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## NONNEGATIVITY OF SCHUR COMPLEMENTS OF NONNEGATIVE IDEMPOTENT MATRICES\*

SHMUEL FRIEDLAND<sup>†</sup> AND ELENA VIRNIK<sup>‡</sup>

**Abstract.** Let  $A$  be a nonnegative idempotent matrix. It is shown that the Schur complement of a submatrix, using the Moore-Penrose inverse, is a nonnegative idempotent matrix if the submatrix has a positive diagonal. Similar results for the Schur complement of any submatrix of  $A$  are no longer true in general.

**Key words.** Nonnegative idempotent matrices, Schur complement, Moore-Penrose inverse, generalized inverse.

**AMS subject classifications.** 15A09, 15A15, 15A48.

**1. Introduction.** Let  $\langle n \rangle := \{1, \dots, n\}$  and assume that  $\alpha \subset \langle n \rangle$ ,  $\alpha^c := \langle n \rangle \setminus \alpha$ ,  $\beta \subset \langle n \rangle$  are three nonempty sets. For  $A \in \mathbb{R}^{n \times n}$ , denote by  $A[\alpha, \beta]$  the submatrix of  $A$  composed of the rows and columns indexed by the sets  $\alpha$  and  $\beta$ , respectively. Assume that  $A[\alpha, \alpha]$  is invertible. Then, the  $\alpha$  Schur complement of  $A$ , which is equal to the Schur complement of  $A[\alpha, \alpha]$ , is given by

$$A(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{-1}A[\alpha, \alpha^c]. \quad (1.1)$$

If  $A[\alpha, \alpha]$  is not invertible we define

$$A_{\text{ginv}}(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{\text{ginv}}A[\alpha, \alpha^c], \quad (1.2)$$

for some semi-inverse  $A[\alpha, \alpha]^{\text{ginv}}$  [1]. The  $\alpha$  Moore-Penrose Schur complement of  $A$  is defined as

$$A_{\dagger}(\alpha) := A[\alpha^c, \alpha^c] - A[\alpha^c, \alpha]A[\alpha, \alpha]^{\dagger}A[\alpha, \alpha^c],$$

where  $A[\alpha, \alpha]^{\dagger}$  is the Moore-Penrose inverse of  $A[\alpha, \alpha]$  [3, 5, 6].

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Assume that  $A$  is a nonnegative idempotent matrix, i.e.,  $A^2 = A \in \mathbb{R}_+^{n \times n}$ . In this note we show that if  $A[\alpha, \alpha]$  has a positive diagonal then  $A_{\dagger}(\alpha)$  is a nonnegative idempotent matrix. We give an example of  $A$ , where  $A[\alpha, \alpha]$  has a nonpositive diagonal, and  $A_{\dagger}(\alpha)$  has positive and negative entries. We show that for certain  $A[\alpha, \alpha]$  with a nonpositive diagonal, which includes the above example, one can define a semi-inverse such that  $A_{\text{ginv}}(\alpha)$  is nonnegative and idempotent. We do not know if this result holds in general. Our results follow from Flor's theorem [4], using manipulations with block matrices. Our study was motivated by the analysis of positive differential-algebraic equations (DAEs) [2, 7].

**2. Main result.** First, we recall the following facts [1]. For  $U \in \mathbb{R}^{m \times n}$ , a matrix  $U^{\text{ginv}} \in \mathbb{R}^{n \times m}$  is called a semi-inverse of  $U$  if the following conditions hold

$$UU^{\text{ginv}}U = U, \quad U^{\text{ginv}}UU^{\text{ginv}} = U^{\text{ginv}}. \quad (2.1)$$

If  $0 \neq U = \mathbf{xy}^{\top}$  then

$$U^{\dagger} = \frac{1}{(\mathbf{x}^{\top}\mathbf{x})(\mathbf{y}^{\top}\mathbf{y})}\mathbf{yx}^{\top}.$$

If we assume that  $U$  is a direct sum of matrices  $U = \bigoplus_{i=1}^s U_i$ , then  $U^{\dagger} = \bigoplus_{i=1}^s U_i^{\dagger}$ .

For our main result we need the following simplification of Flor's theorem [4].

**LEMMA 2.1.** *Any nonzero nonnegative idempotent matrix  $B \in \mathbb{R}_+^{n \times n}$  is permutationally similar to the following  $3 \times 3$  block matrix*

$$P := \begin{bmatrix} J & JG & 0 \\ 0 & 0 & 0 \\ FJ & FJG & 0 \end{bmatrix}, \quad J \in \mathbb{R}_+^{n_1 \times n_1}, G \in \mathbb{R}_+^{n_1 \times n_2}, F \in \mathbb{R}_+^{n_3 \times n_1}, \quad (2.2)$$

where  $n = n_1 + n_2 + n_3, 1 \leq n_1, 0 \leq n_2, 0 \leq n_3$ .  $F, G$  are arbitrary nonnegative matrices, and  $J$  is a direct sum of  $k \geq 1$  rank one positive idempotent matrices  $J_i \in \mathbb{R}_+^{l_i \times l_i}$ , i.e.,

$$J = \bigoplus_{i=1}^k J_i, \quad J_i = \mathbf{u}_i \mathbf{v}_i^{\top}, \quad \mathbf{0} < \mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}_+^{l_i}, \quad \mathbf{v}_i^{\top} \mathbf{u}_i = 1, \quad i = 1, \dots, k. \quad (2.3)$$

*Proof.* Flor's theorem states that  $B$  is permutationally similar to the following block matrix [4]

$$C := \begin{bmatrix} J & JG_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F_1 J & F_1 JG_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here,  $J \in \mathbb{R}_+^{n_1 \times n_1}$  is of the form (2.3),  $G_1 \in \mathbb{R}_+^{n_1 \times m_2}$ ,  $F_1 \in \mathbb{R}_+^{n_3 \times n_1}$  are arbitrary nonnegative matrices, and the last  $m_4$  rows and columns of  $C$  are zero. Hence,  $n_1 + m_2 + n_3 + m_4 = n$  and  $0 \leq m_2, n_3, m_4$ . If  $m_4 = 0$  then  $C$  is of the form (2.2). It remains to show that  $C$  is permutationally similar to  $P$  if  $m_4 > 0$ .

Interchanging the last row and column of  $C$  with the  $(n_1 + m_2 + 1)$ -st row and column of  $C$  we obtain a matrix  $C_1$ . Then, we interchange the  $(n - 1)$ -st row and column of  $C_1$  with the  $(n_1 + m_2 + 2)$ -nd row and column of  $C_1$ . We continue this process until we obtain the idempotent matrix  $P$  with  $n_2 = m_2 + m_4$  zero rows located at the rows  $n_1 + 1, \dots, n_1 + n_2$ . It follows that  $P$  is of the form

$$P := \begin{bmatrix} J & G & 0 \\ 0 & 0 & 0 \\ F & H & 0 \end{bmatrix}, G \in \mathbb{R}_+^{n_1 \times n_2}, F \in \mathbb{R}_+^{n_3 \times n_1}, H \in \mathbb{R}_+^{n_3 \times n_3}.$$

Since  $P^2 = P$  we have that

$$G = JG, F = FJ, H = FG = (FJ)(JG) = FJG.$$

Hence,  $P$  is of the form (2.2).  $\square$

**THEOREM 2.2.** *Let  $A \in \mathbb{R}_+^{n \times n}$  be a nonnegative idempotent matrix. We assume that for  $\emptyset \neq \alpha \subsetneq \langle n \rangle$ , the submatrix  $A[\alpha, \alpha]$  has a positive diagonal. Then  $A_{\dagger}(\alpha)$  is a nonnegative idempotent matrix. Furthermore,*

$$\text{rank } A_{\dagger}(\alpha) = \text{rank } A - \text{rank } A[\alpha, \alpha]. \tag{2.4}$$

*Proof.* Without loss of generality we may assume that  $A$  is of the form (2.2). Since  $A[\alpha, \alpha]$  has a positive diagonal, we deduce that  $A[\alpha, \alpha]$  is a submatrix of  $J$ . First we consider the special case  $A[\alpha, \alpha] = J$ . Using the identity  $JJ^{\dagger}J = J$ , we obtain that  $A_{\dagger}(\alpha) = 0$ . Since  $\text{rank } A = \text{rank } J$ , also the equality in (2.4) holds.

Let  $J, F, G$  be defined as in (2.2) and assume now that  $A[\alpha, \alpha]$  is a strict submatrix of  $J$ . In the following, for an integer  $j$  we write  $j + \langle m \rangle$  for the index set  $\{j + 1, \dots, j + m\}$ . Let  $\alpha' := \langle n_1 \rangle \setminus \alpha$ ,  $\beta := n_1 + \langle n_2 \rangle$  and  $\gamma := n_1 + n_2 + \langle n_3 \rangle$ . Then,

$$\begin{aligned} A[\alpha^c, \alpha]A[\alpha, \alpha]^{\dagger}A[\alpha, \alpha^c] &= \begin{bmatrix} J[\alpha', \alpha] \\ 0 \\ (FJ)[\gamma, \alpha] \end{bmatrix} J[\alpha, \alpha]^{\dagger} \begin{bmatrix} J[\alpha, \alpha'] & (JG)[\alpha, \beta] & 0 \end{bmatrix} \\ &= \begin{bmatrix} J[\alpha', \alpha]J[\alpha, \alpha]^{\dagger}J[\alpha, \alpha'] & J[\alpha', \alpha]J[\alpha, \alpha]^{\dagger}(JG)[\alpha, \beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma, \alpha]J[\alpha, \alpha]^{\dagger}J[\alpha, \alpha'] & (FJ)[\gamma, \alpha]J[\alpha, \alpha]^{\dagger}(JG)[\alpha, \beta] & 0 \end{bmatrix}. \end{aligned}$$

On the other hand, we have

$$A[\alpha^c, \alpha^c] = \begin{bmatrix} J[\alpha', \alpha'] & (JG)[\alpha', \beta] & 0 \\ 0 & 0 & 0 \\ (FJ)[\gamma, \alpha'] & FJG & 0 \end{bmatrix}.$$

Thus, the nonnegativity of  $A_+(\alpha)$  is equivalent to the following, (entrywise), inequalities

$$\begin{aligned} J[\alpha', \alpha'] &\geq J[\alpha', \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'], \\ (JG)[\alpha', \beta] &\geq J[\alpha', \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta], \\ (FJ)[\gamma, \alpha'] &\geq (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'], \\ FJG &\geq (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger (JG)[\alpha, \beta]. \end{aligned}$$

Without loss of generality, we may assume that  $J$  is permuted such that the indices of the first  $q$  blocks  $J_i$  are contained in  $\alpha^c$ , the indices of the following blocks  $J_i$  for  $i = q + 1, \dots, q + p$  are split between  $\alpha$  and  $\alpha^c$  and the indices of the blocks  $J_i$  for  $i = q + p + 1, \dots, q + p + \ell = k$  are contained in  $\alpha$ . Partitioning the vectors  $\mathbf{u}_i, \mathbf{v}_i$  in (2.3) according to  $\alpha$  and  $\alpha^c$  as

$$\mathbf{u}_i^\top = (\mathbf{a}_i^\top, \mathbf{x}_i^\top), \quad \mathbf{v}_i^\top = (\mathbf{b}_i^\top, \mathbf{y}_i^\top) \quad \text{for } i = q + 1, \dots, q + p,$$

we obtain that

$$J[\alpha', \alpha'] = (\oplus_{i=1}^q J_i) \oplus_{i=q+1}^{q+p} \mathbf{a}_i \mathbf{b}_i^\top, \quad J[\alpha, \alpha] = (\oplus_{j=q+1}^{q+p} \mathbf{x}_j \mathbf{y}_j^\top) \oplus_{i=q+p+1}^{q+p+\ell} J_i.$$

Note that

$$q = \text{rank } J - \text{rank } A[\alpha, \alpha] = \text{rank } A - \text{rank } A[\alpha, \alpha]. \quad (2.5)$$

We will only consider the case  $q, p, \ell > 0$ , as other cases follow similarly. We have

$$J[\alpha, \alpha]^\dagger = (\oplus_{i=q+1}^{q+p} \frac{1}{(\mathbf{x}_i^\top \mathbf{x}_i)(\mathbf{y}_i^\top \mathbf{y}_i)} \mathbf{y}_i \mathbf{x}_i^\top) \oplus_{i=q+p+1}^{q+p+\ell} \frac{1}{(\mathbf{u}_i^\top \mathbf{u}_i)(\mathbf{v}_i^\top \mathbf{v}_i)} \mathbf{v}_i \mathbf{u}_i^\top, \quad (2.6)$$

$$J[\alpha, \alpha'] = \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p} \mathbf{x}_i \mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix}, \quad J[\alpha', \alpha] = \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p} \mathbf{a}_i \mathbf{y}_i^\top & 0 \end{bmatrix}, \quad (2.7)$$

and hence,

$$\begin{aligned} J[\alpha', \alpha]J[\alpha, \alpha]^\dagger &= \begin{bmatrix} 0 & 0 \\ \oplus_{i=q+1}^{q+p} \frac{1}{\mathbf{x}_i^\top \mathbf{x}_i} \mathbf{a}_i \mathbf{x}_i^\top & 0 \end{bmatrix}, \\ J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] &= \begin{bmatrix} 0 & \oplus_{i=q+1}^{q+p} \frac{1}{\mathbf{y}_i^\top \mathbf{y}_i} \mathbf{y}_i \mathbf{b}_i^\top \\ 0 & 0 \end{bmatrix}, \\ J[\alpha', \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] &= \begin{bmatrix} 0 & 0 \\ 0 & \oplus_{i=q+1}^{q+p} \mathbf{a}_i \mathbf{b}_i^\top \end{bmatrix}. \end{aligned}$$

Therefore, we obtain

$$J[\alpha', \alpha'] - J[\alpha', \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = \begin{bmatrix} \oplus_{i=1}^q J_i & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$

which proves (2.5).

We now show the inequalities (2.5) and (2.5). First, we observe that  $JG$  and  $FJ$  have the following block form

$$JG = \begin{bmatrix} \mathbf{u}_1 \mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_k \mathbf{g}_k^\top \end{bmatrix}, \quad FJ = [ \mathbf{f}_1 \mathbf{v}_1^\top \quad \cdots \quad \mathbf{f}_k \mathbf{v}_k^\top ], \quad \mathbf{g}_i \in \mathbb{R}_+^{n_2}, \mathbf{f}_i \in \mathbb{R}_+^{n_3} \text{ for } i = 1, \dots, k.$$

Hence, we obtain

$$(JG)[\alpha, \beta] = \begin{bmatrix} \mathbf{x}_{q+1} \mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{x}_{q+p} \mathbf{g}_{q+p}^\top \\ \mathbf{u}_{q+p+1} \mathbf{g}_{q+p+1}^\top \\ \vdots \\ \mathbf{u}_k \mathbf{g}_k^\top \end{bmatrix}, \tag{2.8}$$

$$(JG)[\alpha', \beta] = \begin{bmatrix} \mathbf{u}_1 \mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_q \mathbf{g}_q^\top \\ \mathbf{a}_{q+1} \mathbf{g}_{q+1}^\top \\ \vdots \\ \mathbf{u}_{q+p} \mathbf{g}_{q+p}^\top \end{bmatrix}, \tag{2.9}$$

$$(FJ)[\gamma, \alpha] = [ \mathbf{f}_{q+1} \mathbf{v}_{q+1}^\top \quad \cdots \quad \mathbf{f}_{q+p} \mathbf{v}_{q+p}^\top \quad \mathbf{f}_{q+p+1} \mathbf{v}_{q+p+1}^\top \quad \cdots \quad \mathbf{f}_k \mathbf{v}_k^\top ], \tag{2.10}$$

$$(FJ)[\gamma, \alpha'] = [ \mathbf{f}_1 \mathbf{v}_1^\top \quad \cdots \quad \mathbf{f}_q \mathbf{v}_q^\top \quad \mathbf{f}_{q+1} \mathbf{b}_{q+1}^\top \quad \cdots \quad \mathbf{f}_{q+p} \mathbf{b}_{q+p}^\top ]. \tag{2.11}$$

We use (2.8) to deduce that

$$(FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = [ 0 \quad \cdots \quad 0 \quad \mathbf{f}_{q+1} \mathbf{b}_{q+1}^\top \quad \cdots \quad \mathbf{f}_{q+p} \mathbf{b}_{q+p}^\top ].$$

Therefore, we have

$$(FJ)[\gamma, \alpha'] - (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger J[\alpha, \alpha'] = [ \mathbf{f}_1 \mathbf{v}_1^\top \quad \cdots \quad \mathbf{f}_q \mathbf{v}_q^\top \quad 0 \quad \cdots \quad 0 ]. \tag{2.12}$$

Similarly, using (2.8), we obtain

$$(JG)[\alpha', \beta] - J[\alpha', \alpha]J[\alpha, \alpha]^\dagger(JG)[\alpha, \beta] = \begin{bmatrix} \mathbf{u}_1 \mathbf{g}_1^\top \\ \vdots \\ \mathbf{u}_q \mathbf{g}_q^\top \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence, the inequalities (2.5) and (2.5) hold.

We now show the last inequality (2.5). To this end, we observe that

$$FJG = (FJ)(JG) = \sum_{i=1}^k \mathbf{f}_i \mathbf{g}_i^\top. \quad (2.13)$$

Multiplying (2.6), (2.8) and (2.10) we obtain that

$$(FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger(JG)[\alpha, \beta] = \sum_{i=q+1}^k \mathbf{f}_i \mathbf{g}_i^\top.$$

Hence,

$$FJG - (FJ)[\gamma, \alpha]J[\alpha, \alpha]^\dagger(JG)[\alpha, \beta] = \sum_{i=1}^q \mathbf{f}_i \mathbf{g}_i^\top \geq 0.$$

In particular, this proves that (2.5) holds.

It is left to show that  $A_\dagger(\alpha)$  is an idempotent matrix. Clearly, if  $q = 0$  then  $A_\dagger(\alpha) = 0$ . So  $A_\dagger(\alpha)$  is a trivial idempotent matrix, and (2.5) yields (2.4).

Assuming finally that  $q > 0$ , it follows that  $A_\dagger(\alpha)$  has the block form (2.2) with  $J = \bigoplus_{i=1}^q J_i \oplus 0$ . Hence  $A_\dagger(\alpha)$  is an idempotent matrix whose rank is  $q$ , and (2.5) yields (2.4).  $\square$

**COROLLARY 2.3.** *Let  $A \in \mathbb{R}_+^{n \times n}$ ,  $A \neq 0$  be idempotent. If  $\alpha \subsetneq \langle n \rangle$  is chosen such that  $A[\alpha, \alpha]$  is an invertible matrix, then  $A[\alpha, \alpha]$  is diagonal.*

*Proof.* Note that the number  $\ell$  in the proof of Theorem 2.2 is either zero or the corresponding blocks  $J_i$  are positive  $1 \times 1$  matrices for  $i = q + p + 1, \dots, q + p + \ell$ . Furthermore, for the split blocks, we also have that  $\mathbf{x}_i \mathbf{y}_i^T \in \mathbb{R}^{1 \times 1}$ , for  $i = q + 1, \dots, q + p$ , since  $\mathbf{x}_i \mathbf{y}_i^T$  is of rank 1. Therefore,  $A[\alpha, \alpha]$  is diagonal.  $\square$

**COROLLARY 2.4.** *Let  $A \in \mathbb{R}_+^{n \times n}$ ,  $A \neq 0$  be idempotent. If  $\alpha \subsetneq \langle n \rangle$  is chosen such that  $A[\alpha, \alpha]$  is an invertible matrix, then the standard Schur complement (1.1) is nonnegative.*

COROLLARY 2.5. Let  $A \in \mathbb{R}_+^{n \times n}$ ,  $A \neq 0$  be idempotent. Choose  $\alpha \subsetneq \langle n \rangle$ , such that  $I - A[\alpha, \alpha]$  is invertible. Then,  $\tilde{A}(\alpha)$  defined by

$$\tilde{A}(\alpha) := A[\alpha^c, \alpha^c] + A[\alpha^c, \alpha](I - A[\alpha, \alpha])^{-1}A[\alpha, \alpha^c]$$

is a nonnegative idempotent matrix.

To prove this Corollary 2.5 we need the following fact for idempotent matrices, which is probably known.

LEMMA 2.6. Let  $A \in \mathbb{R}^{n \times n}$ ,  $A \neq 0$  be idempotent given as a  $2 \times 2$  block matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ . Assume that  $I - A_{22} \in \mathbb{R}^{n-m}$  is invertible. Then  $B := A_{11} + A_{12}(I - A_{22})^{-1}A_{21}$  is idempotent.

*Proof.* Let

$$E = (I - A_{22})^{-1}A_{21}, \quad D = A_{21} + A_{22}E, \quad z = \begin{bmatrix} x \\ Ex \end{bmatrix} \in \mathbb{R}^n, \quad x \text{ any vector in } \mathbb{R}^m.$$

Note that  $Az = \begin{bmatrix} Bx \\ Dx \end{bmatrix}$ . As  $A^2z = Az$  and  $x$  is an arbitrary vector, we obtain the equalities

$$A_{11}B + A_{12}D = B, \quad A_{21}B + A_{22}D = D. \quad (2.14)$$

From the second equality of (2.14) we obtain  $D = EB$ . Substituting this equality into the first equality of (2.14) we obtain that  $B^2 = B$ .  $\square$

*Proof of Corollary 2.5.* The assumption that  $I - A[\alpha, \alpha]$  is invertible implies that  $A[\alpha, \alpha]$  does not have an eigenvalue 1, i.e.,  $\rho(A[\alpha, \alpha]) < 1$ . Hence,  $I - A[\alpha, \alpha]$  is an  $M$ -matrix [1] and  $(I - A[\alpha, \alpha])^{-1} \geq 0$ . The assertion of Corollary 2.5 now follows using Lemma 2.6.  $\square$

### 3. Additional results.

**3.1. An example.** In this subsection we assume that the nonnegative idempotent matrix  $A$  is of the special form

$$A := \begin{bmatrix} J & JG \\ 0 & 0 \end{bmatrix}. \quad (3.1)$$

Furthermore, we assume that  $A[\alpha, \alpha]$  has a zero on its main diagonal. We give an example where  $A_{\dagger}(\alpha)$  may fail to be nonnegative. To this end, we first start with the following known result.



LEMMA 3.1. Let  $A \in \mathbb{R}^{n \times n}$  be a singular matrix of the following form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0_{(n-p) \times p} & 0_{(n-p) \times (n-p)} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{p \times p}, A_{12} \in \mathbb{R}^{p \times (n-p)},$$

for some  $1 \leq p < n$ . Then  $(A^\dagger)^\top$  has the same block form as  $A$ .

*Proof.* Let  $r = \text{rank } A$ . So  $r \leq p$ . Then the reduced singular value decomposition of  $A$  is of the form  $U_r \Sigma_r V_r^\top$ , where  $U_r, V_r \in \mathbb{R}^{n \times r}$ ,  $U_r^\top U_r = V_r V_r^\top = I_r$  and  $\Sigma_r$  is a diagonal matrix, whose diagonal entries are the positive singular values of  $A$ .

Clearly,  $AA^\top = \begin{bmatrix} A_{11}A_{11}^\top + A_{12}A_{12}^\top & 0 \\ 0 & 0 \end{bmatrix}$ . Hence all eigenvectors of  $AA^\top$ , corresponding to positive eigenvalues are of the form  $(\mathbf{x}^\top, \mathbf{0}^\top)^\top$ ,  $\mathbf{x} \in \mathbb{R}^p$ . Thus  $U_r^\top = [U_{r1}^\top \ 0_{r \times (n-p)}]$  where  $U_{r1} \in \mathbb{R}^{p \times r}$ . Recall that  $A^\dagger = V_r \Sigma_r^{-1} U_r^\top$ . The above form of  $U_r$  establishes the lemma.  $\square$

In the following example we permute some rows and columns of  $A$ , in order to find the Schur complement of the right lower block.

EXAMPLE 3.2. Consider a nonnegative idempotent matrix in the block form

$$B = \left[ \begin{array}{cc|cc} \mathbf{u}_1 \mathbf{v}_1^\top & 0 & \mathbf{u}_1 \mathbf{s}_1^\top & \mathbf{u}_1 \mathbf{t}_1^\top & 0 \\ 0 & \mathbf{a}_2 \mathbf{b}_2^\top & \mathbf{a}_2 \mathbf{s}_2^\top & \mathbf{a}_1 \mathbf{t}_2^\top & \mathbf{a}_2 \mathbf{y}_2^\top \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x}_2 \mathbf{b}_2^\top & \mathbf{x}_2 \mathbf{s}_2^\top & \mathbf{x}_1 \mathbf{t}_2^\top & \mathbf{x}_2 \mathbf{y}_2^\top \end{array} \right].$$

Then,

$$B[\alpha, \alpha] = \begin{bmatrix} 0 & 0 \\ \mathbf{x}_1 \mathbf{t}_2^\top & \mathbf{x}_2 \mathbf{y}_2^\top \end{bmatrix}, \quad B[\alpha, \alpha]^\dagger = \begin{bmatrix} 0 & \frac{\mathbf{t}_2 \mathbf{x}_2^\top}{(\mathbf{x}_2^\top \mathbf{x}_2)(\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2)} \\ 0 & \frac{\mathbf{y}_2 \mathbf{x}_2^\top}{(\mathbf{x}_2^\top \mathbf{x}_2)(\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2)} \end{bmatrix},$$

and

$$B[\alpha^c, \alpha] B[\alpha, \alpha]^\dagger B[\alpha, \alpha^c] = \begin{bmatrix} 0 & \frac{\mathbf{t}_1^\top \mathbf{t}_2 \mathbf{u}_1 \mathbf{b}_2^\top}{\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2} & \frac{\mathbf{t}_1^\top \mathbf{t}_2 \mathbf{u}_1 \mathbf{s}_2^\top}{\mathbf{t}_2^\top \mathbf{t}_2 + \mathbf{y}_2^\top \mathbf{y}_2} \\ 0 & \mathbf{a}_2 \mathbf{b}_2^\top & \mathbf{a}_2 \mathbf{s}_2^\top \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence  $B_\dagger(\alpha)_{11} > 0$ ,  $B_\dagger(\alpha)_{12} \leq 0$  and the Moore-Penrose inverse Schur complement is neither nonnegative nor nonpositive if  $t_1^\top t_2 > 0$ .

**3.2. Nonnegativity of semi-inverse Schur complement.** In this section we extend the results of Section 2 for idempotent matrices of the form (2.2) for some Schur complements with zero diagonal entries. We start with the following simple observation.

PROPOSITION 3.3. *Let the assumptions of Lemma 3.1 hold. Suppose that*

$$A_{11}(A_{11})^\dagger A_{12} = A_{12}.$$

*Then  $A^{\text{ginv}} = \begin{bmatrix} (A_{11})^\dagger & 0_{p \times (n-p)} \\ 0_{(n-p) \times p} & 0_{(n-p) \times (n-p)} \end{bmatrix}$  is a semi-inverse of  $A$ . In particular any principle submatrix of an idempotent matrix as in (3.1) with at least one zero diagonal element has a semi-inverse of this form.*

*Proof.* The proposition follows by checking the conditions in (2.1).  $\square$  Note that condition  $A_{11}(A_{11})^\dagger A_{12} = A_{12}$  holds in general for idempotent matrices  $A$  of the form as in (3.1).

The following theorem states the general result of this subsection.

THEOREM 3.4. *Let  $A \in \mathbb{R}_+^{n \times n}$  be of the form (2.2), where  $n_2 + n_3 \geq 1$  and the condition in (2.3) holds. Furthermore, let  $\alpha_1 \subset \langle n \rangle$  be of the following form*

$$\begin{aligned} \text{either } & \alpha_1 = \alpha \cup \beta, \emptyset \neq \beta \subseteq n_1 + \langle n_2 \rangle, \\ \text{or } & \alpha_1 = \alpha \cup \gamma, \emptyset \neq \gamma \subseteq n_1 + n_2 + \langle n_3 \rangle, \end{aligned} \tag{3.2}$$

*where  $\alpha \subseteq \langle n_1 \rangle$ . Then, there exists a semi-inverse  $A^{\text{ginv}}[\alpha_1, \alpha_1]$  of  $A[\alpha_1, \alpha_1]$  such that  $A_{\text{ginv}}(\alpha_1)$  as defined in (1.2) is a nonnegative idempotent matrix. The rank of  $A_{\text{ginv}}(\alpha_1)$  is equal to the multiplicity of the eigenvalue 1 in  $A[\alpha', \alpha']$ , where  $\alpha' = \langle n_1 \rangle \setminus \alpha$ . In particular, if 1 is not an eigenvalue of  $A[\alpha', \alpha']$ , then  $A_{\text{ginv}}(\alpha) = 0$ .*

*Proof.* First we consider the case that  $\alpha_1 = \alpha \cup \beta$ . If  $\alpha = \emptyset$ , then  $A[\alpha_1, \alpha_1]$  and  $A[\alpha_1, \alpha_1]^{\text{ginv}}$  are zero matrices for any semi-inverse and  $A_{\text{ginv}}(\alpha_1) = A[\alpha_1^c, \alpha_1^c]$ . Using the proof of Theorem 2.2 we obtain that  $A_{\text{ginv}}(\alpha_1)$  is a nonnegative idempotent matrix of rank  $k$ .

Assuming now that  $\alpha \neq \emptyset$ , we observe that  $A[\alpha_1, \alpha_1]$  satisfies the assumption of Proposition 3.3. Defining  $A[\alpha_1, \alpha_1]^{\text{ginv}}$  as in Proposition 3.3 and following the arguments of the proof of Theorem 2.2 we deduce the theorem in this case.

We assume now that  $\alpha_1 = \alpha \cup \gamma$ . If  $\alpha = \emptyset$  we obtain that  $A_{\text{ginv}}(\alpha_1)$  is a nonnegative idempotent matrix of rank  $k$  as above. Assuming finally that  $\alpha \neq \emptyset$ , we have that  $A[\alpha_1, \alpha_1]^\top$  satisfies the assumption of Proposition 3.3. Define  $(A[\alpha_1, \alpha_1]^\top)^{\text{ginv}}$  as in Proposition 3.3 and let  $A[\alpha_1, \alpha_1]^{\text{ginv}} := ((A[\alpha_1, \alpha_1]^\top)^{\text{ginv}})^\top$ . Repeating the arguments of the proof of Theorem 2.2 we deduce the theorem in this case.  $\square$

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