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ON THE BRUALDI-LIU CONJECTURE FOR THE EVEN PERMANENT

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Abstract. Counterexamples are given to Brualdi and Liu's conjectured even permanent analogue of the van der Waerden-Egorychev-Falikman Theorem.

Key words. Even permanent, Doubly stochastic, Permutation matrix.

AMS subject classifications. 15A15.

For an \( n \times n \) matrix \( M = [m_{ij}] \) consider the sum

\[
\sum_{\sigma} \prod_{i=1}^{n} m_{i \sigma(i)}.
\]

If the sum is taken over all permutations \( \sigma \) of \( [n] = \{1, 2, \ldots, n\} \) then we get \( \text{per}(M) \), the permanent of \( M \). If, however, we only take the sum over all even permutations \( \sigma \) of \( [n] \) then we get \( \text{per}^\text{ev}(M) \), the even permanent of \( M \).

Let \( \Omega_n \) denote the set of doubly stochastic matrices (non-negative matrices with row and column sums 1). It is well known that \( \Omega_n \) consists of all matrices which can be written as a convex combination of permutation matrices of order \( n \). By analogy we define \( \Omega_n^\text{ev} \) to be the set of all matrices which can be written as a convex combination of even permutation matrices of order \( n \).

The famous van der Waerden-Egorychev-Falikman Theorem states that \( \text{per}(M) \geq n!/n^n \) for all \( M \in \Omega_n \) with equality iff every entry of \( M \) equals \( 1/n \). Similarly, Brualdi and Liu [2] conjectured \( \text{per}^\text{ev}(M) \geq \frac{1}{2}n!/n^n \) for all \( M \in \Omega_n^\text{ev} \) with equality iff every entry of \( M \) equals \( 1/n \). They claimed their conjecture was true for \( n \leq 3 \). We show below that their conjecture is false for \( n \in \{4, 5\} \), although we leave open the possibility that it is true for larger \( n \). For background on all of the above, see Brualdi’s new book [1].

Let

\[
C_4 = \begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix}.
\]
Then \( C_4 \in \Omega_5^{ev} \) since
\[
C_4 = \frac{1}{3} \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]

To show that \( C_4 \) is a counterexample we consider the more general problem of finding \( \text{per}^{ev}(C_n) \) where \( C_n \) is the \( n \times n \) matrix with zeroes on the main diagonal and every other entry equal to \( 1/(n-1) \). Clearly \( \text{per}(C_n) = D_n/(n-1)^n \) where
\[
D_n = n! \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} \cdots (-1)^n \frac{1}{n!} \right)
\]
is the number of derangements (fixed point free permutations) of \([n]\). Using the cards-decks-hands method of Wilf \[4\] it can be shown that \( (1-x)^{-k}e^{-yx} \) is a generating function in which the coefficient of \( \frac{x^k}{k!} \) is the number of derangements of \([n]\) with exactly \( k \) cycles. It can then be deduced that the number of even derangements is \( \frac{1}{2} (D_n + (-1)^n(1-n)) \) (this result is probably well-known, certainly it is obtained in \[3\]). Hence
\[
\text{per}^{ev}(C_n) = \frac{\left( D_n + (-1)^n(1-n) \right)}{n!} \left( \frac{n}{n-1} \right)^n \exp \left( \frac{1}{e} - \frac{1}{n(n-2)!} \right) \exp \left( 1 + \frac{1}{2n} \right) > 1
\]
for \( n \geq 5 \). It follows that \( C_n \) is not a counterexample to the Brualdi-Liu conjecture for any \( n \geq 5 \). However, \( \text{per}^{ev}(C_4) = 1/27 < 3/64 \) so \( C_4 \) is a counterexample.

Two further counterexamples arise from the following family of matrices. Let \( T_n \) denote the mean of the \((n-1)(n-2)\) permutation matrices corresponding to 3-cycles which move the point 1. For example,
\[
T_5 = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix},
\]

Then \( T_n \in \Omega_n^{ev} \) by construction. Now given that \( \text{per}^{ev}(T_4) = 5/108 < 3/64 \) and \( \text{per}^{ev}(T_5) = 11/576 < 12/625 \), both \( T_4 \) and \( T_5 \) are counterexamples to the Brualdi-Liu conjecture. That the family \{\( T_n \)\} contains no further counterexamples is easy to show. The permutation matrices corresponding to 3-cycles alone contribute at least
\[
(n-1)(n-2) \frac{1}{(n-1)^2} \frac{1}{n-1} (n-3)^{n-3} \frac{1}{n} = \frac{(n-3)^{n-3}}{(n-1)^{n-1}} \sim \frac{1}{(cm)^2}
\]
to \( \text{per}^{ev}(T_n) \).
REFERENCES


