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THE STRUCTURE OF LINEAR PRESERVERS OF LEFT MATRIX
MAJORIZATION ON $\mathbb{R}^p$ *

FATEMEH KHALOOEI† AND ABBAS SALEMI†

Abstract. For vectors $X, Y \in \mathbb{R}^n$, $Y$ is said to be left matrix majorized by $X$ ($Y \prec_{\ell} X$) if for some row stochastic matrix $R$, $Y = RX$. A linear operator $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ is said to be a linear preserver of $\prec_{\ell}$ if $Y \prec_{\ell} X$ on $\mathbb{R}^p$ implies that $TY \prec_{\ell} TX$ on $\mathbb{R}^n$. The linear operators $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ ($n < p(p-1)$) which preserve $\prec_{\ell}$ have been characterized. In this paper, linear operators $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ which preserve $\prec_{\ell}$ are characterized without any condition on $n$ and $p$.

Key words. Row stochastic matrix, Doubly stochastic matrix, Matrix majorization, Weak matrix majorization, Left (right) multivariate majorization, Linear preserver.

AMS subject classifications. 15A04, 15A21, 15A51.

1. Introduction. Let $M_{nm}$ be the algebra of all $n \times m$ real matrices. A matrix $R = [r_{ij}] \in M_{nm}$ is called a row stochastic (resp., row substochastic) matrix if $r_{ij} \geq 0$ and $\sum_{k=1}^m r_{ik} = 1$ (resp., $\leq 1$) for all $i, j$. For $A, B$ in $M_{nm}$, $A$ is said to be left matrix majorized by $B$ ($A \prec_{\ell} B$), if $A = RB$ for some $n \times n$ row stochastic matrix $R$. These notions were introduced in [11]. If $A \prec_{\ell} B \prec_{\ell} A$, we write $A \prec_{\ell} B$. Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator. $T$ is said to be a linear preserver of $\prec_{\ell}$ if $Y \prec_{\ell} X$ on $\mathbb{R}^p$ implies that $TY \prec_{\ell} TX$ on $\mathbb{R}^n$. For more information about types of majorization see [1], [5] and [10]; for their preservers see [2]-[4], [6] and [9].

We shall use the following conventions throughout the paper: Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a nonzero linear operator and let $[T] = [t_{ij}]$ denote the matrix representation of $T$ with respect to the standard bases $\{e_1, e_2, \ldots, e_p\}$ of $\mathbb{R}^p$ and $\{f_1, f_2, \ldots, f_n\}$ of $\mathbb{R}^n$. If $p = 1$, then all linear operators on $\mathbb{R}^1$ are preservers of $\prec_{\ell}$. Thus, we assume $p \geq 2$. Let $A_i$ be $m_i \times p$ matrices, $i = 1, \ldots, k$. We use the notation $[A_1/A_2/\ldots/A_k]$ to denote the corresponding $(m_1 + m_2 + \ldots + m_k) \times p$ matrix. We let $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^p$, and denote

$$a := \max\{\max T(e_1), \ldots, \max T(e_p)\},$$

$$b := \min\{\min T(e_1), \ldots, \min T(e_p)\}.$$  

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Theorem 1.1. ([9, Theorem 2.2]) Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a nonzero linear preserver of \( \prec \) and suppose \( p \geq 2 \). Then \( p \leq n \), \( b \leq 0 \leq a \) and for each \( i \in \{1, \ldots, p\} \), \( a = \max T(e_i) \) and \( b = \min T(e_i) \). In particular, every column of \([T]\) contains at least one entry equal to \( a \) and at least one entry equal to \( b \).

Definition 1.2. Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator. We denote by $P_i$ (resp., $N_i$) the sum of the nonnegative (resp., non positive) entries in the $i^{th}$ row of $[T]$. If all the entries in the $i^{th}$ row are positive (resp., negative), we define $N_i = 0$ (resp., $P_i = 0$).

We know that $T$ is a linear preserver of \( \prec \) if and only if $\alpha T$ is also a linear preserver of \( \prec \) for some nonzero real number $\alpha$. Without loss of generality we make the following assumption.

Assumption 1.3. Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a nonzero linear preserver of \( \prec \). Let $a$ and $b$ be as in (1.1). We assume that $0 \leq -b \leq 1 = a$.

Definition 1.4. Let $P$ be the permutation matrix such that $P(e_i) = e_{i+1}$, $1 \leq i \leq p - 1$, $P(e_p) = e_1$. Let $I$ denote the $p \times p$ identity matrix, and let $r, s \in \mathbb{R}$ be such that $rs < 0$. Define the $p(p-1) \times p$ matrix $P_p(r,s) = [P_1/P_2/\ldots/P_{p-1}]$, where $P_j = rI + sP^j$, for all $j = 1, 2, \ldots, p - 1$. It is clear that up to a row permutation, the matrices $P_p(r,s)$ and $P_p(s,r)$ are equal. Also define $P_p(r,0) := rI$, $P_p(0,s) := sI$ and $P_p(0,0)$ as a zero row.

The structure of all linear operators $T: M_{nm} \rightarrow M_{nm}$ preserving matrix majorizations was considered in [6, 7, 8]. Also the linear operators $T$ from $\mathbb{R}^p$ to $\mathbb{R}^n$ that preserve the left matrix majorization \( \prec \) were characterized in [9] for $n < p(p-1)$. In the present paper, we will characterize all linear preservers of \( \prec \) mapping $\mathbb{R}^p$ to $\mathbb{R}^n$ without any additional conditions.

2. Left matrix majorization. In this section we obtain a key condition that is necessary for $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ to be a linear preserver of \( \prec \). We first need the following.

Lemma 2.1. Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator such that $\min T(X) \leq \min T(X)$ for all $X \prec Y$. Then $T$ is a preserver of \( \prec \).

Proof. Let $X \prec Y$. It is enough to show that $\max T(X) \leq \max T(Y)$. Since $X \prec Y$, $-X \prec -Y$, and hence $\min T(-Y) \leq \min T(-X)$. This means that $\max T(X) \leq \max T(Y)$. Then $T$ is a preserver of \( \prec \). \[ \square \]

Remark 2.2. Let $T: \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear preserver of \( \prec \) and let $a$ and $b$ be as in Assumption 1.3. By Theorem 1.1 we know that in each column of $[T] = [t_{ij}]$ there is at least one entry equal to $a(= 1)$ and at least one entry equal to $b$. For $1 \leq k \leq p$, ...
we define

\[ I_k = \{i : 1 \leq i \leq n, t_{ik} = 1\}, \quad J_k = \{j : 1 \leq j \leq n, t_{jk} = b\}. \]

Next we state the key theorem of this paper.

**Theorem 2.3.** Let \( T : \mathbb{R}^p \to \mathbb{R}^n \) be a linear preserver of \( \prec \) and let \( a \) and \( b \) be as in Assumption 1.3. Then there exist \( 0 \leq \alpha \leq 1 \) and \( b \leq \beta \leq 0 \) such that \( P_p(1, \beta) \) and \( P_p(\alpha, b) \) are submatrices of \( [T] \), where \( P_p(r, s) \) is as in Definition 1.4.

**Proof.** Let \( 1 \leq k \leq p \) be a fixed number and let \( I_k \) and \( J_k \) be as in Remark 2.2. Since \( T \) is a linear preserver of \( \prec \), it follows that \( I_k \) and \( J_k \) are nonempty sets. Also \( e_k + e_l \prec e_k, l \neq k \). Thus, the other entries in the \( i^{th} \) row, \( i \in I_k \) (resp., \( j^{th} \) row, \( j \in J_k \)) are non positive (resp., nonnegative). Hence, \( t_{il} \leq 0 \), \( t_{jl} \geq 0 \), \( l \neq k \), \( i \in I_k \), and \( j \in J_k \). Let \( \beta_k = \sum_{l \neq k} t_{il} \leq 0 \), \( i \in I_k \) and \( \alpha_k = \sum_{l \neq k} t_{jl} \geq 0 \), \( j \in J_k \).

\[ \beta_k := \min\{\beta_k, i \in I_k\}, \quad \alpha_k := \max\{\alpha_k, j \in J_k\}. \]

Define \( X_k = -(N + 1)e_k + e \). Choose \( N_0 \) large enough such that for all \( N \geq N_0 \) and \( 1 \leq i \leq n \),

\[ \min T(X_k) = -N + \beta_k \leq -Nt_{ik} + \sum_{l \neq k} t_{il} \leq -Nb + \alpha_k = \max T(X_k). \]

We know that \( X_k \sim \prec X_r = -(N + 1)e_r + e \), \( 1 \leq r \leq p \) and \( T \) is a linear preserver of \( \prec \). Hence by (2.2), \( \alpha := \alpha_k = \alpha_r \) and \( \beta := \beta_k = \beta_r \), \( 1 \leq r \leq p \). Also, \( X_k \sim \prec -Ne_i + e_j, i \neq j \). For each \( N \geq N_0 \), there exists \( 1 \leq h \leq n \) such that \( -Nt_{hi} + t_{hj} = \min T(-Ne_i + e_j) = \min T(X_k) = -N + \beta \) and for each \( 1 \leq i \leq p \), \( 1 \leq j \leq p \) and \( N \geq N_0 \), there exists \( 1 \leq h \leq n \) such that \( -N(1 - t_{hi}) = t_{hj} - \beta \). It follows that \( t_{hi} = 1 \) and \( t_{hj} = \beta \). Hence \( P_p(1, \beta) \) is a submatrix of \( [T] \). Similarly, there exists \( N_1 \), such that for each \( N \geq N_1 \) there exists \( 1 \leq h \leq n \) so that \( -Nt_{hi} + t_{hj} = \max T(-Ne_i + e_j) = \max T(X_k) = -Nb + \alpha \) and \( -N(b - t_{hi}) = t_{hj} - \alpha \). Since \( 1 \leq i \neq j \leq p \) was arbitrary, \( P_p(b, \alpha) \) is a submatrix of \( [T] \).

Therefore, \( P_p(1, \beta) \) and \( P_p(b, \alpha) \) are submatrices of \( [T] \).

**Remark 2.4.** Let \( T : \mathbb{R}^p \to \mathbb{R}^n \) and \( \hat{T} : \mathbb{R}^p \to \mathbb{R}^m \) be two linear operators such that \( [T] = [T_1/T_2/\ldots/T_n] \) and let \( [\hat{T}] = [\hat{T}_1/\hat{T}_2/\ldots/\hat{T}_m] \) be the matrix representation of these operators with respect to the standard basis. Let \( R(T) = \{T_1, T_2, \ldots, T_n\} \) be the set of all rows of \( [T] \). If \( R(T) = R(\hat{T}) \), then \( T \) preserves \( \prec \) if and only if \( \hat{T} \) preserves \( \prec \).

**Lemma 2.5.** Let \( T \) be a linear operator on \( \mathbb{R}^p \). If \( [T] = P_p(\alpha, \beta) \), \( \alpha \beta \leq 0 \), then \( T \) is a preserver of \( \prec \).
Proof. Without loss of generality, let $\beta \leq 0 \leq \alpha$ and let $X = (x_1, \ldots, x_p)$, $Y = (y_1, \ldots, y_p) \in \mathbb{R}^p$ such that $X \preceq_{\ell} Y$. Then $y_m = \min Y \leq x_i \leq \max Y = y_M$, for all $1 \leq i \leq p$. It is easy to check that $\alpha y_m + \beta y_M \leq \alpha x_i + \beta x_j$, for all $i \neq j \in \{1, \ldots, p\}$, which implies $\min TY \leq \min TX$. Hence by Lemma 2.1, $TX \preceq_{\ell} TY$. □

3. Left matrix majorization on $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be a linear operator and let $a, b, c$ be as in Assumption 1.3. We consider the square $S = [b, 1] \times [b, 1]$ in $\mathbb{R}^2$.

**Definition 3.1.** Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be a linear operator and let $[T] = [T_1/\ldots/T_n]$, where $T_i = (t_{i1}, t_{i2})$, $1 \leq i \leq n$. Define

$$\Delta := \text{Conv} \{ (t_{i1}, t_{i2}), (t_{i2}, t_{i1}), 1 \leq i \leq n \} \subseteq \mathbb{R}^2.$$ 

Also, let $C(T)$ denote the set of all corners of $\Delta$.

**Lemma 3.2.** Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be a linear preserver of $\preceq_{\ell}$ and $[T] = [T_1/\ldots/T_n]$, where $T_j = (t_{j1}, t_{j2})$, $1 \leq j \leq n$. If for some $1 \leq i \leq n$, $t_{i1}t_{i2} > 0$, then $T_j \notin C(T)$, where $C(T)$ is as in Definition 3.1.

Proof. Assume that, if possible, there exists $1 \leq i \leq n$ such that $T_i \in C(T)$ and $t_{i1}t_{i2} > 0$. By Remark 2.4 we can assume that $[T]$ has no identical rows. Without loss of generality, we assume that there exist $1 \leq i \leq n$ and real numbers $m, M \in \mathbb{R}$ such that $t_{i1} > 0, t_{i2} > 0$ and $mt_{i1} + Mt_{i2} < mt_{j1} + Mt_{j2}, j \neq i$. Choose $\varepsilon > 0$ small enough so that $mt_{i1} + (M + \varepsilon)t_{i2} < mt_{j1} + (M + \varepsilon)t_{j2}, j \neq i$. Since $(m, M)^t \prec_{\ell} (m, M + \varepsilon)^t$, $T(m, M)^t \prec_{\ell} T(m, M + \varepsilon)^t$. But $\min (T(m, M + \varepsilon)^t) = mt_{i1} + (M + \varepsilon)t_{i2} > mt_{i1} + Mt_{i2} = \min (T(m, M)^t)$, a contradiction. □

Next we shall characterize all linear operators $T : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ which preserve $\preceq_{\ell}$.

**Theorem 3.3.** Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ be a linear operator. Then $T$ is a linear preserver of $\preceq_{\ell}$ if and only if $P_2(x, y)$ is a submatrix of $[T]$ and $xy \leq 0$ for all $(x, y) \in C(T)$.

Proof. Let $T$ be a linear preserver of $\preceq_{\ell}$ with $0 \leq -b \leq 1 = a$. Let $(x, y) \in C(T)$, then by Lemma 3.2, $xy \leq 0$. Without loss of generality, let $T_i = (t_{i1}, t_{i2}) \in C(T)$ and $t_{i1}t_{i2} \leq 0$. By Remark 2.4, we assume that $[T]$ has no identical rows. Then there exist real numbers $m, M \in \mathbb{R}$ such that $mt_{i1} + Mt_{i2} < mt_{j1} + Mt_{j2}, j \neq i$. Choose $\varepsilon_0 > 0$ small enough so that $(m-\varepsilon)t_{i1} + (M+\varepsilon)t_{i2} < (m-\varepsilon)t_{j1} + (M+\varepsilon)t_{j2}, j \neq i, 0 < \varepsilon \leq \varepsilon_0$. Since $(m+\varepsilon, m-\varepsilon)^t \sim_{\ell} (m-\varepsilon, M+\varepsilon)^t$, $T(m+\varepsilon, m-\varepsilon)^t \sim_{\ell} T(m-\varepsilon, M+\varepsilon)^t$. Hence, for all $0 < \varepsilon \leq \varepsilon_0$, there exist $1 \leq k \leq n$ such that $T_k = (t_{k1}, t_{k2}) \in C(T)$ and $(m-\varepsilon)t_{i1} + (M+\varepsilon)t_{i2} = \min T(m-\varepsilon, M+\varepsilon)^t = \min T(M+\varepsilon, m-\varepsilon)^t = (M+\varepsilon)t_{i1} + (m-\varepsilon)t_{i2} - t_{i1}$. Therefore, $P_2(t_{i1}, t_{i2})$ is a submatrix of $[T]$.

Conversely, let $P_2(x, y)$ be a submatrix of $[T]$ and suppose for all $(x, y) \in C(T)$,
Since \(xy \leq 0\). Define the linear operator \(\hat{T}\) on \(\mathbb{R}^2\) such that \([\hat{T}] = [P_2(x_1, y_1) / \cdots / P_2(x_r, y_r)]\), where \((x_i, y_i) \in C(T), 1 \leq i \leq r\). By elementary convex analysis, we know that \(\max X = \max \hat{T}(X)\) and \(\min X = \min \hat{T}(X)\) for all \(X \in \mathbb{R}^2\). Hence it is enough to show that \(\hat{T}\) is a linear preserver of \(\prec_t\). By Lemma 2.5, each \(P_2(x, y)\) is a linear preserver of \(\prec_t\). Thus, \(\hat{T}\) is a linear preserver of \(\prec_t\). 

4. Left matrix majorization on \(\mathbb{R}^p\). In this section we shall characterize all linear operators \(T : \mathbb{R}^p \to \mathbb{R}^n\) which preserve \(\prec_t\). We shall prove several lemmas and prove the main theorem of this paper.

**Definition 4.1.** Let \(T : \mathbb{R}^p \to \mathbb{R}^n\) be a linear operator and let \([T] = [T_1 / \cdots / T_n]\). Define

\[\Omega := \text{Conv}\{\{T_i = (t_{i1}, \ldots, t_{ip}), 1 \leq i \leq n\}\} \subseteq \mathbb{R}^p.\]

Also, let \(C(T)\) be the set of all corners of \(\Omega\).

**Lemma 4.2.** Let \(T : \mathbb{R}^p \to \mathbb{R}^n\) be a linear preserver of \(\prec_t\) and \([T] = [T_1 / \cdots / T_n]\), where \(T_i = (t_{i1}, t_{i2}, \ldots, t_{ip}), 1 \leq i \leq n\). Suppose there exists \(1 \leq i \leq n\) such that \(t_{ij} > 0, \forall 1 \leq j \leq p\), or \(t_{ij} < 0, \forall 1 \leq j \leq p\). Then \(T_i \notin C(T)\), where \(C(T)\) is as in Definition 4.1.

**Proof.** Assume that, if possible, there exists \(1 \leq i \leq n\) such that \(T_i \in C(T)\) and \(t_{ij} > 0\), for all \(1 \leq j \leq p\), or \(t_{ij} < 0\), for all \(1 \leq j \leq p\). By Remark 2.4, without loss of generality, we can assume that \([T]\) has no identical rows and there exists \(1 \leq i \leq n\) such that \(t_{ij} > 0\), for all \(1 \leq j \leq p\). Since \(T_i \in C(T)\), there exists \(X = (x_1, \ldots, x_p)^t\) such that \(x_1 t_{i1} + x_2 t_{i2} + \cdots + x_p t_{ip} < x_1 t_{j1} + x_2 t_{j2} + \cdots + x_p t_{jp}, j \neq i\). Let \(x_k = \max\{x_i, 1 \leq i \leq p\}\). Choose \(\varepsilon > 0\) small enough so that \(x_1 t_{i1} + \cdots + (x_k + \varepsilon) t_{ik} + \cdots + x_p t_{ip} < x_1 t_{j1} + \cdots + (x_k + \varepsilon) t_{jk} + \cdots + x_p t_{jp}, j \neq i\). Define \(\tilde{X} = (x_1, \ldots, x_k + \varepsilon, \ldots, x_p)^t\). Since \(t_{ik} > 0\), hence \(\min T(X) = x_1 t_{i1} + x_2 t_{i2} + \cdots + x_p t_{ip} < x_1 t_{i1} + \cdots + (x_k + \varepsilon) t_{ik} + \cdots + x_p t_{ip} = \min T(\tilde{X})\). But \(X \prec_t \tilde{X}\), a contradiction.

Let \(T : \mathbb{R}^p \to \mathbb{R}^n\) be a linear operator. Without loss of generality, we assume that \([T] = [T^P / T^n / T]\), where all entries of \(T^p\) (resp., \(T^n\)) are positive (resp., negative) and each row of \(T\) has nonnegative and non positive entries.

**Corollary 4.3.** Let \(T\) and \(\tilde{T}\) be as above. Then \(T\) preserves \(\prec_t\) if and only if \(C(T) = C(\tilde{T})\) and \(\tilde{T}\) preserves \(\prec_t\), where \(C(T)\) is as in Definition 4.1.

**Proof.** Let \(T\) preserve \(\prec_t\). By Lemma 4.2, \(C(T) = C(\tilde{T})\). Thus, if \(X \in \mathbb{R}^p\), then \(\max T(X) = \max \tilde{T}(X)\) and \(\min T(X) = \min \tilde{T}(X)\). Therefore \(\tilde{T}\) preserves \(\prec_t\). Conversely, let \(C(T) = C(\tilde{T})\). Then \(\max T(X) = \max \tilde{T}(X)\) and \(\min T(X) = \min \tilde{T}(X)\). Since \(\tilde{T}\) preserves \(\prec_t\), \(T\) preserves \(\prec_t\).
DEFINITION 4.4. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. Define

$$\Delta = \text{Conv} \{(P_i, N_i) : 1 \leq i \leq n\},$$

where $P_i, N_i$ be as in (1.2). Let $E(T) = \{(P_i, N_i) : (P_i, N_i) \text{ is a corner of } \Delta\}$. Let $1 \leq i \leq n$, define $[i] = \{j : 1 \leq j \leq n, P_i = P_j \text{ and } N_i = N_j\}$.

LEMMA 4.5. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and let $C(T), E(T)$ be as in Definitions 4.1, 4.4, respectively. If $(P_r, N_r) \in E(T)$ for some $1 \leq r \leq n$, then there exists $k \in [r]$ such that $T_k \in C(T)$.

Proof. Suppose there exist $1 \leq r \leq n$ such that $(P_r, N_r) \in E(T)$. Then there exists $m \leq M$ such that

$$P_r m + N_r M < P_j m + N_j M, \quad j \notin [r].$$

Let $X \in \mathbb{R}^p$ such that $\min(X) = m$ and $\max(X) = M$. Then there exists $1 \leq k \leq n$ such that $\min TX = \sum_{l=1}^p t_{kl} x_l$. Hence

$$P_r m + N_r M \leq P_k m + N_k M \leq \sum_{l=1}^p t_{kl} x_l = \min T(X).$$

Define $Y \in \mathbb{R}^p$ by $y_l = m$, if $t_{rl} > 0$ and $y_l = M$, if $t_{rl} \leq 0$. Obviously $Y \prec_\ell X$. Since $T$ preserves $\prec_\ell$, $TY \prec_\ell TX$ which implies that

$$P_k m + N_k M \leq \sum_{l=1}^p t_{kl} x_l = \min TX \leq \min TY \leq P_r m + N_r M.$$

Now, by (4.2) and (4.3), we have $P_r m + N_r M = P_k m + N_k M$. Thus by (4.1), $k \in [r]$ and $\min TX = \sum_{l=1}^p t_{kl} x_l$. Hence $T_k \in C(T)$ for some $k \in [r]$. \qed

Next we state the main result in this paper.

THEOREM 4.6. Let $T$ and $E(T)$ be as in Definition 4.4. Then $T$ preserves $\prec_\ell$ if and only if $P_{\alpha, \beta}(\alpha, \beta)$ is a submatrix of $[T]$ for all $(\alpha, \beta) \in E(T)$.

Proof. Let $T$ be a preserver of $\prec_\ell$ and let $(P_r, N_r) \in E(T)$. Then there exists $m \leq M$ such that $P_r m + N_r M < P_j m + N_j M, \quad j \notin [r]$. Choose $\varepsilon > 0$ small enough so that for all $0 < \varepsilon < \varepsilon_0$,

$$P_r (m - \varepsilon) + N_r (M + \varepsilon) < P_j (m - \varepsilon) + N_j (M + \varepsilon), \quad j \notin [r].$$

If $j \in [r]$, then $P_j = P_r$ and $N_j = N_r$. Thus

$$P_r (m - \varepsilon) + N_r (M + \varepsilon) \leq P_j (m - \varepsilon) + N_j (M + \varepsilon), \quad 1 \leq j \leq n.$$
Let $0 < \varepsilon < \varepsilon_0$, be fixed and let $X^\varepsilon = (x_1^\varepsilon, \ldots, x_p^\varepsilon)^t \in \mathbb{R}^p$ with $\min X^\varepsilon = m - \varepsilon$ and $\max X^\varepsilon = M + \varepsilon$. As in the proof of Lemma 4.5, there exists $k \in [r]$ such that

$$P_r(m - \varepsilon) + N_r(M + \varepsilon) = \min T(X^\varepsilon) = \sum_{l=1}^{p} t_{kl}x_{l}^\varepsilon.$$

Fix $i \neq j \in \{1, \ldots, p\}$ and define $Y^\varepsilon = (y_1^\varepsilon, \ldots, y_p^\varepsilon)^t \in \mathbb{R}^p$ such that $y_i^\varepsilon = m - \varepsilon$, $y_j^\varepsilon = M + \varepsilon$ and $y_l^\varepsilon = \gamma_l$, $m - \varepsilon < \gamma_l < M + \varepsilon$, $l \neq i, j$. Since $X^\varepsilon \sim T Y^\varepsilon$, $TX^\varepsilon \sim T Y^\varepsilon$, there exists $q \in [r]$ such that $t_{qj}(m - \varepsilon) + t_{qi}(M + \varepsilon) + \sum_{l \neq i, j} \gamma_l t_{ql} = P_r(m - \varepsilon) + N_r(M + \varepsilon)$. Since $0 < \varepsilon < \varepsilon_0$ and $m - \varepsilon \leq \gamma_l \leq M + \varepsilon$, $l \neq r, s$ are arbitrary, it is easy to show that there exists $s \in [r]$ such that $t_{si} = P_r$ and $t_{sj} = N_r$ and $t_{sl} = 0, l \neq i, j$. Therefore $[T]$ has $P_r(P_r, N_r)$ as a submatrix.

Conversely, Let $E(T) = \{(P_{i_1}, N_{i_1}), \ldots, (P_{i_k}, N_{i_k})\}$. Then up to a row permutation $[T] = [P_p(P_{i_1}, N_{i_1})/\ldots/P_p(P_{i_k}, N_{i_k})]/Q]$.

Let $\hat{T}$ be the operator on $\mathbb{R}^p$ such that $[\hat{T}] = [P_p(P_{i_1}, N_{i_1})/\ldots/P_p(P_{i_k}, N_{i_k})]$.

Let $T_{l} \in Q$ and suppose there exists $X \in \mathbb{R}^p$ such that

$$\min T(X) = \sum_{l=1}^{p} t_{il}x_l \leq \sum_{l=1}^{p} t_{jl}x_l, 1 \leq j \leq n.$$

Obviously, $P_l m + N_l M \leq \sum_{l=1}^{p} t_{il}x_l \leq \sum_{l=1}^{p} t_{jl}x_l, 1 \leq j \leq n$, where $m = \min X$ and $M = \max X$. We know that $(P_l, N_l) \in \Delta$ and $\Delta$ is convex. Hence there is $1 \leq k \leq n$ such that $(P_k, N_k) \in E(T)$ and $P_k m + N_k M \leq P_l m + N_l M$. As in the proof of Lemma 4.5, $\min TX = P_k m + N_k M$. Then $\min \hat{T} X \leq \min TX$. But we know that $\min T(X) \leq \min \hat{T} X$ and thus $\min \hat{T} X = \min TX$. Similarly, $\max \hat{T} X = \max TX$. Therefore, $T$ is a preserver of $\prec_{\ell}$ if and only if $\hat{T}$ preserves $\prec_{\ell}$. By Lemma 2.5 each $P_p(P_{i_l}, N_{i_l})$ is a preserver of $\prec_{\ell}, 1 \leq l \leq k$. Hence $\hat{T}$ is a preserver of $\prec_{\ell}$ and the theorem is proved. \qed

Next we state necessary conditions for $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ to be a linear preserver of $\prec_{\ell}$. We use the notation of Theorem 2.3 in the following corollary.

**Corollary 4.7.** Let $T : \mathbb{R}^p \rightarrow \mathbb{R}^n$ be a linear operator and let $a$ and $b$ be as given in (1.1). If the following conditions hold, then $T$ is a linear preserver of $\prec_{\ell}$.

- $[T]$ has $[P_p(a, 0)/P_p(0, b)/P_p(a, b)]$ as a submatrix.
- $0 \leq P_i \leq a$ and $b \leq N_i \leq 0$, $1 \leq i \leq n$,

where $P_i$ and $N_i$, $1 \leq i \leq n$ are as in Definition 1.2.

**Proof.** It is clear that $E(T) = \{(a, 0), (0, b), (a, b)\}$. Since $[T]$ has $P_p(a, 0), P_p(0, b)$ and $P_p(a, b)$ as submatrices, it follows by Theorem 4.6 that $T$ is a linear preserver of $\prec_{\ell}$. \qed
Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$, and let $[T] = [T^1|T^2|\ldots|T^p]$, where $T^i$ is the $i^{th}$ column of $[T]$. For $i \neq j \in \{1,\ldots,p\}$ define $T^{ij} : \mathbb{R}^2 \to \mathbb{R}^n$ such that $[T^{ij}] = [T^i|T^j]$.

**Lemma 4.8.** Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$, and let $T^{ij}$ be as above. Then $T^{ij}$ is a linear preserver of $\prec_\ell$ for all $i \neq j \in \{1,\ldots,p\}$.

**Proof.** Let $i \neq j \in \{1,\ldots,p\}$ and let $x = (x_1,x_2)^t$, $y = (y_1,y_2)^t \in \mathbb{R}^2$ such that $x \prec_\ell y$. Define $X,Y \in \mathbb{R}^p$ such that $X_i = x_1$, $X_j = x_2$, $Y_i = y_1$, $Y_j = y_2$ and $X_k = Y_k = 0$, for all $k \neq i,j$. It is obvious that $X \prec_\ell Y$ in $\mathbb{R}^p$ and hence $TX \prec_\ell TY$ in $\mathbb{R}^n$. But $T^{ij}x = x_1 T^i + x_2 T^j = TX \prec_\ell TY = y_1 T^i + y_2 T^j = T^{ij}y$. Therefore, $T^{ij}$ is a linear preserver of $\prec_\ell$. \(\square\)

The following example shows that the converse of Lemma 4.8 is not necessarily true.

**Example 4.9.** Assume $[T] = [P_3(1,-0.5)/ 0.25 0.25 0.25]$. Consider $X = (-1,-1,-1)^t$ and $Y = (-1,-1,-0.75)^t$, we know that $X \prec_\ell Y$ and $\min TX < \min TY$. Thus $T$ is not a linear preserver of $\prec_\ell$. However, by Corollary 4.7, for all $i \neq j \in \{1,2,3\}$, $T^{ij}$ preserves $\prec_\ell$.

5. **Additional results.** In this section we give short proofs of some Theorems from [6, 9].

**Theorem 5.1.** [6] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. Then $T$ preserves $\prec_\ell$ if and only if $T$ has the form $T(X) = (aI + bP)X$ for all $X \in \mathbb{R}^2$, where $P$ is the $2 \times 2$ permutation matrix not equal to $I$, and $ab \leq 0$.

**Proof.** Let $T$ be a preserver of $\prec_\ell$. By Assumption 1.3, $a = 1$. By Theorem 2.3, there exist $0 \leq \alpha \leq 1$ and $b \leq \beta \leq 0$ such that $P(1,\beta)$ and $P(b,\alpha)$ are submatrices of $[T]$. Since $[T]$ is a $2 \times 2$ matrix, $\beta = b$ and $\alpha = 1$. Therefore, $[T] = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ and hence $T(X) = (I + bP)X$, for all $X \in \mathbb{R}^2$. Conversely, up to a row permutation, $[T] = P_2(1,b)$ and by Lemma 2.5, $T$ preserves $\prec_\ell$. \(\square\)

**Theorem 5.2.** [6] Let $p \geq 3$. Then $T : \mathbb{R}^p \to \mathbb{R}^p$ is a linear preserver of left matrix majorization if and only if $T$ is of the form $X \mapsto aPX$ for some $a \in \mathbb{R}$ and some permutation matrix $P$.

**Proof.** By Assumption 1.3, we have $a = 1$. Let $T$ be a preserver of $\prec_\ell$. By Theorem 2.3, $b = 0$ and $[T]$ has $P_p(1,0)$ as a submatrix; hence, up to a row permutation, $[T] = P_p(1,0) = I$. Conversely, by a row permutation, $[T] = P_p(1,0)$; hence by Lemma 2.5, $T$ preserves $\prec_\ell$. \(\square\)
Theorem 5.3. ([9, Theorem 3.1]) For a linear preserver $T$ of $\mathbb{R}^p$ to $\mathbb{R}^n$ the following assertions hold.

(a) If $n < 2p$ and $p \geq 3$, then $T$ is nonnegative.

(b) If $T$ is nonnegative, then there exists an $n \times n$ permutation matrix $Q$ such that $[T] = Q[I/W]$, where $W$ is a (possibly vacuous) $(n - p) \times p$ matrix of one of the following forms (i), (ii) or (iii):

(i) $W$ is row stochastic;

(ii) $W$ is row substochastic and has a zero row;

(iii) $W = [(cI)/B]$, where $0 < c < 1$ and $B$ is an $(n - 2p) \times p$ row substochastic matrix with row sums at least $c$.

(c) Let $Q$ be an $n \times n$ permutation matrix, and let $W$ be an $(n - p) \times p$ matrix of the form (i), (ii), or (iii) in part (b). Then the operator $X \mapsto Q[X/(WX)]$ from $\mathbb{R}^p$ into $\mathbb{R}^n$ is a nonnegative linear preserver of $\prec_n$.

Proof. Assume that, if possible, $b < 0$. By Theorem 2.3 $n \geq p(p - 1)$. Since $p \geq 3$, $n \geq 2p$, a contradiction.

(b) Since $T$ is nonnegative, $N_i = 0, 1 \leq i \leq n$, and $0 \leq P_i \leq 1$. By Theorem 2.3, $[T]$ has $P_p(1,0)$ as its submatrix and therefore up to a row permutation $[T] = [I/W]$. Let $c = \min\{P_i, 1 \leq i \leq n\}$. Then $E(T) = \{(1,0),(c,0)\}$. By Theorem 4.6, $P_p(c,0)$ is a submatrix of $[T]$. If $c = 1$ then (i) holds; if $c = 0$ then (ii) holds and if $0 < c < 1$, then (iii) holds.

(c) Let $[T] = [I/W]$, where $W$ is an $(n - p) \times p$ matrix of the form (i), (ii), or (iii) in part (b). Then $E(T) = \{(1,0),(c,0)\}$. By Theorem 4.6, $T$ is a nonnegative linear preserver of $\prec_n$.

Theorem 5.4. ([9, Theorem 4.5]) Assume $T : \mathbb{R}^p \to \mathbb{R}^n$ is a linear preserver of $\prec_n$, $b < 0$ and $2p \leq n < p(p - 1)$. Let $P_i$ (resp., $N_i$) denote the sum of the positive (resp., negative) entries of the $i^{th}$ row of $[T]$. Then, up to a row permutation, $[T] = [I/bI/B]$ and $\min(N_i + bP_i) = b$, $(i = 1, 2, \ldots, n)$.

Proof. By Theorem 2.3, $P_p(1,\beta)$ and $P_p(\alpha,b)$ are submatrices of $[T]$. Since $n < p(p - 1)$, $\beta = \alpha = 0$ and $E(T) = \{(1,0),(0,b)\}$, where $E(T)$ is as in Definition 4.4. Then up to a row permutation, $[T] = [I/bI/B]$ and $\min\{(bx + y) : (x,y) \in \Delta\} = \min\{(bx + y) : (x,y) \in E(T)\} = b$. Therefore, $\min(N_i + bP_i) = b$, $(i = 1, 2, \ldots, n)$.
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