permanent analogue of the van der Waerden-Egorychev-Falikman Theorem.

Key words. Even permanent, Doubly stochastic, Permutation matrix.

AMS subject classifications. 15A15.

For an $n \times n$ matrix $M = [m_{ij}]$ consider the sum

$$
\sum_{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)}.
$$

If the sum is taken over all permutations $\sigma$ of $[n] = \{1, 2, \ldots, n\}$ then we get $\text{per}(M)$, the permanent of $M$. If, however, we only take the sum over all even permutations $\sigma$ of $[n]$ then we get $\text{per}^{ev}(M)$, the even permanent of $M$.

Let $\Omega_n$ denote the set of doubly stochastic matrices (non-negative matrices with row and column sums 1). It is well known that $\Omega_n$ consists of all matrices which can be written as a convex combination of permutation matrices of order $n$. By analogy we define $\Omega_n^{ev}$ to be the set of all matrices which can be written as a convex combination of even permutation matrices of order $n$.

The famous van der Waerden-Egorychev-Falikman Theorem states that $\text{per}(M) \geq n!/n^n$ for all $M \in \Omega_n$ with equality iff every entry of $M$ equals $1/n$. Similarly, Brualdi and Liu [2] conjectured $\text{per}^{ev}(M) \geq \frac{1}{2} n!/n^n$ for all $M \in \Omega_n^{ev}$ with equality iff every entry of $M$ equals $1/n$. They claimed their conjecture was true for $n \leq 3$. We show below that their conjecture is false for $n \in \{4, 5\}$, although we leave open the possibility that it is true for larger $n$. For background on all of the above, see Brualdi’s new book [1].

Let

$$
C_4 = \begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix}.
$$

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Then $C_4 \in \Omega^e_4$ since

$$C_4 = \frac{1}{3} \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.$$

To show that $C_4$ is a counterexample we consider the more general problem of finding $\per^e(C_n)$ where $C_n$ is the $n \times n$ matrix with zeroes on the main diagonal and every other entry equal to $1/(n-1)$. Clearly $\per(C_n) = D_n/(n-1)^n$ where

$$D_n = n! \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} \cdots (-1)^n \frac{1}{n!} \right)$$

is the number of derangements (fixed point free permutations) of $[n]$. Using the cards-decks-hands method of Wilf [4] it can be shown that $\frac{x^n}{e^x}$ is a generating function in which the coefficient of $\frac{x^n}{n^n}$ is the number of derangements of $[n]$ with exactly $k$ cycles. It can then be deduced that the number of even derangements is $\frac{1}{2} \left( D_n + (-1)^n (1-n) \right)$ (this result is probably well-known, certainly it is obtained in [3]). Hence

$$\frac{\per^e(C_n)}{2n!/n^n} = \frac{D_n + (-1)^n (1-n)}{n!} \left( \frac{n}{n-1} \right)^n \frac{1}{e} \left( \frac{1}{n} - \frac{1}{n-1} \right) \exp \left( 1 + \frac{1}{2n} \right) > 1$$

for $n \geq 5$. It follows that $C_n$ is not a counterexample to the Brualdi-Liu conjecture for any $n \geq 5$. However, $\per^e(C_4) = 1/27 < 3/64$ so $C_4$ is a counterexample.

Two further counterexamples arise from the following family of matrices. Let $T_n$ denote the mean of the $(n-1)(n-2)$ permutation matrices corresponding to 3-cycles which move the point 1. For example,

$$T_5 = \begin{bmatrix}
0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{bmatrix}.$$ 

Then $T_n \in \Omega^e_n$ by construction. Now given that $\per^e(T_4) = 5/108 < 3/64$ and $\per^e(T_5) = 11/576 < 12/625$, both $T_4$ and $T_5$ are counterexamples to the Brualdi-Liu conjecture. That the family $\{T_n\}$ contains no further counterexamples is easy to show. The permutation matrices corresponding to 3-cycles alone contribute at least

$$(n-1)(n-2) \frac{1}{(n-1)^2} \left( \frac{n-3}{n-1} \right)^{n-3} \frac{1}{(n-1)(n-2)} = \frac{(n-3)^{n-3}}{(n-1)^{n-1}} \sim \frac{1}{(\text{em})^2}$$

to $\per^e(T_n)$. 

REFERENCES


