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DESCRIPTION OF ALL SOLUTIONS OF A LINEAR COMPLEMENTARITY PROBLEM

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Abstract. Description of all solutions of an \( n \times n \) linear complementarity problem \( x^+ = Mx^- + q \) in terms of \( 2^n \) matrices and their Moore-Penrose inverses is given. The result is applied to describe all solutions of the absolute value equation \( Ax + B|x| = b \).

Key words. Linear complementarity problem, Moore-Penrose inverse, Verified solution, Absolute value equation.

AMS subject classifications. 90C33.

1. Introduction. In this paper we consider a linear complementarity problem (LCP) in the form

\[
x^+ = Mx^- + q,
\]

(1.1)

where \( M \in \mathbb{R}^{n \times n} \), \( q \in \mathbb{R}^n \); for \( x = (x_i)_{i=1}^n \) the vectors \( x^+ \) and \( x^- \) are defined by \( x^+ = (\max(x_i, 0))_{i=1}^n \), \( x^- = (\max(-x_i, 0))_{i=1}^n \), so that \( x^+ \geq 0, x^- \geq 0 \), \( (x^+)^T x^- = 0 \),

\[
x = x^+ - x^-
\]

(1.2)

and

\[
|x| = x^+ + x^-,
\]

(1.3)

where \( |x| = (|x_i|)_{i=1}^n \). The linear complementarity problem has been much studied in the last forty years, as evidenced in the monographs by Cottle, Pang and Stone [2], Murty [4] and Schäfer [7]. The traditional approach, as demonstrated e.g., in Lemke’s algorithm [3], looks for some solution of (1.1). On the contrary, we are interested here in the description of all solutions of (1.1). This is done in full generality in Theorem 2.2 of Section 2, where the Moore-Penrose inverses of \( 2^n \) matrices \( F_z \) are employed for this purpose. In Proposition 2.4 we show that the description essentially simplifies
if all the matrices $F_z$ are nonsingular, in which case the LCP (1.1) has at most $2^n$ solutions. Section 3 contains a $10 \times 10$ example which has exactly $2^{10} = 1024$ solutions. In Section 4 we show how the main ideas behind the proof of Theorem 2.2 can be used for the description of all solutions of the absolute value equation $Ax + B|x| = b$ (see Theorem 4.1).

We use the following notation. $I$ is the unit matrix and $e = (1, \ldots, 1)^T$ is the vector of all ones. $Z_n = \{z \mid |z| = e\}$ is the set of all $\pm 1$-vectors in $\mathbb{R}^n$, so that its cardinality is $2^n$. For each $z \in Z_n$ we denote

$$T_z = \text{diag}(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{pmatrix}.$$ 

For a matrix $F$, $F^\dagger$ denotes its Moore-Penrose inverse (see [1], [5]). We shall utilize its property

$$FF^\dagger F = F. \quad (1.4)$$

2. Main result. The core of our approach consists in reformulating the LCP (1.1) as an absolute value equation.

**Proposition 2.1.** A vector $x \in \mathbb{R}^n$ is a solution of the linear complementarity problem (1.1) if and only if it solves the equation

$$\frac{1}{2}(I + M)x + \frac{1}{2}(I - M)|x| = q. \quad (2.1)$$

**Proof.** Let $x$ solve (1.1). From (1.2) and (1.3) we have $x^+ = \frac{1}{2}(|x| + x)$ and $x^- = \frac{1}{2}(|x| - x)$, which, when substituted into (1.1), gives (2.1). Conversely, (2.1) in the light of (1.2) and (1.3) implies (1.1). \qed

Let us denote the solution set of the LCP (1.1) by $X$, i.e.,

$$X = \{ x \mid x^+ = Mx^- + q \}.$$ 

Our main result below gives a description of the solution set in the general case. It builds on the ideas of Penrose’s description of the solution set of a system of linear equations [6].

**Theorem 2.2.** For any $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$ the solution set $X$ of (1.1) is given by

$$X = \{ F_z^\dagger q + G_z y \mid T_z G_z y \geq -T_z F_z^\dagger q, H_z q = 0, y \in \mathbb{R}^n, z \in Z_n \}, \quad (2.2)$$
where

\[
F_z = \frac{1}{2}((I + M) + (I - M)T_z), \quad (2.3)
\]

\[
G_z = I - F_z^\dagger F_z, \quad (2.4)
\]

\[
H_z = I - F_z F_z^\dagger \quad (2.5)
\]

for each \( z \in Z_n \).

Proof. Let \( x \) solve (1.1), then, by Proposition 2.1, it also solves (2.1). Set \( z_i = 1 \) if \( x_i \geq 0 \) and \( z_i = -1 \) otherwise \((i = 1, \ldots, n)\), then \( z \in Z_n \) and \( T_z x = (z_i x_i)_{i=1}^n = (|x_i|)_{i=1}^n = |x| \geq 0 \). Substituting \( |x| = T_z x \) into (2.1), we get that \( x \) satisfies

\[
\frac{1}{2}((I + M) + (I - M)T_z)x = q, \quad (2.6)
\]

i.e.,

\[
F_z x = q, \quad (2.7)
\]

where \( F_z \) is given by (2.3). Then

\[
H_z q = (I - F_z F_z^\dagger)q = (I - F_z F_z^\dagger)F_z x = (F_z - F_z F_z^\dagger F_z) x = 0 \quad (2.8)
\]

because of (1.4). Now set \( y = x - F_z^\dagger q \), then we have

\[
F_z y = F_z x - F_z F_z^\dagger q = q - F_z F_z^\dagger q = H_z q = 0
\]

by (2.8), therefore \( x \) can be written as

\[
x = F_z^\dagger q + y = F_z^\dagger q + (I - F_z^\dagger F_z)y = F_z^\dagger q + G_z y,
\]

and \( T_z x \geq 0 \) implies that \( y \) satisfies

\[
T_z G_z y \geq -T_z F_z^\dagger q.
\]

In this way we have proved that

\[
X \subseteq \{ F_z^\dagger q + G_z y \mid T_z G_z y \geq -T_z F_z^\dagger q, H_z q = 0, y \in \mathbb{R}^n, z \in Z_n \}
\]

holds. To prove the converse inclusion, let \( x \) be of the form \( x = F_z^\dagger q + G_z y \) for some \( y \in \mathbb{R}^n \) and \( z \in Z_n \) satisfying \( T_z G_z y \geq -T_z F_z^\dagger q \) and \( H_z q = 0 \). Then

\[
F_z x = F_z F_z^\dagger q + F_z G_z y = q - H_z q + (F_z - F_z F_z^\dagger F_z)y = q
\]

and

\[
T_z x = T_z F_z^\dagger q + T_z G_z y \geq 0,
\]
hence \( x \) solves (2.6) and since \( T_z x = |x| \), it satisfies (2.1) and thus also (1.1). This concludes the proof of (2.2). \( \square \)

Let us note that the columns of a matrix \( F_z \) can be expressed by

\[
(F_z)_{j} = \begin{cases} 
I_j & \text{if } z_j = 1, \\
M_j & \text{if } z_j = -1 
\end{cases} \quad (j = 1, \ldots, n).
\]

Taking into account the singularity/nonsingularity of \( F_z \), we can bring the description of \( X \) to a more complex, but also a more specific form.

**Proposition 2.3.** For any \( M \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \) the solution set \( X \) of (1.1) is given by

\[
X = \left\{ F_z^\dagger q + G_z y \mid F_z \text{ singular}, T_z G_z y \geq -T_z F_z^\dagger q, H_z q = 0, y \in \mathbb{R}^n, z \in \mathbb{Z}^n \right\} \cup \\
\left\{ F_z^{-1} q \mid F_z \text{ nonsingular}, T_z F_z^{-1} q \geq 0, z \in \mathbb{Z}^n \right\},
\]

where \( F_z, G_z, H_z \) are as in Theorem 2.2.

**Proof.** If \( F_z \) is nonsingular for some \( z \in \mathbb{Z}^n \), then we have \( F_z^\dagger = F_z^{-1} \), \( G_z = 0 \) and \( H_z = 0 \), hence (2.2) becomes (2.9). \( \square \)

In particular, if each matrix \( F_z, z \in \mathbb{Z}^n \), is nonsingular, we have the following simplified description of \( X \).

**Proposition 2.4.** Let \( M \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \). If each \( F_z, z \in \mathbb{Z}^n, \) is nonsingular, then

\[
X = \left\{ F_z^{-1} q \mid T_z F_z^{-1} q \geq 0, z \in \mathbb{Z}^n \right\}.
\]

Hence, if each \( F_z, z \in \mathbb{Z}^n, \) is nonsingular, then the linear complementarity problem (1.1) has a finite number of solutions (at most \( 2^n \)). It is easy to show that this upper bound can really be attained. Consider the LCP

\[
x^+ = -x^- + e,
\]

which, in view of (1.3), is equivalent to

\[
|x| = e.
\]

This shows that the solution set \( X \) of (2.11) consists of all the \( \pm 1 \)-vectors, i.e., \( X = \mathbb{Z}^n \).

A less obvious example is given in the next section.

Finally we give a sufficient condition for (1.1) to have infinitely many solutions.

**Proposition 2.5.** Let \( M \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \). If \( T_z F_z^\dagger q > 0, G_z \neq 0 \) and \( H_z q = 0 \) for some \( z \in \mathbb{Z}^n \), then (1.1) has infinitely many solutions.
Proof. Under the assumptions, the inequality $T_z G_z y \geq -T_z F_z^T q$ has not only the solution $y = 0$, but also a whole neighborhood of it, hence by Theorem 2.2 there are infinitely many solutions to (1.1). \]

3. Example. At the author’s web page [10] there is a freely available verification software package VERSOFT written in MATLAB, currently consisting of more than 50 verification programs. One of them, called VERLCPALL, is dedicated to the present problem; it can be directly assessed at [9]. Its syntax is

\[
[X, all] = \text{verlcpall}(M, q)
\]

where $M, q$ are the data of (1.1) and $X$ is a matrix whose columns are interval vectors each of whom is guaranteed to contain a solution of (1.1). The parameter \texttt{all} satisfies \texttt{all} = 1 if it is verified that all solutions have been found, and \texttt{all} = \texttt{-1} otherwise.

Consider the following example with random (but somewhat structured) data

```matlab
>> n=10; M=-eye(n,n)+0.03*(2*rand(n,n)-1); M=round(100*M)
M =
   -100   -1   -2    2   -1    3    2   -1    2    3
   -2   -99    2    1    2   -1    1    1    0
   -2    2   -1  -101    0   -3   -1    2   -2   -2
   -1    3    2    1  -100   -1   -1    0    2   -2
   -1    1    0   -3   -2  -102    3    1    2   -1
   -1    0    2    0    0  -100   -1    1    2    3
   -1    2   -3    1    2    0    1  -101   -2    2
   -3    1   -1   -2    0    3    1    1  -102   -2
   -3    2    0    0    2   -3   -2   -2    1   -97

>> q=rand(n,1); q=round(100*q)
q =
   32
    21
    74
    41
    48
    32
    10
    87
    86
    96
```

Running the program gives the following result:
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```matlab
>> tic, [X,all]=verlcpall(M,q); sols=size(X,2), all, toc
sols =
   1024
all =
   1
Elapsed time is 9.563235 seconds.
```

Here, `sols` is the number of columns of `X`, i.e., the number of solutions found. We see that `sols = 1024 = 2^{10}`, and “all = 1” indicates that all the solutions have been found, all of them verified. The rather long computation time is due to the verification procedure involved. We have suppressed the output of `X` since it is a $10 \times 1024$ interval matrix. But we can look e.g., at the last solution:

```matlab
>> X(:,1024)
intval ans =
[-0.32628969214138, -0.32628969214137]
[-0.24832082740008, -0.24832082740006]
[-0.71247684314471, -0.71247684314469]
[-0.38263062365052, -0.38263062365051]
[-0.52000025193934, -0.52000025193933]
[-0.31502584347121, -0.31502584347120]
[-0.12420810850764, -0.12420810850763]
[-0.82951280893732, -0.82951280893731]
[-0.84650089698698, -0.84650089698697]
[ 93.55847130433784,  93.55847130433789]
```

This interval vector is guaranteed to contain a solution of (1.1). Observe the high accuracy of the result.

4. **The equation** $Ax + B|x| = b$. The method employed in the proof of Theorem 2.2 can be extended to describe all solutions of the absolute value equation

$$Ax + B|x| = b$$

(\(A, B \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n\)), which is more general but less frequently used than (1.1).

Denote

$$X_a = \{ x \mid Ax + B|x| = b \}.$$

Then we have this description.

**Theorem 4.1.** For any $A,B \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ the solution set $X_a$ of (4.1) is given by

$$X_a = \{ F^b_1 + G_y \mid T_z G_y \geq -T_z F^b_1, H_y b = 0, y \in \mathbb{R}^n, z \in Z_n \},$$
where

\[ F_{z} = A + BT_{z}, \]
\[ G_{z} = I - F_{z}^{†}F_{z}, \]
\[ H_{z} = I - F_{z}F_{z}^{†} \]

for each \( z \in Z_{n} \).

Proof. If \( x \) solves (4.1), then it satisfies \((A + BT_{z})x = b\), where \( z \) is the sign vector of \( x \), hence

\[ F_{z}x = b. \]

The rest of the proof runs exactly as in Theorem 2.2 (with \( q \) replaced by \( b \)) because from (2.7) on, it does not depend on the actual form of \( F_{z} \).

We do not formulate here the analogues of Propositions 2.3, 2.4 and 2.5 as they are obvious. We note in passing that VERSOFT [10] also contains a program VER-ABSVALEQNALL for finding and verifying all solutions of (4.1); it works similarly to VERLCPALL and can be directly assessed at [8].

REFERENCES