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ISOTROPIC SUBSPACES FOR PAIRS OF HERMITIAN MATRICES*

LEIBA RODMAN†

Abstract. The maximal dimension of a subspace which is neutral with respect to two hermitian matrices simultaneously, is identified (in many cases) in terms of inertia of linear combinations of the matrices.

Key words. Neutral subspace, Isotropic subspace, Hermitian matrices.

AMS subject classifications. 15A57.

1. Introduction.

Let F be the real field R, the complex field C, or the skew field of real quaternions H. We denote by $\mathsf{F}^{m\times n}$ the set of $m \times n$ matrices with entries in F. If $A \in \mathsf{F}^{n\times n}$ then we say that an F-subspace $\mathcal{M} \subseteq \mathsf{F}^{n\times 1}$ (understood as the right vector space in the quaternionic case) is A-*neutral* (or A-*isotropic*) if $x^*Ay = 0$ for all vectors $x, y \in \mathcal{M}$. Here, X^* stands for the conjugate transpose (=transpose if $\mathsf{F} = \mathsf{R}$) of the matrix or vector X. We will use the notion of neutral subspace for hermitian (=symmetric if $\mathsf{F} = \mathsf{R}$) matrices A. Denoting by $i_+(A)$, resp. $i_-(A)$, the number of nonnegative, resp. nonpositive, eigenvalues of a hermitian matrix A counted with multiplicities, we have the following well-known properties for a hermitian matrix A:

(a) If $\mathcal{M}$ is A-neutral, then

$$\dim \mathcal{M} \leq \min\{i_+(A), i_-(A)\} \tag{1.1}$$

(the dimension is understood in the sense of F; thus, $\mathsf{H}^{n\times 1}$ has dimension n)

(b) The maximal dimension of an A-neutral subspace is equal to

$$\min\{i_+(A), i_-(A)\}.$$

For a proof, see for example [5] or [1] (for the quaternionic case).

If $A, B \in \mathsf{F}^{n\times n}$ are two hermitian matrices, it is of interest to study subspaces that are (A, B)-*neutral*, i.e., simultaneously A-neutral and B-neutral. In the context of selfadjoint complex matrices with respect to indefinite inner products, such subspaces play a key role in many problems of symmetric factorization and other applications [9, 14, 15, 8, 13]. It easily follows from (a) that

$$\dim \mathcal{M} \leq \min_{0\leq\theta<2\pi}\{i_+((\cos\theta)A + (\sin\theta)B)\} \tag{1.2}$$

for every (A, B)-neutral subspace $\mathcal{M}$. Denoting by $\gamma(A, B)$ the maximal dimension of (A, B)-neutral subspaces, and by $\mathrm{mi}_+(A, B)$ the right hand side of (1.2), we therefore have

$$\gamma(A, B) \leq \mathrm{mi}_+(A, B). \tag{1.3}$$

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Note that also

$$\mathrm{mi}_+(A,B) = \min_{a,b\in \mathsf{R}} \{i_+(aA+bB)\} = \min_{0\le\theta<2\pi} \{i_-((\cos\theta)A + (\sin\theta)B)\}.$$

A natural question arises whether or not the equality persists in (1.3) for all pairs of hermitian matrices A and B. Using the techniques of higher rank numerical ranges, it was proved in [12] that this is indeed the case if $\mathsf{F}=\mathsf{C}$. However, in the real case a strict inequality may occur in (1.3): Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then the right hand side of (1.3) is 1, but the only real (A,B)-neutral subspace is the zero subspace. Of course, in accordance with [12] there exist complex, as well as quaternionic, 1-dimensional (A,B)-neutral subspaces. Namely, the complex (A,B)-neutral subspaces (there are only two of them) are given by $\mathrm{Span}_{\mathsf{C}}\left\{\begin{bmatrix} 1 \\ \pm\mathsf{i}\end{bmatrix}\right\}$, and a continuum of quaternionic (A,B)-neutral subspaces is given by $\mathrm{Span}_{\mathsf{H}}\left\{\begin{bmatrix} 1 \\ b\mathsf{i}+c\mathsf{j}+d\mathsf{k}\end{bmatrix}\right\}$, where $b,c,d\in\mathsf{R}$ are such that $b^2+c^2+d^2=1$. Here and in the sequel, we denote by $\mathrm{Span}_{\mathsf{F}}\{x_1,\dots,x_p\}$ the F-subspace spanned by the vectors $x_1,\dots,x_p$.

In this paper we study the relation between $\gamma(A,B)$ and $\mathrm{mi}_+(A,B)$ for a pair of hermitian matrices A and B. Our approach is based on (essentially known) unified canonical form for pairs of hermitian matrices over $\mathsf{R},\mathsf{C},\mathsf{H}$. In particular, we provide another proof of equality in (1.3) in the complex case, and show that the equality holds in the quaternionic case as well as in many situations when $\mathsf{F}=\mathsf{R}$. We state the main result in the next section. The proof will be given in Section 3. In the last Section 4 we prove some partial results concerning the cases of inequality in (1.3).

We conclude the introduction with notation to be used throughout the paper. We denote by $\mathrm{diag}\,(X_1,X_2,\dots,X_p)$, or by $X_1\oplus X_2\oplus\cdots\oplus X_p$, the block diagonal matrix with diagonal blocks $X_1,\dots,X_p$ (in that order). The notation A^T and A^* stand for the transpose and the conjugate transpose, respectively, of the matrix A. The vector $e_j\in\mathsf{R}^{k\times 1}$ is the unit coordinate vector: 1 in the jth position, zeros elsewhere. The $r\times r$ identity and zero matrix are denoted I_r and 0_r, respectively. Finally, we introduce real symmetric matrices in special forms:

$$F_m := \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} = F_m^{-1} \in \mathsf{R}^{m\times m}, \tag{1.4}$$

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$$G_m := \begin{bmatrix} 0 & \cdots & \cdots & 1 & 0 \\ \vdots & & & 0 & 0 \\ \vdots & & \iddots & & \vdots \\ 1 & 0 & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} F_{m-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathsf{R}^{m\times m}, \tag{1.5}$$

$$H_{2m} = \begin{bmatrix} 0 & \dots & & & 1 & 0 \\ & & & & 0 & -1 \\ & & 1 & 0 & & \\ & & 0 & -1 & & \\ \vdots & \iddots & \iddots & & & \vdots \\ 1 & 0 & & & & \\ 0 & -1 & & \dots & & 0 \end{bmatrix} \in \mathsf{R}^{2m\times 2m},$$

$$Z_{2m}(t,\,\mu,\,\nu) := (t+\mu)F_{2m} + \nu H_{2m} + \begin{bmatrix} F_{2m-2} & 0 \\ 0 & 0_2 \end{bmatrix} \in \mathsf{R}^{2m\times 2m}, \tag{1.6}$$

where μ,ν are real and $\nu > 0$. Here t is a real parameter.

2. Main result: equality of $\gamma(A,B)$ and $\mathrm{mi}_+(A,B)$.

We start with the canonical form which will be convenient to recast in terms of matrix pencils. Let F be one of R, C, H. Two $n\times n$ matrix pencils A_1+tB_1 and A_2+tB_2, where $A_1, A_2, B_1, B_2 \in \mathsf{F}^{n\times n}$ are hermitian matrices, are said to be F-*congruent*, if

$$S^*(A_1+tB_1)S = A_2+tB_2$$

for some invertible matrix $S \in \mathsf{F}^{n\times n}$.

Theorem 2.1. *Every hermitian matrix pencil $A+tB$, $A=A^*, B=B^* \in \mathsf{F}^{n\times n}$, is F-congruent to a real hermitian matrix pencil of the form*

$$\begin{aligned} 0_u \,\oplus\, &\left(t\begin{bmatrix} 0 & 0 & F_{\varepsilon_1} \\ 0 & 0 & 0 \\ F_{\varepsilon_1} & 0 & 0 \end{bmatrix} + G_{2\varepsilon_1+1}\right) \oplus\cdots\oplus \left(t\begin{bmatrix} 0 & 0 & F_{\varepsilon_p} \\ 0 & 0 & 0 \\ F_{\varepsilon_p} & 0 & 0 \end{bmatrix} + G_{2\varepsilon_p+1}\right) \\ &\oplus\, ((\sin\theta_1)F_{k_1} - (\cos\theta_1)G_{k_1} + t((\cos\theta_1)F_{k_1} + (\sin\theta_1)G_{k_1})) \oplus \\ &\cdots\oplus ((\sin\theta_r)F_{k_r} - (\cos\theta_r)G_{k_r} + t((\cos\theta_r)F_{k_r} + (\sin\theta_r)G_{k_r})) \\ &\oplus\, Z_{2m_1}(t,\,\mu_1,\,\nu_1) \oplus\cdots\oplus Z_{2m_s}(t,\,\mu_s,\,\nu_s). \end{aligned} \tag{2.1}$$

Here $0 \le \theta_j < 2\pi$, μ_j, ν_j are real numbers, and $\nu_j > 0$.

The form (2.1) is uniquely determined by $A+tB$ up to a permutation of the blocks.

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The form (2.1) is well known for the case $\mathsf{F} = \mathsf{R}$; see [18, 10], for example, and references there, and [19], also [16], for the version presented in Theorem 2.1 that includes the sine/cosine functions. It is also well known that the canonical form of pairs of hermitian quaternionic matrices can be taken in the set of complex matrices [7, 16, 4, 3]. For the complex case, the form (2.1) originates in author's discussions with Bella and Olshevsky [2]; it can be obtained from the standard canonical form for complex hermitian pencils under C-congruence (as presented in [10], for example) using the following observation: Any complex matrix that is selfadjoint with respect to a nondegenerate inner product in $\mathsf{C}^{n\times 1}$, is similar to a real matrix.

We now state the main result:

Theorem 2.2. *Let $A, B \in \mathsf{F}^{n\times n}$ be hermitian matrices. Assume in addition that in case $\mathsf{F} = \mathsf{R}$ only, the following condition is satisfied:*

(A) *For every fixed ordered pair μ_0, ν_0, where $\mu_0 \in \mathsf{R}$ and $\nu_0 > 0$, the number of blocks $Z_{2m}(t, \mu_0, \nu_0)$ with odd m in the canonical form of $A + tB$ is even (perhaps zero).*

Then

$$\gamma(A, B) = \mathrm{mi}_+(A, B). \tag{2.2}$$

Condition (A) can be interpreted in terms of the Kronecker form (the canonical form under strict equivalence $A + tB \longrightarrow S(A + tB)T$, where S and T are invertible real matrices) of the real matrix pencil $A + tB$. Namely, (A) holds if and only if for every complex nonreal eigenvalue λ of $A + tB$, the number of odd multiplicities associated with λ is even.

The next section is devoted to the proof of Theorem 2.2.

3. Proof of Theorem 2.2.

We start with recalling a version of the well known interlacing property for eigenvalues of real symmetric matrices:

Proposition 3.1. *Let $X \in \mathsf{R}^{p\times p}$, $Y \in \mathsf{R}^{q\times q}$ be real symmetric matrices, with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_p$ and $\mu_1 \geq \cdots \geq \mu_q$, respectively. Assume $p \geq q$. Then there exists an isometry $Q \in \mathsf{R}^{p\times q}$, i.e., $Q^T Q = I$, such that $Q^T X Q = Y$ if and only if the interlacing inequalities*

$$\lambda_1 \geq \mu_1 \geq \lambda_{p-q+1},\ \lambda_2 \geq \mu_2 \geq \lambda_{p-q+2},\ \ldots,\ \lambda_q \geq \mu_q \geq \lambda_p$$

are satisfied.

Proposition 3.1 is valid also in the complex and quaternionic cases (replacing transposition with conjugate transposition), but only the real version will be used in the present paper. The "only if" part is the interlacing property of the eigenvalues; see, e.g., [11, Chapter 8] or [6, Chapter 4]. The "if" part (for the special case of $q = p - 1$ from which the general case follows easily by induction) is found in [17, Chapter 10] or [6, Chapter 4].

To complete preparations for the proof of Theorem 2.2, we state and prove a lemma:

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Lemma 3.2. *Let $A_1, B_1 \in \mathsf{R}^{m\times m}$ and $A_2, B_2 \in \mathsf{R}^{\ell\times\ell}$ be hermitian matrices, and let*

$$A = A_1 \oplus A_2, \quad B = B_1 \oplus B_2.$$

Assume furthermore that for some integer k, $0 \le k \le m/2$, there exists an $(m-k)$-dimensional (A_1, B_1)-neutral subspace $\mathcal{M} \subseteq \mathsf{F}^{m\times 1}$, where F is either R, C, or H, and for some θ_0, $0 \le \theta_0 < 2\pi$, the rank of the hermitian matrix $(\cos\theta_0)A_1 + (\sin\theta_0)B_1$ is equal to $2k$. Then

$$\mathrm{mi}_+(A, B) = m - k + \mathrm{mi}_+(A_2, B_2). \tag{3.1}$$

Proof. First observe that for every $\theta \in [0, 2\pi)$ the number k' of negative eigenvalues (counted with multiplicities) of $(\cos\theta)A_1 + (\sin\theta)B_1$ does not exceed k; otherwise, there would be a nonzero intersection of the $(m-k)$-dimensional $(\cos\theta)A_1 + (\sin\theta)B_1$-neutral subspace and the negative k'-dimensional $(\cos\theta)A_1 + (\sin\theta)B_1$-invariant subspace, which is impossible. Analogously, the number of negative eigenvalues of $(\cos\theta)A_1 + (\sin\theta)B_1$ does not exceed k.

Thus,

$$i_+((\cos\theta)A_1 + (\sin\theta)B_1) \ge m - k, \qquad \forall\ \theta \in [0, 2\pi).$$

For every fixed θ we have

$$\begin{aligned} i_+((\cos\theta)A + (\sin\theta)B) &= i_+((\cos\theta)A_1 + (\sin\theta)B_1) + i_+((\cos\theta)A_2 + (\sin\theta)B_2) \\ &\ge m - k + i_+((\cos\theta)A_2 + (\sin\theta)B_2). \end{aligned}$$

Taking first minimum of the right hand side with respect to $\theta \in [0, 2\pi)$, and then minimum of the left hand side, the inequality $\ge$ in (3.1) follows.

To prove the opposite inequality, first of all observe that for every $\theta \in [0, 2\pi)$, after a suitable unitary similarity the matrix $(\cos\theta)A_1 + (\sin\theta)B_1$ has a top left $(m-k)\times(m-k)$ zero block, and therefore the rank of $(\cos\theta)A_1 + (\sin\theta)B_1$ cannot exceed $2k$. Since by the hypotheses, this rank is actually equal to $2k$ for some θ_0, it easily follows that

$$\mathrm{rank}\,((\cos\theta)A_1 + (\sin\theta)B_1) = 2k$$

for all $\theta \in [0, 2\pi)$ save at most a finite set of values θ. On the other hand, because of continuity of the spectrum, if $\theta_1 \in [0, 2\pi)$ is such that

$$\mathrm{mi}_+(A_2, B_2) = i_+((\cos\theta_1)A_2 + (\sin\theta_1)B_2),$$

then we also have

$$\mathrm{mi}_+(A_2, B_2) = i_+((\cos\theta')A_2 + (\sin\theta')B_2)) \tag{3.2}$$

for all θ' sufficiently close to θ_1. Now select θ' so that (3.2) holds and also

$$\mathrm{rank}\,((\cos\theta')A_1 + (\sin\theta')B_1) = 2k.$$

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We must have then that the number of positive, resp. negative, eigenvalues of $(\cos\theta')A_1 + (\sin\theta')B_1$ is exactly k, and therefore $i_+((\cos\theta')A_1 + (\sin\theta')B_1) = m - k$. Now

$$i_+((\cos\theta')A + (\sin\theta')B) = i_+((\cos\theta')A_1 + (\sin\theta')B_1) + i_+((\cos\theta')A_2 + (\sin\theta')B_2)$$

$$= m - k + \mathrm{mi}_+(A_2, B_2),$$

and the inequality $\leq$ in (3.1) follows. □

The following corollary is easily obtained from Lemma 3.2:

Corollary 3.3. *Under the hypotheses of Lemma* 3.2*, if* $\gamma(A_2, B_2) = \mathrm{mi}_+(A_2, B_2)$*, then also* $\gamma(A, B) = \mathrm{mi}_+(A, B)$.

Indeed, if $\mathcal{M}_2$ is an (A_2, B_2)-neutral subspace of dimension $\gamma(A_2, B_2)$, then $\mathcal{M} \oplus \mathcal{M}_2$ is an (A, B)-neutral subspace of dimension $m - k + \mathrm{mi}_+(A_2, B_2)$, so

$$\gamma(A, B) \geq \mathrm{mi}_+(A, B),$$

whereas the opposite inequality is obvious (see (1.3)).

Proof of Theorem 2.2. Without loss of generality we may assume that the pencil $A + tB$ is given by the right hand side of (2.1). In particular, A and B are both real (but the (A, B)-neutral subspaces we are after, are still in $\mathsf{F}^{n\times 1}$).

We use induction on the size n. So we are done if one (or more) of the blocks in (2.1) has the properties of (A_1, B_1) of Lemma 3.2. It is easy to see that such are the blocks 0_u (with $k = 0$),

$$t\begin{bmatrix} 0 & 0 & F_\varepsilon \\ 0 & 0 & 0 \\ F_\varepsilon & 0 & 0 \end{bmatrix} + G_{2\varepsilon_1}$$

(with $k = \varepsilon$),

$$(\sin\theta)F_m - (\cos\theta)G_m + t((\cos\theta)F_m + (\sin\theta)G_m)$$

for m even (with $k = m/2$), and $Z_{2p}(t,\, \mu,\, \nu)$ (with $k = p$), where for the real case p must be even. For example, if $\mathsf{F} = \mathsf{C}$ or $\mathsf{F} = \mathsf{H}$, and if p is odd, then

$$\mathrm{Span}_{\mathsf{F}}\,\{e_p + \mathsf{i}e_{p+1}, e_{p+2}, \ldots, e_{2p}\}$$

is a p-dimensional (A_1, B_1)-neutral subspace, where the matrices A_1 and B_1 are defined by $A_1 + tB_1 := Z_{2p}(t,\, \mu,\, \nu)$. Also, if (in the real case only) we have a pair of blocks

$$A_1 + tB_1 := Z_{2p}(t,\, \mu_0,\, \nu_0) \oplus Z_{2q}(t,\, \mu_0,\, \nu_0)$$

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with odd p and q, then it is easy to verify that the real subspace

$$\mathrm{Span}_{\mathsf{R}}\{e_{p+2}, e_{p+3}, \ldots, e_{2p}, e_{2p+q+2}, e_{2p+q+3}, \ldots, e_{2p+2q}, e_p \mp e_{2p+q+1}, e_{p+1} \pm e_{2p+q}\}$$

is $(p+q)$-dimensional and (A_1, B_1)-neutral.

Thus, the proof is reduced to the case when

$$\begin{aligned} A + tB = &((\sin\theta_1)F_{k_1} - (\cos\theta_1)G_{k_1} + t((\cos\theta_1)F_{k_1} + (\sin\theta_1)G_{k_1})) \oplus \\ &\cdots \oplus ((\sin\theta_r)F_{k_r} - (\cos\theta_r)G_{k_r} + t((\cos\theta_r)F_{k_r} + (\sin\theta_r)G_{k_r})), \end{aligned} \tag{3.3}$$

where k_j's are odd positive integers. So assume A and B are given by (3.3). We reduce the proof further to the case when all k_j's are equal 1. (If $k_j = 1$, the matrix G_{k_j} disappears from (3.3).) Denote

$$A' + tB' = (\sin\theta_1 F_1 + t\cos\theta_1 F_1) \oplus \cdots \oplus (\sin\theta_r F_1 + t\cos\theta_r F_1) \in \mathsf{R}^{r\times r}.$$

Assuming Theorem 2.2 is already proved for the pair A', B', let

$$w = \mathrm{mi}_+(A', B'),$$

and let

$$\mathcal{M}' := \mathrm{Span}_{\mathsf{F}}\left\{\sum_{q=1}^{r} x_{1q}e_q, \sum_{q=1}^{r} x_{2q}e_q, \ldots, \sum_{q=1}^{r} x_{wq}e_q\right\}, \quad x_{sq} \in \mathsf{F},$$

be a w-dimensional (A', B')-neutral subspace. Let $\mathcal{M}$ be the subspace spanned (over F) by the following vectors in $\mathsf{F}^{k\times 1}$, where $k = k_1 + k_2 + \cdots + k_r$:

$$e_{(k_1+1)/2+1}, \ldots, e_{k_1}, e_{k_1+(k_2+1)/2+1}, \ldots, e_{k_1+k_2}, \ldots, e_{k_1+\cdots+k_{r-1}+(k_r+1)/2+1}, \ldots e_k,$$

$$\sum_{q=1}^{r} x_{1q}e_{k_1+\cdots+k_{q-1}+((k_q+1)/2)}, \ \sum_{q=1}^{r} x_{2q}e_{k_1+\cdots+k_{q-1}+((k_q+1)/2)}, \ldots,$$

$$\sum_{q=1}^{r} x_{wq}e_{k_1+\cdots+k_{q-1}+((k_q+1)/2)}.$$

(We take $k_0 = 0$.) One verifies that $\mathcal{M}$ is (A, B)-neutral and $\dim\mathcal{M} = \mathrm{mi}_+(A, B)$.

Finally, we prove Theorem 2.2 for the pair A, B, where

$$A = \mathrm{diag}\,(\sin\theta_1, \sin\theta_2, \ldots, \sin\theta_r), \quad B = \mathrm{diag}\,(\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_r),$$

with $0 \le \theta_1 \le \cdots \le \theta_r < 2\pi$. We leave aside the trivial case when A and B are linearly dependent. Note that we may replace A and B with $aA + bB$ and $cA + dB$, respectively, where the real numbers a, b, c, d are such that $ad - bc \neq 0$. Therefore, without loss of generality we may assume that

$$i_+(A) = \mathrm{mi}_+(A, B).$$

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Furthermore, replacing if necessary A with $A + \varepsilon B$ for sufficiently small positive ε, we may assume that A is invertible. Applying simultaneous congruence to A and B with a suitable permutation congruence matrix, we may further assume that

$$A = \begin{bmatrix} I_s & 0 \\ 0 & -I_{r-s} \end{bmatrix}, \tag{3.4}$$

where

$$s = \mathrm{mi}_+(A, B). \tag{3.5}$$

Obviously, $s \le r - s$. We also replace B with $A + \varepsilon' B$ for sufficiently small positive ε'; thus, B will have the form

$$B = \begin{bmatrix} \mathrm{diag}\,(d_1, \ldots, d_s) & 0 \\ 0 & \mathrm{diag}\,(-c_1, \ldots, -c_{r-s}) \end{bmatrix}, \tag{3.6}$$

where $d_1 \ge d_2 \ge \cdots \ge d_s$, $c_1 \ge c_2 \ge \cdots \ge c_{r-s}$ are positive numbers. For A and B given by (3.4), (3.6), we seek an s-dimensional (A, B)-neutral subspace in the form of the column space of the matrix $\begin{bmatrix} I \\ Q \end{bmatrix}$, for some $Q \in \mathsf{R}^{(r-s)\times s}$. The (A, B)-neutrality of the subspace amounts to equalities

$$Q^T Q = I, \quad Q^T \mathrm{diag}\,(c_1, \ldots, c_{r-s}) Q = \mathrm{diag}\,(d_1, \ldots, d_s).$$

By Proposition 3.1 we need only to verify the interlacing inequalities

$$c_1 \ge d_1 \ge c_{r-2s+1}, \;\; c_2 \ge d_2 \ge c_{r-2s+2}, \;\ldots,\; c_s \ge d_s \ge c_{r-s}. \tag{3.7}$$

Indeed, suppose (3.7) does not hold, say $c_j < d_j$ for some index j. Let $x > 0$ be such that $xd_j > 1 > xc_j$. Then a straightforward computation shows that $A - xB$ has less than s nonnegative eigenvalues, a contradiction with (3.5). Analogously, suppose $d_j < c_{r-2s+j}$ for some j. Let $y > 0$ be such that $yd_j < 1 < yc_{r-2s+j}$. Then $-A + yB$ has at most $s - 1$ nonnegative eigenvalues, a contradiction again. This completes the proof of Theorem 2.2.

4. The exceptional case. In this section we consider briefly the exceptional case not covered in Theorem 2.2, i.e., $\mathsf{F} = \mathsf{R}$ and the condition (A) is not satisfied. In this case we have a lower bound on $\gamma(A, B)$:

Theorem 4.1. *Assume $\mathsf{F} = \mathsf{R}$. Let $A, B \in \mathsf{R}^{n\times n}$ be symmetric matrices, and let β be the number of ordered pairs (μ_0, ν_0) such that $\mu_0 \in \mathsf{R}$, $\nu_0 > 0$, and the number of blocks $Z_{2m}(t, \mu_0, \nu_0)$ with odd m in the canonical form of $A + tB$ is odd. Then*

$$\gamma(A, B) \ge \mathrm{mi}_+(A, B) - \beta + \lfloor \beta/2 \rfloor, \tag{4.1}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Proof. We let $A + tB$ be given by the right hand side of (2.1). Partition A and B into two parts:

$$A + tB = (A_1 + tB_1) \oplus (A_2 + tB_2),$$

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where

$$A_1 + tB_1 = \oplus_{j=1}^{\beta} Z_{2m_j}(t, \mu_j, \nu_j), \qquad \mu_j \in \mathsf{R}, \quad \nu_j > 0,$$

with $m_1, \ldots, m_\beta$ odd positive integers, and where the pair (A_2, B_2) satisfies the hypotheses of Theorem 2.2. By Theorem 2.2 we have

$$\gamma(A_2, B_2) = \mathrm{mi}_+(A_2, B_2).$$

Also, we show (as in the proof of Lemma 3.2) that

$$\mathrm{mi}_+(A, B) = \mathrm{mi}_+(A_1, B_1) + \mathrm{mi}_+(A_2, B_2) = m_1 + \cdots + m_\beta + \mathrm{mi}_+(A_2, B_2).$$

On the other hand, one can easily find an (A_1, B_1)-neutral subspace $\mathcal{M}_1$ of dimension $\lfloor \beta/2 \rfloor + \sum_{j=1}^{\beta}(m_j - 1)$; for example, if $\mu_1 = \cdots = \mu_\beta = 1$ and β is even, then

$$\mathcal{M}_1 := \mathrm{Span}_{\mathsf{R}} \{\sqrt{\nu_2} e_2 + \sqrt{\nu_1} e_3, \sqrt{\nu_4} e_6 + \sqrt{\nu_3} e_7, \ldots, \sqrt{\nu_\beta} e_{2\beta-2} + \sqrt{\nu_{\beta-1}} e_{2\beta-1}\} \subseteq \mathsf{R}^{2\beta \times 1}$$

will do. If $\mathcal{M}_2$ is any (A_2, B_2)-neutral subspace of dimension $\mathrm{mi}_+(A_2, B_2)$, then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is an (A, B)-neutral subspace of dimension $\mathrm{mi}_+(A, B) - \beta + \lfloor \beta/2 \rfloor$, and (4.1) follows. □

We say that a pair of $n \times n$ real symmetric matrices A, B is *totally invertible* if $\det((\cos\theta)A + (\sin\theta)B) \neq 0$ for all $\theta \in [0, 2\pi)$. This class can be characterized in several ways:

Proposition 4.2. *The following statements are equivalent for symmetric $A, B \in \mathsf{R}^{n \times n}$:*

(1) *The pair (A, B) is totally invertible.*
(2) *Only blocks of the form $Z_{2m}(t, \mu, \nu)$ are present in the canonical form of (A, B).*
(3) *n must be even and $i_+((\cos\theta)A + (\sin\theta)B) = n/2$ for all θ.*
(4) *The matrix B is invertible, and $B^{-1}A$ has only nonreal eigenvalues.*

Moreover, if $(1) - (4)$ hold, then $\mathrm{mi}_+(A, B) = n/2$, and if in addition n is not divisible by 4, then $\gamma(A, B) < n/2$.

Proof. All but the last statement follow easily from the canonical form (2.1). Suppose now n is not divisible by 4. If (A, B) is a totally invertible pair of real symmetric matrices, and if there exists an $n/2$-dimensional (A, B)-neutral subspace, then we may apply simultaneous congruence to A and B and assume that

$$A = \begin{bmatrix} 0_{n/2} & A_1 \\ A_1^T & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0_{n/2} & B_1 \\ B_1^T & B_2 \end{bmatrix},$$

where $A_1, A_2, B_1, B_2 \in \mathsf{R}^{n/2 \times n/2}$. Since (A, B) is totally invertible, the matrix A_1 is invertible and so is $tA_1 + B_1$ for all real t. But this is impossible, because the real odd size matrix $A_1^{-1} B_1$ must have a real eigenvalue. □

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Example. Let $A, B \in \mathsf{R}^{4\times 4}$ be a totally invertible pair of symmetric matrices. Then

$$\gamma(A,B) = \begin{cases} 1 & \text{if } B^{-1}A \text{ has 4 distinct eigenvalues;} \\ 2 & \text{otherwise.} \end{cases}$$

For verification assume that the pair (A, B) is in the canonical form and seek a 2-dimensional (A, B)-neutral subspace (if such exists) in the form $\operatorname{Im} \begin{bmatrix} I \\ Q \end{bmatrix}$ for a suitable 2×2 matrix Q. We omit details.

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