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ON A CLASSIC EXAMPLE IN THE NONNEGATIVE INVERSE EIGENVALUE PROBLEM*

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Abstract. This paper presents a construction of nonnegative matrices with nonzero spectrum $\tau = (3+t, 3-t, -2, -2, -2)$ for $t > 0$. The result presented gives a constructive proof of a result of Boyle and Handelman in this special case. This example exhibits a surprisingly fast convergence of the spectral gap of τ to zero as a function of the number of zeros that are added to the spectrum.

Key words. Nonnegative matrices, Nonnegative inverse eigenvalue problem, Spectral Gap, Companion Matrices.

AMS subject classifications. 15A48,15A18, 15A29.

1. Introduction.

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions for a given list of complex numbers

$$\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$$

to be the spectrum of some entrywise nonnegative matrix. If for a given list σ a nonnegative matrix A with the spectrum σ exists, we will say that σ is realizable and that the matrix A realizes σ.

A classical result of Perron and Frobenius tells us that the spectrum of a nonnegative matrix has to contain a positive real number that is greater than or equal to the absolute value of any other number in the list. This number is called the Perron eigenvalue. Since for any nonnegative matrix A the trace of A^k, $k = 1, 2, \ldots,$ is nonnegative, the following condition must hold:

$$s_k(\sigma) = \lambda_1^k + \lambda_2^k + \ldots + \lambda_n^k \geq 0.$$

Some other necessary conditions for the NIEP are known, some of which we will mention later. However, known necessary conditions are far from being sufficient.

The problem of characterizing spectra of nonnegative matrices was posed by Suleimanova in [14]. Suleimanova considered a special case when σ contains only

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real numbers, $\lambda_1 > 0$ and $\lambda_i \leq 0$, $i = 1, 2, \ldots, n$. In this case it turns out that the condition $s_1(\sigma) = \lambda_1 + \lambda_2 + \ldots + \lambda_n \geq 0$ is sufficient for realizability of σ. An elegant proof of this result was given in [2], where it is shown that in this special case the companion matrix of the polynomial $f(x) = (x - \lambda_1) \ldots (x - \lambda_n)$ is nonnegative.

A major breakthrough on the nonnegative inverse eigenvalue problem was achieved by Boyle and Handelman in [1]. They considered the problem of characterizing the nonzero spectra of nonnegative matrices. We say that a list of complex numbers σ is the nonzero spectrum of a nonnegative matrix, if there exists a nonnegative integer N such that σ together with N zeros added to it, is the spectrum of some nonnegative matrix. A primitive matrix is a square nonnegative matrix for which some power is strictly positive. Boyle and Handelman proved the following result.

Theorem 1.1. ([1]) *A list of complex numbers σ is the nonzero spectrum of some primitive matrix if and only if the following conditions hold:*

1. *σ contains a positive number strictly greater than the modulus of any other entry in σ.*
2. *σ is closed under complex conjugation.*
3. *For all positive integers n and k,*

$$s_n(\sigma) \geq 0,$$

and $s_n(\sigma) > 0$ implies $s_{nk}(\sigma) > 0$.

A question that stems from the above theorem is: how many zeros do we need to add to a given list of complex numbers σ, with $s_k(\sigma) > 0$ for $k = 1, 2, \ldots,$ in order to achieve realizability. Furthermore, the proof of Theorem 1.1 presented in [1] is not constructive. The problem of finding a constructive proof of this result is still open.

An advance on this questions was made in [8] where a complete and constructive proof of the nonnegative inverse eigenvalue problem was given for a list of complex numbers where all but the Perron eigenvalue have negative real parts.

Theorem 1.2. ([8]) *Let $\lambda_2, \lambda_3, \ldots, \lambda_n$ be nonzero complex numbers with real parts less than or equal to zero and let λ_1 be a positive real number. Then the list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the nonzero spectrum of a nonnegative matrix if the following assumptions are satisfied:*

1. *The list σ is closed under the complex conjugation.*
2. $\mathrm{s}_1(\sigma) \geq 0$.
3. $\mathrm{s}_2(\sigma) > 0$.

The minimal number of zeros that need to be added to σ to make it realizable is the

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smallest nonnegative integer N for which the following inequality is satisfied:

$$\mathrm{s}_1^2(\sigma) \leq (n+N)\mathrm{s}_2(\sigma). \tag{1.1}$$

Furthermore, the list $(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, \ldots, 0)$ can be realized by a nonnegative matrix of the form $C + \alpha I$, where C is a companion matrix with trace zero, α is a nonnegative scalar and I is the identity matrix of the appropriate size.

Note that the construction in the above theorem yields a primitive matrix. In the special case considered, the necessary condition that governs the number of zeros needed for realizability, is the inequality:

$$\mathrm{s}_1(\sigma)^2 \leq (n+N)\mathrm{s}_2(\sigma). \tag{1.2}$$

This inequality is one of the so called JLL inequalities that were proved to be necessary for realizability in [11] and independently in [3]. A more general statement of the JLL inequalities is that a realizable list of n complex numbers σ satisfies

$$n^{k-1}\mathrm{s}_{km}(\sigma) \geq \mathrm{s}_m^k(\sigma) \tag{1.3}$$

for all positive integers k and m.

The example that is frequently referred to in papers on the NIEP is the question of realizability of the list of the form

$$\tau(t) = (3+t, 3-t, -2, -2, -2)$$

or its slight modification

$$\tau'(t) = (3+t, 3, -2, -2, -2).$$

Neither $\tau(0)$ nor $\tau'(0)$ are realizable. Neither of them can even be the nonzero spectrum of a nonnegative matrix. The smallest t for which $\tau(t)$ is realizable is $t = \sqrt{16\sqrt{6} - 39}$, [7]. The question of finding the smallest t for which $\tau'(t)$ is realizable is still open. Currently is known that this t lies in the interval

$$0.396711738... \leq t \leq 0.51931098....$$

These bounds were found in [13].

Note that for any $t > 0$, $\tau(t)$ has a Perron eigenvalue $3+t$, $\mathrm{s}_k(\tau(t)) > 0$ for $k = 1, 2, \ldots$, and all the JLL inequalities hold. Theorem 1.1 tells us that $\tau(t)$ is the nonzero spectrum of some nonnegative matrix for all $t > 0$. When we add zeros to the list $\tau(t)$, the smallest t that makes the extended list realizable must tend to zero as the number of zeros added tends to infinity.

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Starting from an $(n+N)\times(n+N)$ nonnegative matrix B_N with spectrum $\tau(t)$ with N zeros attached, one can construct, using results in [15], a nonnegative matrix B_N' with spectrum $\tau'(2t)$ with N zeros attached.

Examples $\tau(t)$ and $\tau'(t)$ were studied also in the context of the symmetric inverse eigenvalue problem (SNIEP) ([10, 12, 9]), which asks when is the list of real numbers the spectrum of an $n\times n$ symmetric nonnegative matrix. Hartwig and Loewy showed in an unpublished work that the smallest t for which $\tau'(t)$ is realizable by a symmetric nonnegative matrix is $t = 1$. The smallest t for which $\tau(t)$ is the spectrum of a symmetric nonnegative matrix is $t = 1$. This was first shown by Loewy and a different proof can be found in [12]. The case when we add one zero to $\tau(t)$ was considered in [9], where it was shown that $\tau(1/3)$ with one zero added is realizable by a symmetric nonnegative matrix. This shows that adding zeros to the list improves realizability also in the symmetric case. While in the general case arbitrarily large numbers of zeros may need to be added to the nonzero spectrum to obtain realization, in the symmetric case the number of zeros needed is bounded in terms of the number of nonzero elements in the list [4].

In this paper we present a method that gives a constructive proof of Theorem 1.1 for the list $\tau(t) = (3+t, 3-t, -2, -2, -2)$. This proof also gives a bound on the number of zeros needed for realizability for any given $t > 0$. This example was used as a test sequence by many authors with references going back at least to 1977 [2]. Our method uses block companion type matrices presented in [6] and is an explicit variation of the power series method developed by Kim, Ormes and Roush [5]. It enables us to explicitly present bounds on the number of zeros needed and to discuss the convergence rate with which t tends to zero as the number of zeros added to the spectrum is increased. The methods presented in this paper can be applied to a much more general setting, and this will be developed in a forthcoming paper.

2. Main Result. Let

$$\sigma_N(t) = (3+t, 3-t, -2, -2, -2, \underbrace{0, \ldots, 0}_{N}).$$

We are looking for a nonnegative matrix $A_N(t)$ with characteristic polynomial

$$w_N(x) = (x-3-t)(x-3+t)(x+2)^3 x^N.$$

Let $g(x) = (x-3)(x^2+3x+3) = x^3 - 6x - 9$ and $h(x) = x^2 - 3$. Then we write

$$w_N(x) = g(x)q_N(x) + r_N(x)$$

and

$$q_N(x) = (x^2-3)v_N(x) + p_N(x),$$

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where $r_N(x) = a_N x^2 + b_N x + c_N$ and $p_N(x) = e_N x + f_N$ are polynomials of degree (at most) 2 and 1, respectively. For the polynomial

$$v_N(x) = x^N + \alpha_1 x^{N-1} + \alpha_2 x^{N-2} + \ldots + \alpha_N$$

let $C(v_N)$ denote the companion matrix of v_N :

$$C(v_N) = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \\ -\alpha_N & -\alpha_{N-1} & -\alpha_{N-2} & -\alpha_{N-3} & \ldots & -\alpha_1 \end{pmatrix}.$$

Then the matrix

$$A_N(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 9 & 6 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & C(v_N) & & \\ 0 & 0 & 0 & 0 & 0 & & & & \\ -c_N & -b_N & -a_N & -f_N & -e_N & & & & \end{pmatrix} \tag{2.1}$$

has characteristic polynomial $w_N(x)$ [6]. Clearly, a_N, b_N, c_N, e_N, f_N and $v_N(x)$ depend on t and N. We will show that for every $t > 0$ there exists a positive integer N, so that a_N, b_N, c_N, e_N, f_N will be greater than or equal to zero and the companion matrix of v_N will be nonnegative. In other words, we claim that for each $t > 0$ there exists N so that $A_N(t) \geq 0$.

To begin let us consider the case where $N = 0$. We compute

$$a_0 = -1 - 6t^2 \tag{2.2}$$
$$b_0 = 6 - 18t^2 \tag{2.3}$$
$$c_0 = -9 - 17t^2 \tag{2.4}$$
$$e_0 = 0 \tag{2.5}$$
$$f_0 = -6 - t^2 \tag{2.6}$$
$$v_0(x) = 1. \tag{2.7}$$

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The form of $A_0(t)$ is different than form of $A_N(t)$ for $N \geq 1$:

$$A_0(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 9 & 6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 9+17t^2 & -6+18t^2 & 1+6t^2 & 9+t^2 & 0 \end{pmatrix}.$$

This implies that for $t_0 = \frac{1}{\sqrt{3}}$ we have $A_0(t_0) \geq 0$.

Now we will find recursive relations for the expressions we are interested in. From $w_{N+1}(x) = g(x)q_{N+1}(x) + r_{N+1}(x)$ and

$$\begin{aligned} w_{N+1}(x) &= xw_N(x) && (2.8)\\ &= xg(x)q_N(x) + xr_N(x) && (2.9)\\ &= g(x)(xq_N(x) + a_N) + b_N x^2 + (6a_N + c_N)x + 9a_N, && (2.10) \end{aligned}$$

we get

$$\begin{aligned} q_{N+1}(x) &= xq_N(x) + a_N && (2.11)\\ a_{N+1} &= b_N && (2.12)\\ b_{N+1} &= 6a_N + c_N && (2.13)\\ c_{N+1} &= 9a_N. && (2.14) \end{aligned}$$

Moreover, we have

$$\begin{aligned} q_{N+1}(x) &= xq_N(x) + a_N && (2.15)\\ &= x(x^2-3)v_N(x) + xp_N(x) + a_N && (2.16)\\ &= (x^2-3)(xv_N(x) + e_N) + xf_N + 3e_N + a_N. && (2.17) \end{aligned}$$

This gives us

$$\begin{aligned} v_{N+1}(x) &= xv_N(x) + e_N && (2.18)\\ e_{N+1} &= f_N && (2.19)\\ f_{N+1} &= 3e_N + a_N && (2.20) \end{aligned}$$

Initial condition $v_0(x) = 1$ and equation (2.18) imply that to prove that $C(v_N)$ is nonnegative, it is sufficient to prove that $e_N \leq 0$ for all positive integers N and $t > 0$.

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The recursive equations

$$a_{N+1} = b_N \tag{2.21}$$
$$b_{N+1} = 6a_N + c_N \tag{2.22}$$
$$c_{N+1} = 9a_N \tag{2.23}$$
$$e_{N+1} = f_N \tag{2.24}$$
$$f_{N+1} = 3e_N + a_N \tag{2.25}$$

together with initial conditions

$$a_0 = -1 - 6t^2 \tag{2.26}$$
$$b_0 = 6 - 18t^2 \tag{2.27}$$
$$c_0 = -9 - 17t^2 \tag{2.28}$$
$$e_0 = 0 \tag{2.29}$$
$$f_0 = -6 - t^2. \tag{2.30}$$

give us the following solutions:

$$a_N = \frac{3^{N/2}}{21}(-3^{N/2}125t^2 - (21+t^2)\cos(\tfrac{5N\pi}{6}) - 3\sqrt{3}(-21+t^2)\sin(\tfrac{5N\pi}{6}))$$
$$b_N = \frac{3^{N/2}}{7}(-3^{N/2}125t^2 - (-42+t^2)\cos(\tfrac{5N\pi}{6}) + \frac{\sqrt{3}}{3}(-84+5t^2)\sin(\tfrac{5N\pi}{6}))$$
$$c_N = \frac{3^{N/2}}{7}(-3^{N/2}125t^2 + (-63+6t^2)\cos(\tfrac{5N\pi}{6}) + \sqrt{3}(-105+4t^2)\sin(\tfrac{5N\pi}{6})).$$

The solutions for e_N and f_N split into two cases: the case when N is even and the case when N is odd. Moreover, since $e_{N+1} = f_N$ it is sufficient to study e_N.

$$\begin{aligned} e_{2N_1} &= 3^{N_1}\Big(\frac{1}{3}(-5 + 5\cos(\tfrac{5N_1\pi}{3}) - \sqrt{3}\sin(\tfrac{5N_1\pi}{3})) \\ &\quad + \frac{1}{126}t^2(-3^{N_1}125 + 133 - 8\cos(\tfrac{5N_1\pi}{3}) + 4\sqrt{3}\sin(\tfrac{5N_1\pi}{3}))\Big) \\ e_{2N_1+1} &= 3^{N_1}\Big(-3 + \frac{5}{\sqrt{3}}\cos(\tfrac{5(2N_1+1)\pi}{6}) - \sin(\tfrac{5(2N_1+1)\pi}{6}) \\ &\quad + \frac{3}{126}t^2(-3^{N_1}125 - \frac{8\sqrt{3}}{3}\cos(\tfrac{5(2N_1+1)\pi}{6}) + 77 + 4\sin(\tfrac{5(2N_1+1)\pi}{6}))\Big) \end{aligned}$$

Let us consider expressions for a_N, b_N and c_N as a function of N for some fixed $t > 0$. Observe that the term $-3^{N/2}125t^2$ appears in a_N, b_N and c_N. Since other terms are multiplied by either $\cos(\frac{5N\pi}{6})$ or $\sin(\frac{5N\pi}{6})$ and are otherwise independent of N, we see that the term $-3^{N/2}125t^2$ will dominate the expressions for sufficiently large N. Hence, we can always find sufficiently large N that will, for a given t, make expressions for a_N, b_N and c_N less than or equal to zero.

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It is left to prove that e_N is less than or equal to zero for all $t > 0$ and all positive integers N. Let us look at the expressions for $\frac{1}{3^{N_1}}e_{2N_1}$ and $\frac{1}{3^{N_1}}e_{2N_1+1}$ as polynomials of degree two in t. The constant terms in those expressions are periodic functions in N_1 that for all positive integers N_1 take values that are less than or equal to zero. The coefficient of t is zero in both cases and it is not difficult to see that the coefficients of t^2 are less that or equal to zero for all positive integers N_1. This shows that e_N is less than or equal to zero for all $t > 0$ and all positive integers N, which also implies that f_N is less than or equal to zero and that companion matrix of v_N is nonnegative for all $t > 0$ and all positive integers N. We now state our main result.

THEOREM 2.1. *Let*

$$\sigma_N(t) = (3+t, 3-t, -2, -2, -2, \underbrace{0, \ldots, 0}_{N}).$$

Then if $t \geq 3^{-N/4}\sqrt{2}$, $\sigma_N(t)$ can be realized by a matrix $A_N(t)$ of the form (2.1).

To complete the proof, we need to show that $A_N(t)$ is nonnegative for $t \geq 3^{-N/4}\sqrt{2}$. This is done in the next section.

COROLLARY 2.2. *The list*

$$\sigma'_N(t) = (3+t, 3, -2, -2, -2, \underbrace{0, \ldots, 0}_{N})$$

is realizable for $t \geq 3^{-N/4}2\sqrt{2}$.

3. Proof of the bound and comments on spectral gap. From the last section, it follows that $A_N(t)$ is nonnegative as long as a_N, b_N and c_N are less than or equal to zero. Let for a given positive integer N, t_N denote the smallest $t > 0$ for which a_N, b_N and c_N are less than or equal to zero.

From the formulae for a_N, b_N and c_N it is possible to explicitly compute t_N for

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any given N. Below we have computed some examples.

$$t_0 = \frac{1}{\sqrt{3}} = 0.57735 \tag{3.1}$$

$$t_1 = \frac{1}{\sqrt{3}} = 0.57735 \tag{3.2}$$

$$t_2 = \frac{1}{\sqrt{3}} = 0.57735 \tag{3.3}$$

$$t_3 = \frac{1}{\sqrt{6}} = 0.408248 \tag{3.4}$$

$$t_4 = \frac{1}{\sqrt{6}} = 0.408248 \tag{3.5}$$

$$t_5 = \sqrt{\frac{3}{161}} = 0.136505 \tag{3.6}$$

$$t_{10} = \sqrt{\frac{3}{1085}} = 0.052583 \tag{3.7}$$

$$t_{20} = \sqrt{\frac{15}{1054447}} = 0.00377167 \tag{3.8}$$

Since $-3^{N/2}125t^2$ is the dominating term in expressions for a_N, b_N and c_N, we see that t_N will converge to zero surprisingly rapidly, with the rate of $3^{-N/4}$.

In fact, using rough estimates one can check that

$$t_N \leq 3^{-N/4}\sqrt{2}.$$

Let $3^{-N/2}2 \leq t^2 \leq 1$, then

$$\begin{aligned}
21 \cdot 3^{-N/2} a_N &\leq -125 \cdot 2 + (21 + t^2) + 3\sqrt{3}(21 - t^2) \leq -250 + 22 + 63\sqrt{3} \leq 0, \\
7 \cdot 3^{-N/2} b_N &\leq -125 \cdot 2 + (42 - t^2) + \frac{1}{\sqrt{3}}(84 - 5t^2) \leq -250 + 42 + \frac{84}{\sqrt{3}} \leq 0 \\
7 \cdot 3^{-N/2} c_N &\leq -125 \cdot 2 + (63 - 6t^2) + \sqrt{3}(105 - 4t^2) \leq -250 + 63 + 105\sqrt{3} \leq 0
\end{aligned}$$

This yields the bound presented in Theorem 2.1

The difference between the largest and the modulus of the second largest eigenvalue of a nonnegative matrix, is often called the spectral gap. The spectral gap for the realizable list $\sigma_N(t_N)$ is equal to $2t_N$. The example considered in this paper exhibits surprisingly fast convergence of the spectral gap to zero as we add zeros to the spectrum.

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