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ON A NEW CLASS OF STRUCTURED MATRICES RELATED TO THE DISCRETE SKEW-SELF-ADJOINT DIRAC SYSTEMS[*]

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Abstract. A new class of the structured matrices related to the discrete skew-self-adjoint Dirac systems is introduced. The corresponding matrix identities and inversion procedure are treated. Analogs of the Schur coefficients and of the Christoffel-Darboux formula are studied. It is shown that the structured matrices from this class are always positive-definite, and applications for an inverse problem for the discrete skew-self-adjoint Dirac system are obtained.

Key words. Structured matrices, Matrix identity, Schur coefficients, Christoffel-Darboux formula, Transfer matrix function, Discrete skew-self-adjoint Dirac system, Weyl function, Inverse problem.

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1. Introduction. It is well-known that Toeplitz and block Toeplitz matrices are closely related to a discrete system of equations, namely to Szegő recurrence. This connection have been actively studied during the last decades. See, for instance, [1]–[5], [12, 25] and numerous references therein. The connections between block Toeplitz matrices and Weyl theory for the self-adjoint discrete Dirac system were treated in [11]. (See [26] for the Weyl theory of the discrete analog of the Schrödinger equation.) The Weyl theory for the skew-self-adjoint discrete Dirac system

$$(1.1) \quad W_{k+1}(\lambda) - W_k(\lambda) = -\frac{i}{\lambda} C_k W_k(\lambda), \quad C_k = C_k^* = C_k^{-1}, \quad k = 0, 1, \ldots$$

was developed in [14, 18]. Here C_k are $2p \times 2p$ matrix functions. When $p = 1$, system (1.1) is an auxiliary linear system for the isotropic Heisenberg magnet model. Explicit solutions of the inverse problem were constructed in [14]. A general procedure to construct the solutions of the inverse problem for system (1.1) was given in [18], using a new class of structured matrices S, which satisfy the matrix identity

$$(1.2) \qquad\qquad AS - SA^* = i\Pi\Pi^*.$$

Here, S and A are $(n+1)p \times (n+1)p$ matrices and Π is an $(n+1)p \times 2p$ matrix. The block matrix A has the form

$$(1.3) \qquad A := A(n) = \left\{ a_{j-k} \right\}_{k,j=0}^{n}, \quad a_r = \begin{cases} 0 & \text{for} \quad r > 0 \\ \dfrac{i}{2} I_p & \text{for} \quad r = 0 \\ i\, I_p & \text{for} \quad r < 0 \end{cases},$$

where I_p is the $p \times p$ identity matrix. The matrix $\Pi = [\Phi_1 \;\; \Phi_2]$ consists of two block columns of the form

$$(1.4) \qquad \Phi_1 = \begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} \alpha_0 \\ \alpha_0 + \alpha_1 \\ \vdots \\ \alpha_0 + \alpha_1 + \cdots + \alpha_n \end{bmatrix}.$$

DEFINITION 1.1. *The class of the block matrices S determined by the matrix identity (1.2) and formulas (1.3) and (1.4) is denoted by Ω_n.*

Notice that the blocks α_k in [18] are Taylor coefficients of the Weyl functions and that the matrices C_n $(0 \le n \le l)$ in (1.1) are easily recovered from the expressions $\Pi(n)^* S(n)^{-1} \Pi(n)$ $(0 \le n \le l)$ (see Theorem 3.4 of [18]). In this way, the structure of the matrices S determined by the matrix identity (1.2) and formulas (1.3) and (1.4), their inversion and conditions of invertibility prove essential. Recall that the self-adjoint block Toeplitz matrices satisfy [15]–[17] the identity $AS - SA^* = i\Pi J \Pi^*$ $\left(J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}\right)$, which is close to (1.2)–(1.4). We refer also to [20]–[24] and references therein for the general method of the operator identities. The analogs of various results on the Toeplitz matrices and j-theory from [6]–[11] can be obtained for the class Ω_n, too.

2. Structure of the matrices from Ω_n. Consider first the block matrix $S = \left\{ s_{kj} \right\}_{k,j=0}^{n}$ with the $p \times p$ entries s_{kj}, which satisfies the identity

$$(2.1) \qquad\qquad AS - SA^* = iQ, \quad Q = \left\{ q_{kj} \right\}_{k,j=0}^{n}.$$

One can easily see that the equality

$$(2.2) \qquad\qquad q_{kj} = s_{kj} + \sum_{r=0}^{k-1} s_{rj} + \sum_{r=0}^{j-1} s_{kr}$$

follows from (2.1). Sometimes we add comma between the indices and write $s_{k,j}$. Putting $s_{-1,j} = s_{k,-1} = q_{-1,j} = q_{k,-1} = 0$, from (2.2) we have

$$(2.3) \quad s_{k+1,j+1} - s_{kj} = q_{kj} + q_{k+1,j+1} - q_{k+1,j} - q_{k,j+1}, \quad -1 \le k, j \le n-1.$$

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Now, putting $Q = i\Pi\Pi^*$ and taking into account (2.3), we get the structure of S.

PROPOSITION 2.1. *Let $S \in \Omega_n$. Then we have*

$$(2.4) \qquad s_{k+1,j+1} - s_{kj} = \alpha_{k+1}\alpha_{j+1}^* \quad (-1 \le k, j \le n-1),$$

excluding the case when $k = -1$ and $j = -1$ simultaneously. For that case, we have

$$(2.5) \qquad s_{00} = I_p + \alpha_0\alpha_0^*.$$

Notice that for the block Toeplitz matrix, the equalities $s_{k+1,j+1} - s_{kj} = 0$ ($0 \le k, j \le n-1$) hold. Therefore, Toeplitz and block Toeplitz matrices can be used to study certain homogeneous processes and appear as a result of discretization of homogeneous equations. From this point of view, the matrix $S \in \Omega_n$ is perturbed by the simplest inhomogeneity.

The authors are grateful to the referee for the next interesting remark.

REMARK 2.2. *From (1.2)–(1.4) we get another useful identity, namely,*

$$(2.6) \qquad S - NSN^* = \widehat{\Pi}\widehat{\Pi}^*,$$

where

$$(2.7) \qquad N = \{\delta_{k-j-1}I_p\}_{k,j=0}^n = \begin{bmatrix} 0 & & & 0 \\ I_p & & & 0 \\ & \ddots & & \vdots \\ & & I_p & 0 \end{bmatrix}, \quad \widehat{\Pi} = \begin{bmatrix} I_p & \alpha_0 \\ 0 & \alpha_1 \\ \vdots & \vdots \\ 0 & \alpha_n \end{bmatrix}.$$

Indeed, it is easy to see that $(I_{(n+1)p} - N)A = \frac{i}{2}(I_{(n+1)p} + N)$. Hence, the identity

$$i(S - NSN^*) = i(I_{(n+1)p} - N)\Pi\Pi^*(I_{(n+1)p} - N^*)$$

follows from (1.2). By (2.7), we have $(I_{(n+1)p} - N)\Pi = \widehat{\Pi}$, and so (2.6) is valid. Relations (2.4) and (2.5) are immediate from (2.6).

PROPOSITION 2.3. *Let $S = \left\{s_{kj}\right\}_{k,j=0}^n \in \Omega_n$. Then S is positive and, moreover, $S \ge I_{(n+1)p}$. We have $S > I_{(n+1)p}$ if and only if $\det \alpha_0 \ne 0$.*

Proof. From (2.5) it follows that $S(0) = s_{00} \ge I_p$ and that $S(0) > I_p$, when $\det \alpha_0 \ne 0$. The necessity of $\det \alpha_0 \ne 0$, for the inequality $S > I_{(n+1)p}$ to be true, follows from (2.5), too. We shall prove that $S \ge I_{(n+1)p}$ and that $S > I_{(n+1)p}$, when $\det \alpha_0 \ne 0$, by induction.

Suppose that $S(r-1) = \left\{s_{kj}\right\}_{k,j=0}^{r-1} \ge I_{rp}$ ($r \ge 1$). According to (2.6), we can

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present $S(r) = \left\{s_{kj}\right\}_{k,j=0}^{r}$ in the form $S(r) = S_1 + S_2$,

$$(2.8) \qquad S_1 := \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_r^* \end{bmatrix}, \quad S_2 := \begin{bmatrix} I_p & 0 \\ 0 & S(r-1) \end{bmatrix}.$$

By the assumption of induction, it is immediate that $S(r) \geq S_2 \geq I_{(r+1)p}$. Hence, we get $S = S(n) \geq I_{(n+1)p}$.

Suppose that $\det \alpha_0 \neq 0$ and $S(r-1) > I_{(n+1)p}$. Let $S(r)f = f$ $(f \in BC^{(r+1)p})$, i.e., let $f^*\big(S(r) - I_{(r+1)p}\big)f = 0$. By (2.8), we have $S_1 \geq 0$, and by the assumption of induction, we have $S_2 - I_{(r+1)p} \geq 0$. So, it follows from $f^*\big(S(r) - I_{(r+1)p}\big)f = 0$ that $f^*S_1 f = 0$ and $f^*\big(S_2 - I_{(r+1)p}\big)f = 0$. Hence, as $\alpha_0 \alpha_0^* > 0$ and $S(r-1) > I_{rp}$, we derive $f = 0$. In other words, $S(r)f = f$ implies $f = 0$, that is, $\det(S(r) - I_{(r+1)p}) \neq 0$. From $\det(S(r) - I_{(r+1)p}) \neq 0$ and $S(r) \geq I_{(r+1)p}$, we get $S(r) > 0$. So, the condition $\det \alpha_0 \neq 0$ implies $S(n) > I_{(n+1)p}$ by induction. $\square$

REMARK 2.4. *Using formula (2.5) and representations* $S(r) = S_1(r) + S_2(r)$ $(0 < r \leq n)$, *where* $S_1(r)$ *and* $S_2(r)$ *are given by (2.8), one easily gets*

$$(2.9) \quad S = I_{(n+1)p} + \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_n^* \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} \begin{bmatrix} 0 & \alpha_0^* & \cdots & \alpha_{n-1}^* \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \alpha_0^* \end{bmatrix}$$

$$= I_{(n+1)p} + V_\alpha V_\alpha^*, \quad V_\alpha := \begin{bmatrix} \alpha_0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_0 \end{bmatrix}.$$

Here, V_α is a triangular block Toeplitz matrix, and formula (2.9) is another way to prove Proposition 2.3. Further, we will be interested in a block triangular factorization of the matrix S itself, namely, $S = V_-^{-1}(V_-^*)^{-1}$, where V_- is a lower triangular matrix.

Similar to the block Toeplitz case (see [13] and references therein) the matrices $S \in \Omega_n$ admit the matrix identity of the form $A_1 S - S A_1 = Q_1$, where Q_1 is of low

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rank, $A_1 := \{\delta_{k-j+1}I_p\}_{k,j=0}^n = N^*$ and N is given in (2.7). The next proposition follows easily from (2.4).

PROPOSITION 2.5. *Let* $S \in \Omega_n$. *Then we have*

$$(2.10)\ A_1 S - S A_1 = y_1 y_2^* + y_3 y_4^* + y_5 y_6^*, \quad A_1^* S - S A_1^* = -(y_2 y_1^* + y_4 y_3^* + y_6 y_5^*),$$

where

$$(2.11)\quad y_1 = \begin{bmatrix} s_{10} \\ s_{20} \\ \vdots \\ s_{n0} \\ 0 \end{bmatrix}, \quad y_3 = - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix}, \quad y_5 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ 0 \end{bmatrix}, \quad y_6 = \begin{bmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix},$$

$$(2.12)\quad y_2^* = \begin{bmatrix} I_p & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad y_4^* = \begin{bmatrix} 0 & s_{n0} & s_{n1} & \cdots & s_{n,n-1} \end{bmatrix}.$$

Differently than the block Toeplitz matrix case, the rank of $A_1 S - S A_1$ is in general situation larger than the rank of $AS - SA^*$, where A is given by (1.3). (To see this compare (1.2)–(1.4) and (2.10)–(2.12).)

3. Transfer matrix function and Weyl functions. Introduce the $(r+1)p \times (n+1)p$ matrix

$$(3.1)\qquad\qquad P_k := \begin{bmatrix} I_{(r+1)p} & 0 \end{bmatrix}, \quad r \le n.$$

It follows from (1.3) that $P_r A(n) = A(r) P_r$. Hence, using (1.2) we derive

$$(3.2)\qquad A(r)S(r) - S(r)A(r)^* = i\Pi(r)\Pi(r)^*, \quad \Pi(r) := P_r \Pi.$$

As $S > 0$, it admits a block triangular factorization

$$(3.3)\qquad\qquad S = V_-^{-1}(V_-^*)^{-1},$$

where $V_-^{\pm 1}$ are block lower triangular matrices. It is immediate from (3.3) that

$$(3.4)\qquad S(r) = V_-(r)^{-1}\big(V_-(r)^*\big)^{-1}, \quad V_-(r) := P_r V_- P_r^*.$$

Recall that S-node [21, 23, 24] is the triple $\big(A(r),\ S(r),\ \Pi(r)\big)$ that satisfies the matrix identity (3.2) (see also [21, 23, 24] for a more general definition of the S-node). Following [21, 23, 24], introduce the transfer matrix function corresponding to the S-node:

$$(3.5)\qquad w_A(r, \lambda) = I_{2p} - i\Pi(r)^* S(r)^{-1}\big(A(r) - \lambda I_{(r+1)p}\big)^{-1}\Pi(r).$$

In particular, taking into account (3.4) and (3.5), we get

$$(3.6) \qquad w_A(0,\lambda) = I_{2p} - \frac{2i}{i-2\lambda}\beta(0)^*\beta(0), \quad \beta(0) = V_-(0)\Pi(0).$$

By the factorization theorem 4 from [21] (see also [23, p. 188]), we have

$$(3.7) \quad w_A(r,\lambda) = \Big(I_{2p} - i\Pi(r)^*S(r)^{-1}P^*\big(PA(r)P^* - \lambda I_p\big)^{-1}\big(PS(r)^{-1}P^*\big)^{-1}$$
$$\times PS(r)^{-1}\Pi(r)\Big)w_A(r-1,\lambda), \quad P = [\,0 \ \cdots \ 0 \ I_p\,].$$

According to (1.3), we obtain

$$(3.8) \qquad\qquad \big(PA(r)P^* - \lambda I_p\big)^{-1} = (\tfrac{i}{2} - \lambda)^{-1}I_p.$$

Using (3.4), we derive

$$(3.9) \quad PS(r)^{-1}P^* = (V_-(r))^*_{rr}(V_-(r))_{rr}, \quad PS(r)^{-1}\Pi(r) = (V_-(r))^*_{rr}PV_-(r)\Pi(r),$$

where $(V_-(r))_{rr}$ is the block entry of $V_-(r)$ (the entry from the r-th block row and the r-th block column). In view of (3.8) and (3.9), we rewrite (3.7) in the form

$$(3.10) \qquad\qquad w_A(r,\lambda) = \Big(I_{2p} - \frac{2i}{i-2\lambda}\beta(r)^*\beta(r)\Big)w_A(r-1,\lambda),$$
$$(3.11) \qquad\qquad \beta(r) = PV_-(r)\Pi(r) = (V_-\Pi)_r, \quad 0 < r \le n.$$

Here, $(V_-\Pi)_r$ is the r-th $p \times 2p$ block of the block column vector $V_-\Pi$. Moreover, according to (3.9) and definitions (3.6), (3.11) of β, we have

$$(3.12) \qquad\qquad \big(PS(r)^{-1}P^*\big)^{-\frac{1}{2}}PS(r)^{-1}\Pi(r) = u(r)\beta(r),$$
$$u(r) := \big(PS(r)^{-1}P^*\big)^{-\frac{1}{2}}(V_-(r))^*_{rr}, \quad u(r)^*u(r) = I_p.$$

As u is unitary, the properties of $\big(PS(r)^{-1}P^*\big)^{-\frac{1}{2}}PS(r)^{-1}\Pi(r)$ proved in [18, p. 2098] imply the next proposition.

PROPOSITION 3.1. *Let $S \in \Omega_n$ and let $\beta(k)$ $(0 \le k \le n)$ be given by (3.3), (3.4), (3.6) and (3.11). Then we have*

$$(3.13) \qquad \begin{cases} \beta(k)\beta(k)^* = I_p \quad (0 \le k \le n), \\ \det \beta(k-1)\beta(k)^* \ne 0 \quad (0 < k \le n), \\ \det \beta_1(0) \ne 0, \end{cases}$$

where $\beta_1(k)$, $\beta_2(k)$ are $p \times p$ blocks of $\beta(k)$.

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REMARK 3.2. *Notice that the lower triangular factor V_- is not defined by S uniquely. Hence, the matrices $\beta(k)$ are not defined uniquely, too. Nevertheless, in view of (3.12), the matrices $\beta(k)^*\beta(k)$ are uniquely defined, which suffices for our considerations.*

When $p = 1$ and $C_k \neq \pm I_2$, the matrices $C_k = C_k^* = C_k^{-1}$ (i.e., the potential of the system (1.1)) can be presented in the form $C_k = I_2 - 2\beta(k)^*\beta(k)$, where $\beta(k)\beta(k)^* = 1$. Therefore, it is assumed in [18] for the system (1.1) on the interval $0 \leq k \leq n$, that

$$(3.14) \qquad C_k = I_{2p} - 2\beta(k)^*\beta(k),$$

where $\beta(k)$ are $p \times 2p$ matrices and (3.13) holds. Relation (3.14) implies $C_k = U_k j U_k^*$, where $j = \begin{bmatrix} -I_p & 0 \\ 0 & I_p \end{bmatrix}$ and U_k are unitary $2p \times 2p$ matrices. The equalities $C_k = C_k^* = C_k^{-1}$ follow. Consider the fundamental solution $W_r(\lambda)$ of the system (1.1) normalized by $W_0(\lambda) = I_{2p}$. Using (3.6) and (3.10), one easily derives

$$(3.15) \qquad W_{r+1}(\lambda) = \left(\frac{\lambda - i}{\lambda}\right)^{r+1} w_A\left(r, \frac{\lambda}{2}\right), \quad 0 \leq r \leq n.$$

Similar to the continuous case, the Weyl functions of the system (1.1) are defined via Möbius (linear-fractional) transformation

$$(3.16) \quad \varphi(\lambda) = \left(\mathcal{W}_{11}(\lambda)R(\lambda) + \mathcal{W}_{12}(\lambda)Q(\lambda)\right)\left(\mathcal{W}_{21}(\lambda)R(\lambda) + \mathcal{W}_{22}(\lambda)Q(\lambda)\right)^{-1},$$

where $\mathcal{W}_{ij}$ are $p \times p$ blocks of $\mathcal{W}$ and

$$(3.17) \qquad \mathcal{W}(\lambda) = \{\mathcal{W}_{ij}(\lambda)\}_{i,j=1}^2 := W_{n+1}(\overline{\lambda})^*.$$

Here, R and Q are any $p \times p$ matrix functions analytic in the neighborhood of $\lambda = i$ and such that

$$(3.18) \qquad \det\left(\mathcal{W}_{21}(i)R(i) + \mathcal{W}_{22}(i)Q(i)\right) \neq 0.$$

One can easily verify that such pairs always exist (see [18, p. 2090]). A matrix function $\varphi(\lambda)$ of order p, analytic at $\lambda = i$, generates a matrix $S \in \Omega_n$ via the Taylor coefficients

$$(3.19) \qquad \varphi\left(i\frac{1+z}{1-z}\right) = -(\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n) + O(z^{n+1}) \quad (z \to 0)$$

and identity (1.2). By Theorem 3.7 in [18], such φ is a Weyl function of some system (1.1) if and only if S is invertible. Now, from Proposition 2.3 it follows that $S > 0$, and the next proposition is immediate.

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PROPOSITION 3.3. *Any $p \times p$ matrix function φ, which is analytic at $\lambda = i$, is a Weyl function of some system (1.1) on the interval $0 \le k \le n$, such that (3.13) and (3.14) hold.*

Moreover, from the proof of the statement (ii) of Theorem 3.7 in [18], the Corollary 3.6 in [18] and our Proposition 3.3, we get:

PROPOSITION 3.4. *Let the $p \times p$ matrix function φ be analytic at $\lambda = i$ and admit expansion (3.19). Then φ is a Weyl function of the system (1.1) $(0 \le k \le n)$, where C_k are defined by the formulas (1.2)–(1.4), $\Pi = [\Phi_1 \quad \Phi_2]$, (3.3), (3.11) and (3.14). Moreover, any Weyl function of this system admits expansion (3.19).*

4. Schur coefficients and Christoffel-Darboux formula. The sequence $\{\alpha_k\}_{k=0}^n$ uniquely determines via formulas (1.2)–(1.4) or (1.3), (1.4), (2.4) and (2.5) the S-node (A, S, Π). Then, using (3.3), (3.11) and (3.14), we uniquely recover the system (1.1) $(0 \le k \le n)$, or equivalently, we recover the sequence $\{\beta_k^* \beta_k\}_{k=0}^n$, such that (3.13) holds. By Proposition 3.4, one can use Weyl functions of this system to obtain the sequence $\{\alpha_k\}_{k=0}^n$.

REMARK 4.1. *Thus, there are one to one correspondences between the sequences $\{\alpha_k\}_{k=0}^n$, the S-nodes (A, S, Π) satisfying (1.2), the systems (1.1) $(0 \le k \le n)$ with C_k of the form (3.14) and the sequences $\{\beta_k^* \beta_k\}_{k=0}^n$, such that (3.13) holds.*

Next, we consider a correspondence between $\{\beta_k^* \beta_k\}_{k=0}^n$ and some $p \times p$ matrices $\{\rho_k\}_{k=0}^n$ ($\|\rho_k\| \le 1$). Notice that $0 \le \beta_1(k)\beta_1(k)^* \le I_p$, and suppose that these inequalities are strict:

$$(4.1) \qquad\qquad 0 < \beta_1(k)\beta_1(k)^* < I_p \quad (0 \le k \le n).$$

In view of the first relation in (3.13) and inequalities (4.1), we have $\det \beta_1(k) \ne 0$ and $\det \beta_2(k) \ne 0$. So, we can put

$$(4.2) \qquad\qquad \rho_k := \left(\beta_2(k)^* \beta_2(k)\right)^{-\frac{1}{2}} \beta_2(k)^* \beta_1(k).$$

It follows from (4.2) that

$$\rho_k \rho_k^* = \left(\beta_2(k)^* \beta_2(k)\right)^{-\frac{1}{2}} \beta_2(k)^* \left(I_p - \beta_2(k)\beta_2(k)^*\right) \beta_2(k) \left(\beta_2(k)^* \beta_2(k)\right)^{-\frac{1}{2}}$$
$$(4.3) \qquad = I_p - \beta_2(k)^* \beta_2(k).$$

By (4.2) and (4.3), we obtain

$$(4.4) \qquad\qquad [\rho_k \quad (I_p - \rho_k \rho_k^*)^{\frac{1}{2}}] = u_k \beta(k), \quad \|\rho_k\| < 1,$$

where

$$(4.5) \qquad\qquad u_k := \left(\beta_2(k)^* \beta_2(k)\right)^{-\frac{1}{2}} \beta_2(k)^*, \quad u_k u_k^* = I_p,$$

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i.e., u_k is unitary.

REMARK 4.2. *Under condition (4.1), according to (4.4) and (4.5), the sequence $\{\beta_k^*\beta_k\}_{k=0}^n$ is uniquely recovered from the sequence $\{\rho_k\}_{k=0}^n$ ($\|\rho_k\| \le 1$):*

$$(4.6) \qquad \beta_k^*\beta_k = \begin{bmatrix} \rho_k^* \\ (I_p - \rho_k\rho_k^*)^{\frac{1}{2}} \end{bmatrix} [\rho_k \quad (I_p - \rho_k\rho_k^*)^{\frac{1}{2}}].$$

By Remark 4.1 this means that the S-node can be recovered from the sequence $\{\rho_k\}_{k=0}^n$. Therefore, similar to the Toeplitz case, we call ρ_k the Schur coefficients of the S-node (A, S, Π).

Besides Schur coefficients, we obtain an analog of the Christoffel-Darboux formula.

PROPOSITION 4.3. *Let $S \in \Omega_n$, let $w_A(r, \lambda)$ be introduced by (3.5) for $r \ge 0$ and put $w_A(-1, \lambda) = I_{2p}$. Then we have*

$$\sum_{k=-1}^{n-1} w_A(k, \mu)^*\beta(k+1)^*\beta(k+1)w_A(k, \lambda)$$

$$(4.7) \qquad = \frac{(2\lambda - i)(2\overline{\mu} + i)}{4i(\overline{\mu} - \lambda)}\Big(w_A(n, \mu)^*w_A(n, \lambda) - I_{2p}\Big).$$

Proof. From (3.10) it follows that

$$w_A(k+1, \mu)^*w_A(k+1, \lambda) - w_A(k, \mu)^*w_A(k, \lambda) =$$
$$w_A(k, \mu)^*\left(\left(I_{2p} - \frac{2i}{2\overline{\mu}+i}\beta(k+1)^*\beta(k+1)\right)\right.$$
$$(4.8) \qquad \times \left(I_{2p} + \frac{2i}{2\lambda-i}\beta(k+1)^*\beta(k+1)\right) - I_{2p}\Big) w_A(k, \lambda).$$

Using $\beta(k)\beta(k)^* = I_p$, we rewrite (4.8) in the form

$$w_A(k+1, \mu)^*w_A(k+1, \lambda) - w_A(k, \mu)^*w_A(k, \lambda)$$
$$(4.9) \qquad = \frac{4i(\overline{\mu} - \lambda)}{(2\lambda - i)(2\overline{\mu} + i)}w_A(k, \mu)^*\beta(k+1)^*\beta(k+1)w_A(k, \lambda).$$

Equality (4.7) follows from (4.9). □

5. Inversion of $S \in \Omega_n$. To recover the system (1.1) from $\{\alpha_k\}_{k=0}^n$, it is convenient to use formula (3.11). The matrices $V_-(r)$ ($r \ge 0$) in this formula can be constructed recursively.

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PROPOSITION 5.1. *Let* $S = V_-^{-1}(V_-^*)^{-1} \in \Omega_n$. *Then* $V_-(r+1)$ $(0 \le r < n)$ *can be constructed by the formula*

$$(5.1) \qquad V_-(r+1) = \begin{bmatrix} V_-(r) & 0 \\ -t(r)S_{21}(r)V_-(r)^*V_-(r) & t(r) \end{bmatrix},$$

where $S_{21}(r) = [s_{r+1,0} \quad s_{r+1,1} \quad \cdots \quad s_{r+1,r}],$

$$(5.2) \qquad t(r) = \left(s_{r+1,r+1} - S_{21}(r)V_-(r)^*V_-(r)S_{21}(r)^* \right)^{-\frac{1}{2}}.$$

Proof. To prove the proposition it suffices to assume that $V_-(r)$ satisfies (3.4) and prove $S(r+1) = V_-(r+1)^{-1}(V_-(r+1)^*)^{-1}$. In view of Proposition 2.3 and (3.4), we have $s_{r+1,r+1} - S_{21}(r)V_-(r)^*V_-(r)S_{21}(r)^* > 0$, i.e., formula (5.2) is well defined. Now, it is easily checked that $S(r+1)^{-1} = V_-(r+1)^*V_-(r+1)$ (see formula (2.7) in [17]). □

Put $T = \{t_{kj}\}_{k,j=0}^n = S^{-1}$,

$$(5.3) \qquad \widehat{Q} = \{\widehat{q}_{kj}\}_{k,j=0}^n = T\Pi\Pi^*T, \quad X = T\Phi_1, \quad Y = T\Phi_2,$$

where t_{kj} and $\widehat{q}_{kj}$ are $p \times p$ blocks of T and $\widehat{Q}$, respectively. Similar to [15, 16, 20, 22] and references therein, we get the next proposition.

PROPOSITION 5.2. *Let* $S \in \Omega_n$. *Then* $T = S^{-1}$ *is recovered from* X *and* Y *by the formula*

$$(5.4) \qquad t_{kj} = \widehat{q}_{kj} + \widehat{q}_{k+1,j+1} - \widehat{q}_{k+1,j} - \widehat{q}_{k,j+1} + t_{k+1,j+1},$$

or, equivalently, by the formula

$$(5.5) \qquad t_{kj} = \widehat{q}_{kj} + 2\sum_{r=1}^{n-k} \widehat{q}_{k+r,j+r} - \sum_{r=1}^{n-k} \widehat{q}_{k+r,j+r-1} - \sum_{r=1}^{n-k+1} \widehat{q}_{k+r-1,j+r},$$

where we fix $t_{kj} = 0$ *and* $\widehat{q}_{kj} = 0$ *for* $k > n$ *or* $j > n$, *and*

$$(5.6) \qquad \widehat{Q} = XX^* + YY^*.$$

The block vectors X *and* Y *are connected by the relations*

$$\sum_{r=0}^{n}(X_r - X_r^*) = 0, \quad \sum_{r=0}^{n-k} X_{n-r} = \sum_{r=0}^{n-k} \widehat{q}_{k+r,r} \quad (k \ge 0),$$

$$(5.7) \qquad\qquad\qquad\qquad \sum_{r=0}^{n-k} X_{n-r}^* = \sum_{r=0}^{n-k} \widehat{q}_{r,k+r} \quad (k > 0).$$

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Proof. From the identity (1.2) and formula (5.3), it follows that

$$(5.8) \qquad TA - A^*T = i\widehat{Q},$$

where $\widehat{Q}$ satisfies (5.6). The identity $TA - A^*T = i\widehat{Q}$ yields (5.4), which, in its turn, implies (5.5).

To derive (5.7), we rewrite (5.8) in the form

$$
(5.9) \qquad
\begin{aligned}
&\left(A^* - \lambda I_{(n+1)p}\right)^{-1} T - T\left(A - \lambda I_{(n+1)p}\right)^{-1} \\
&= i\left(A^* - \lambda I_{(n+1)p}\right)^{-1} \widehat{Q}\left(A - \lambda I_{(n+1)p}\right)^{-1},
\end{aligned}
$$

and multiply both sides of (5.9) by Φ_1 from the right and by Φ_1^* from the left. Taking into account (5.3), we get

$$
(5.10) \qquad
\begin{aligned}
&\Phi_1^*\left(A^* - \lambda I_{(n+1)p}\right)^{-1} X - X^*\left(A - \lambda I_{(n+1)p}\right)^{-1} \Phi_1 \\
&= i\Phi_1^*\left(A^* - \lambda I_{(n+1)p}\right)^{-1} \widehat{Q}\left(A - \lambda I_{(n+1)p}\right)^{-1} \Phi_1.
\end{aligned}
$$

It is easily checked (see formula (1.10) in [17]) that

$$
(5.11) \qquad
\begin{aligned}
(A - \lambda I_{(n+1)p})^{-1}\Phi_1 &= \left(\frac{i}{2} - \lambda\right)^{-1} \mathrm{col}[\, I_p \;\; \zeta^{-1}I_p \;\; \cdots \;\; \zeta^{-n} \;\; I_p\,], \\
\Phi_1^*(A^* - \lambda I_{(n+1)p})^{-1} &= -\left(\frac{i}{2} + \lambda\right)^{-1} [\, I_p \;\; \zeta I_p \;\; \cdots \;\; \zeta^n I_p\,],
\end{aligned}
$$

where col means column,

$$(5.12) \qquad \zeta = \frac{\lambda - \frac{i}{2}}{\lambda + \frac{i}{2}}, \quad \frac{i}{2} - \lambda = \frac{i\zeta}{\zeta - 1}, \quad -\frac{i}{2} - \lambda = \frac{i}{\zeta - 1}.$$

Notice that we have

$$(5.13) \qquad \Phi_1^* T \Phi_1 = \Phi_1^* X = X^* \Phi_1,$$

which implies the first equality in (5.7). Multiply both sides of (5.10) by $\lambda^2 + \frac{1}{4}$ and use (5.11), (5.12) and the first equality in (5.7) to rewrite the result in the form

$$
(5.14) \qquad
\begin{aligned}
&\frac{i}{\zeta - 1}\Big([\, (\zeta - 1)I_p \;\; (\zeta^2 - 1)I_p \;\; \cdots \;\; (\zeta^n - 1)I_p\,]X \\
&\quad + X^* \mathrm{col}[\, 0 \;\; \zeta^{-1}(\zeta - 1)I_p \;\; \cdots \;\; \zeta^{-n}(\zeta^n - 1)I_p\,] \Big) \\
&= i[\, I_p \;\; \zeta I_p \;\; \cdots \;\; \zeta^n I_p\,]\widehat{Q}\,\mathrm{col}[\, I_p \;\; \zeta^{-1}I_p \;\; \cdots \;\; \zeta^{-n}I_p\,].
\end{aligned}
$$

The equalities for the coefficients corresponding to the same degrees of ζ on the left-hand side and on the right-hand side of (5.14) imply the second and the third relations in (5.7). $\Box$

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6. Factorization and similarity conditions. The block matrix

$$
(6.1) \qquad K = \begin{bmatrix} K_0 \\ K_1 \\ \vdots \\ K_n \end{bmatrix},
$$

where K_j are $p \times (n+1)p$ matrices of the form

$$
(6.2) \qquad K_j = i\beta(j)[\,\beta(0)^* \ \ \beta(1)^* \ \cdots \ \beta(j-1)^* \ \ \beta(j)^*/2 \ \ 0 \ \cdots \ 0\,],
$$

plays an essential role in [18]. From the proof of Theorem 3.4 in [18] the following result is immediate.

PROPOSITION 6.1. *Let a $(n+1)p \times (n+1)p$ matrix K be given by formulas (6.1) and (6.2), and let conditions (3.13) hold. Then K is similar to A:*

$$
(6.3) \qquad K = V_- A V_-^{-1},
$$

where $V_-^{\pm 1}$ are block lower triangular matrices.

Proposition 6.1 is a discrete analog of the theorem on similarity to the integration operator [19].

REMARK 6.2. *Note that V_-^{-1} can be chosen so that*

$$
(6.4) \qquad V_-^{-1} \begin{bmatrix} \beta_1(0) \\ \vdots \\ \beta_1(n) \end{bmatrix} = \Phi_1.
$$

Moreover, V_-^{-1} is a factor of S, i.e., $S = V_-^{-1}(V_-^*)^{-1} \in \Omega_n$. Any matrix $S \in \Omega_n$ can be obtained in this way.

An analogue of Proposition 6.1 for the self-adjoint discrete Dirac system and block Toeplitz matrices S follows from the proof of Theorem 5.2 in [11].

PROPOSITION 6.3. *Let a $(n+1)p \times (n+1)p$ matrix K be given by formulas (6.1) and*

$$
(6.5) \quad K_j = i\beta(j)J[\,\beta(0)^* \ \cdots \ \beta(j-1)^* \ \ \beta(j)^*/2 \ \ 0 \ \cdots \ 0\,], \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix},
$$

where $\beta(k)$ are $p \times 2p$ matrices. Let conditions $\beta(k)J\beta(k)^* = I_p$ $(0 \le k \le n)$ hold. Then K is similar to A: $K = V_- A V_-^{-1}$, where $V_-^{\pm 1}$ are block lower triangular matrices. Moreover, V_- can be chosen so that $S = V_-^{-1}(V_-^*)^{-1}$ is a block Toeplitz matrix.

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