2009

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Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1310

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POLYNOMIAL NUMERICAL HULLS OF ORDER 3∗

H.R. AFSHIN†, M.A. MEHRJOOFARD†, AND A. SALEMI‡

Dedicated to Professor Chandler Davis for his outstanding contributions to Mathematics

Abstract. In this note, analytic description of \( V^3(A) \) is given for normal matrices of the form
\[
A = A_1 \oplus iA_2 \text{ or } A = A_1 \oplus e^{\frac{i2\pi}{3}}A_2 \oplus e^{\frac{i4\pi}{3}}A_3,
\]
where \( A_1, A_2, A_3 \) are Hermitian matrices. The new concept “\( k^{th} \) roots of a convex set” is used to study the polynomial numerical hulls of order \( k \) for normal matrices.

Key words. Polynomial numerical hull, Numerical order, \( K^{th} \) roots of a convex set.

AMS subject classifications. 15A60, 15A18, 15A57.

1. Introduction. Let \( A \in M_n(C) \), where \( M_n(C) \) denotes the set of all \( n \times n \) complex matrices. The numerical range of \( A \) is denoted by
\[
W(A) := \{ x^*Ax : \|x\| = 1 \}.
\]
Let \( p(\lambda) \) be any complex polynomial. Define
\[
V_p(A) := \{ \lambda : |p(\lambda)| \leq \|p(A)\| \}.
\]
If \( p \) is not constant, \( V_p(A) \) is a compact convex set which contains \( \sigma(A) \) (for more details see [5]). The polynomial numerical hull of \( A \) of order \( k \), denoted by \( V^k(A) \) is defined by
\[
V^k(A) := \bigcap_{p} V_p(A),
\]
where the intersection is taken over all polynomials \( p \) of degree at most \( k \).

The intersection over all polynomials is called the polynomial numerical hull of \( A \) and is denoted by
\[
V(A) := \bigcap_{k=1}^{\infty} V^k(A).
\]

∗Received by the editors December 28, 2008. Accepted for publication April 17, 2009. Handling Editor: Bit-Shun Tam.
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The following proposition due to O. Nevanlinna states the relationship between polynomial numerical hull of order one and the numerical range of a bounded operator.

**Proposition 1.1.** Let $A$ be a bounded linear operator on a Hilbert space $H$, then $V^1(A) = W(A)$ (see [5, 4]).

In the finite dimensional case $V^1(A) = W(A)$. If $A \in M_n(\mathbb{C})$ and the degree of the minimal polynomial of $A$ is $k$, then $V^i(A) = \sigma(A)$ for all $i \geq k$. The integer $m$ is called the numerical order of $A$ and is denoted by $\text{num}(A)$ provided that $V^m(A) = V(A)$ and $V^{m-1}(A) \neq V(A)$. So the numerical order of $A$ is less than or equal to the degree of the minimal polynomial of $A$. Nevanlinna in [6] proved the following result and Greenbaum later in [4] showed this proposition with a shorter proof.

**Proposition 1.2.** Let $A \in M_n(\mathbb{C})$ be Hermitian. Then $\text{num}(A) \leq 2$ and $V^2(A) = \sigma(A)$.

The joint numerical range of $(A_1, \ldots, A_m) \in M_n \times \cdots \times M_n$ is denoted by

$$W(A_1, \ldots, A_m) = \{(x^*A_1x, x^*A_2x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.$$ 

By the result in [3] (see also [1]),

$$V^k(A) = \{\xi \in \mathbb{C} : (0, \ldots, 0) \in \text{conv}\left(W\left((A - \xi I), (A - \xi I)^2, \ldots, (A - \xi I)^k\right)\right)\}$$

where $\text{conv}(X)$ denotes the convex hull of $X \subseteq \mathbb{C}^k$.

Throughout this paper all direct sums are assumed to be orthogonal and we fix the following notations. Define $i[a, b] = \{it : a \leq t \leq b\}$ and $i(a, b) = \{it : a < t < b\}$, where $a$ and $b$ are real numbers. Also $|AB|$ means the length of the line segment $AB$, and $S^{\frac{\pi}{2}} = \{z \in \mathbb{C} : z^n \in S\}$. Let $k \in \mathbb{N}$. Define

$$R_k^j := \left\{re^{i\theta} : r \geq 0, \frac{j\pi}{k} \leq \theta \leq \frac{(j+1)\pi}{k}, 0 \leq j \leq 2k - 1\right\}.$$
In Section 2, we give an analytic description of $V^3(A)$ for any matrix $A \in M_n$ of the form $A = A_1 \oplus iA_2$, where $A_1^2 = A_2^2 = 0$. Section 3 concerns matrices of the form $A = A_1 \oplus e^{i\frac{\pi}{2}}A_2 \oplus e^{i\frac{3\pi}{2}}A_3$, where $A_1^2 = A_1, A_2^2 = A_2, A_3^2 = A_3$. Additional results and remarks about the polynomial numerical hulls of order $k$ of normal matrices are given by a new concept “$k$th roots of a convex set” in section 4.

2. Matrices of the form $A = A_1 \oplus iA_2$. In this section we shall characterize $V^3(A)$, where

$$(2.1) \quad A = A_1 \oplus iA_2, \quad A_1^\ast = A_1, \quad A_2^\ast = A_2.$$ 

**Lemma 2.1.** Let $H$ be a semi-definite Hermitian matrix and $k \geq 2$ be an integer such that $X^kH^kX = (X^kHX)^k$ for some unit vector $X = (x_1, \ldots, x_n)^t$. Then $X^kHX \in \sigma(H)$.

**Proof.** Without loss of generality, we assume that $H = \text{diag}(h_1, h_2, \ldots, h_n)$, where $h_i \geq 0, i = 1, \ldots, n$. Define $P_i = (h_i, h_i^k) \in \mathbb{R}^2, i = 1, \ldots, n$. Let $\mu = X^kHX$. By assumption $\mu^k = (X^kHX)^k = X^kH^kX$. Hence $\|x_1\|^2 (h_1, h_1^k) + \cdots + \|x_n\|^2 (h_n, h_n^k) = (\mu, \mu^k) \in \mathbb{R}^2$. Since the graph of the function $y = x^k, x \geq 0$ is convex, we have $\mu = h_i$ for some $i = 1, \ldots, n$. Consequently, $\mu \in \sigma(A)$.

**Theorem 2.2.** Let $A$ be of the form (2.1) and $A_1$ be a semi-definite matrix. Then $V^3(A) = \sigma(A)$.

**Proof.** Without loss of generality, we assume that $A_2$ is a positive definite matrix. By [2, Theorem 2.2], we know that

$$V^3(A) \subseteq V^2(A) \subseteq \sigma(A_1) \cup \{i\gamma : 0 \leq \gamma \leq r(A_2)\},$$

where $r(A_2)$ is the spectral radius of $A_2$. Then, $V^3(A) \cap \mathbb{R} \subseteq \sigma(A)$. Now, let $i\eta \in V^3(A) \cap \mathbb{R}$. Thus there exists a unit vector $x = x_1 \oplus x_2$ such that

$$\|x_1\|^2 + \|x_2\|^2 = 1,$$

$$x_1^\ast A_1 x_1 + ix_2 A_2 x_2 = i\mu,$$

$$x_1^\ast A_2^2 x_1 - x_2^\ast A_2^2 x_2 = -\mu^2,$$

$$x_1^\ast A_3 x_1 - ix_2 A_2 A_3 x_2 = -i\mu^3.$$ 

The above relations imply that $(\mu, \mu^3) = (x_2^\ast A_2 x_2, x_2^\ast A_3 x_2)$. Define $H = 0 \oplus A_2$, where 0 is the zero matrix of the same size as $A_1$. Hence $H \geq 0$ and $X^kH^kX = (X^kHX)^3$. By Lemma 2.1, $\mu \in \sigma(H)$. Hence $\mu = 0$ or $\mu \in \sigma(A_2) \subseteq \sigma(A)$. It is enough to show that if $\mu = 0$, then $\mu \in \sigma(A)$. By[2, Lemma 2.3] we know that $0 \in \sigma(A)$ if and only if $0 \in V^2(A)$. Since $0 \in V^3(A) \subseteq V^2(A)$, we obtain $\mu = 0 \in \sigma(A)$.

**Corollary 2.3.** Let $A = \text{diag}(\alpha, -\beta, 0, i\gamma)$, where $\alpha, \beta$ and $\gamma$ are positive numbers. Then $V^3(A) = \sigma(A)$ and therefore $\text{num}(A) = 3$. 

COROLLARY 2.4. Let \( A = \text{diag}(\alpha, -\beta, i\gamma, i\theta) \) such that \( \alpha > 0, \beta > 0 \) and \( 0 \leq \gamma < \theta \). Then \( V^3(A) = \sigma(A) \).

THEOREM 2.5. Let \( A = \text{diag}(\alpha, -\beta, i\gamma, -i\theta) \) and \( \alpha, \beta, \gamma \) and \( \theta \) be positive numbers. Then

(a) \( \alpha = \beta \) and \( \gamma = \theta \) if and only if \( V^3(A) = \sigma(A) \cup \{0\} \).

(b) If \( \alpha = \beta \) and \( \gamma \neq \theta \), then \( V^3(A) = \sigma(A) \cup \left\{ \left\{ \frac{\alpha^2(\theta-\gamma)}{\alpha^2+\theta^2} \right\} i \cap W(A) \right\} \).

(c) If \( \alpha \neq \beta \) and \( \gamma = \theta \), then \( V^3(A) = \sigma(A) \cup \left\{ \left\{ \frac{\gamma^2(\beta-\alpha)}{\gamma^2+\beta\alpha} \right\} \cap W(A) \right\} \).

(d) If \( \alpha \neq \beta \) and \( \gamma \neq \theta \), then \( V^3(A) = \sigma(A) \).

Proof. (a) Let \( \alpha = \beta \) and \( \gamma = \theta \). Define \( X = (x, y, z, t)^t \), where

\[
x = \left( \frac{\gamma^2 + \theta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}}, \quad y = \left( \frac{\gamma^2 + \theta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}}, \\
z = \left( \frac{\alpha^2 + \beta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}}, \quad t = \left( \frac{\alpha^2 + \beta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}}.
\]

It is easy to show that \( X \) is a unit vector and \( X^*AX = X^*A^2X = X^*A^3X = 0 \) and hence \( 0 \in V^3(A) \).

Now, let \( \eta \in V^3(A) \). Then there exists a unit vector \( X = (x, y, z, t)^t \) such that

\[
|x|^2 + |y|^2 + |z|^2 + |t|^2 = 1, \tag{2.2}
\]

\[
X^*AX = \alpha |x|^2 - \beta |y|^2 + i\gamma |z|^2 - i\theta |t|^2 = \eta, \tag{2.3}
\]

\[
X^*A^2X = \alpha^2 |x|^2 + \beta^2 |y|^2 - \gamma^2 |z|^2 - \theta^2 |t|^2 = \eta^2, \tag{2.4}
\]

\[
X^*A^3X = \alpha^3 |x|^2 - \beta^3 |y|^2 - i\gamma^3 |z|^2 + i\theta^3 |t|^2 = \eta^3. \tag{2.5}
\]

Conversely, let \( \eta = 0 \). The relations (2.3) and (2.5) imply that \( \beta = \alpha \) or \( |x|^2 = |y|^2 = 0 \) and \( (\theta = \gamma \) or \( |z|^2 = |t|^2 = 0 \). Since \( \alpha, \beta, \gamma, \theta \) are positive numbers and \( X \neq 0 \), by (2.4), we obtain \( \alpha = \beta \) and \( \gamma = \theta \).

(b) By [3, Theorem 2.6], we know that \( V^2(A) \subseteq [-\alpha, \alpha] \cup i[-\gamma, \gamma] \). Let \( \eta \in V^3(A) \), then \( \eta \in [-\alpha, \alpha] \) or \( \eta \in i[-\gamma, \gamma] \). If \( \eta \in \mathbb{R} \), then the relations (2.3) and (2.5) imply that \( |z|^2 = |t|^2 = 0 \). Therefore, \( |x|^2 + |y|^2 = 1 \) and hence \( \eta = \pm \alpha \). Thus, \( V^3(A) \cap \mathbb{R} = \{-\alpha, \alpha\} \subseteq \sigma(A) \).
Let $i\eta \in V^3(A) \cap i\mathbb{R}$. Then $\eta \in [-\theta, \gamma]$. By (2.3) and (2.5), we obtain

$$|x|^2 = |y|^2 = \frac{-\eta^2 + \gamma^2 |z|^2 + \theta^2 |t|^2}{2\alpha^2}, \quad |z|^2 = \frac{\eta (\eta^2 - \theta^2)}{\gamma (\gamma^2 - \theta^2)}, \quad |t|^2 = \frac{\eta (\eta^2 - \gamma^2)}{\theta (\gamma^2 - \theta^2)}.$$ 

Now, replacing the above equations in (2.2), we can write

$$1 = \frac{[\gamma \theta + \alpha^2] \eta^3 - [\gamma \theta (\gamma - \theta)] \eta^2 - [\gamma^2 \theta^2 + \theta^2 - \alpha^2 \gamma \theta - \alpha^2 \gamma^2] \eta}{\alpha^2 \gamma \theta (\gamma - \theta)}.$$ 

Define $P(\eta) := [\gamma \theta + \alpha^2] \eta^3 - [\gamma \theta (\gamma - \theta)] \eta^2 - [\gamma^2 \theta^2 + \theta^2 - \alpha^2 \gamma \theta - \alpha^2 \gamma^2] \eta - \alpha^2 \gamma \theta (\gamma - \theta)$

Since $\{i\gamma, -i\theta\} \subseteq V^3(A)$, the polynomial $P(\eta)$ is divided by $(\eta - \gamma)(\eta + \theta)$. Hence

$$(2.6) \quad P(\eta) = (\eta - \gamma)(\eta + \theta)(\gamma \theta + \alpha^2) \eta - (\theta - \gamma) \alpha^2].$$

Therefore, $V^3(A) \cap i\mathbb{R} \subseteq \{i\gamma, -i\theta, i\frac{(\theta - \gamma) \alpha^2}{\alpha^2 + \gamma^2} \}$. We are looking to find $\eta \in \mathbb{R}$ such that $P(\eta) = 0$ and

$$(2.7) \quad \frac{-\eta^2 + \gamma^2 |z|^2 + \theta |t|^2}{2\alpha^2} \geq 0, \quad \frac{\eta (\eta^2 - \theta^2)}{\gamma (\gamma^2 - \theta^2)} \geq 0, \quad \frac{\eta (\eta^2 - \gamma^2)}{\theta (\gamma^2 - \theta^2)} \geq 0.$$ 

Let $\eta = i\frac{(\theta - \gamma) \alpha^2}{\alpha^2 + \gamma^2} \in [-\theta, \gamma]$. It is readily seen that the relations in (2.7) hold and by (2.6), $P(\eta) = 0$. Therefore, $V^3(A) \cap i\mathbb{R} = \{i\gamma, -i\theta\} \cup \left\{i \frac{\alpha^2 (\theta - \gamma)}{\alpha^2 + \gamma^2} \cap i[-\theta, \gamma] \right\}$. 

(c) It is enough to consider $iA$ instead of $A$.

(d) Let $\eta \in V^3(A) \cap \mathbb{R}$. Then, there exists a unit vector $X$ such that $X^*AX = \eta, X^*A^2X = \eta^2$ and $X^*A^3X = \eta^3$. These relations imply that $|x|^2 = \frac{-\eta^2 + \gamma^2 |z|^2 + \theta^2 |t|^2}{2\alpha^2}, |y|^2 = \frac{\alpha^2 \eta}{\alpha^2 + \gamma^2}$, and $|z|^2 = |t|^2 = 0$. Also, we have $\eta^2 + (\beta - \alpha) \eta - \alpha \beta = 0$. Therefore, $\eta = \beta$ or $\eta = \alpha$ which are in $\sigma(A)$. Similarly, if $\eta \in V^3(A) \cap i\mathbb{R}$ is pure imaginary, then $\eta = -i\theta$ or $i\gamma$ which are in $\sigma(A)$. \(\Box\)
Remark 2.6. In the above Figure, we find a geometric interpretations for the 5th point in $V^3(A)$, where $A$ is a $4 \times 4$ normal matrix as in Theorem 2.5(b), see [1, Theorem 5.1]. The points $M$ and $K$ are the orthocenters of the triangles $ABC$ and $ABD$, respectively. Let $L$ be the intersection of the line $CD$ and the line passing through $A$ and perpendicular to $HJ$. It is readily seen that the slope of the lines $HJ$ and $AP$ are $\cot(\psi - \varphi)$ and $-\tan(\psi - \varphi)$, respectively. Also, $-\tan(\psi - \varphi) = \frac{\tan(\varphi) - \tan(\psi)}{1 + \tan(\varphi)\tan(\psi)} = \frac{\theta/\alpha - \gamma/\alpha}{1 + (\gamma/\alpha)(\theta/\alpha)}$. Hence $L = \left(0, \frac{\alpha^2(\theta - \gamma)}{\alpha^2 + \theta \gamma}\right)$.

For a $3 \times 3$ normal matrix $A$, the 4th point in $V^2(A)$ (if any) is the orthocenter of the triangle generated by $\sigma(A)$. It is interesting that if $\gamma \to \infty$, then $i\frac{\alpha^2(\theta - \gamma)}{\alpha^2 + \gamma \theta} \to i\frac{\alpha^2}{\theta}$, where $i\frac{\alpha^2}{\theta}$ is the orthocenter of the triangle generated by $\{\alpha, -\alpha, -i\theta\}$ [2, Theorem 2.4].

3. Matrices of the form $A = A_1 \oplus e^{i\frac{2\pi}{3}}A_2 \oplus e^{i\frac{4\pi}{3}}A_3$. In this section, we study the polynomial numerical hull of order 3 of matrices of the form

\begin{equation}
A = A_1 \oplus e^{i\frac{2\pi}{3}}A_2 \oplus e^{i\frac{4\pi}{3}}A_3, \quad A_1^* = A_1, \; A_2^* = A_2 \; \text{and} \; A_3^* = A_3.
\end{equation}

Theorem 3.1. Let $A$ be a normal matrix such that $\sigma(A) \subseteq R_1^3 \cup R_3^3 \cup R_5^3$. Then $V^3(A) \subseteq R_1^3 \cup R_3^3 \cup R_5^3$.

Proof. We know that $z \in R_1^3 \cup R_3^3 \cup R_5^3$ if and only if $z^3 \in R_1^1$ (lower half plane), whereas $\sigma(A^3) = \{z^3 : z \in \sigma(A)\}$ and $\sigma(A) \subseteq R_1^3 \cup R_3^3 \cup R_5^3$. Then $\sigma(A^3) \subseteq R_1^1$ and hence $W(A^3) = \text{conv}(\sigma(A^3)) \subseteq R_1^1$. Thus, $V^3(A) \subseteq R_1^3 \cup R_3^3 \cup R_5^3$. \(\square\)
COROLLARY 3.2. Let A be a normal matrix such that \( \sigma(A) \subseteq S = \mathbb{R} \cup e^{i\frac{2\pi}{3}} \mathbb{R} \cup e^{i\frac{4\pi}{3}} \mathbb{R} \). Then \( V^3(A) \subseteq S \).

Proof. Since \( \sigma(A) \subseteq S \) and \( S = (R_0 \cup R_2 \cup R_4) \cap (R_3 \cup R_5 \cup R_7) \), by Theorem 3.1, we obtain \( V^3(A) \subseteq S \).

REMARK 3.3. Let A be as in (3.1). Then \( V^3(A) \subseteq \mathbb{R} \cup e^{i\frac{2\pi}{3}} \mathbb{R} \cup e^{i\frac{4\pi}{3}} \mathbb{R} \). Since \( V^3(e^{i\frac{2\pi}{3}} A) \cap \mathbb{R} = V^3(A) \cap e^{i\frac{2\pi}{3}} \mathbb{R} \), it is enough to find \( V^3(A) \cap \mathbb{R} \).

LEMMA 3.4. Let A be as in (3.1). Then

\[
V^3(A) \cap \mathbb{R} = \left\{ \eta = x_1^*A_1x_1 - x_2^*A_2x_2 : \begin{cases}
    x_1^*x_1 + x_2^*x_2 + x_3^*x_3 = 1, \\
    x_1^*A_1x_1 - x_2^*A_2x_2 = x_1^*A_3x_1,
\end{cases} \right. \]

Proof. Suppose that \( x = x_1 + x_2 + x_3 \) and \( \eta = x^*Ax \in V^3(A) \cap \mathbb{R} \). So

\[
\begin{align*}
    x_1^*x_1 + x_2^*x_2 + x_3^*x_3 &= x^*x = 1, \\
    \eta &= x^*Ax = x_1^*A_1x_1 + e^{i\frac{2\pi}{3}}x_2^*A_2x_2 + e^{i\frac{4\pi}{3}}x_3^*A_3x_3, \\
    \eta^2 &= x^*A^2x = x_1^*A_1^2x_1 + e^{i\frac{4\pi}{3}}x_2^*A_2^2x_2 + e^{i\frac{2\pi}{3}}x_3^*A_3^2x_3, \\
    \eta^3 &= x^*A^3x = x_1^*A_1^3x_1 + x_2^*A_2^3x_2 + x_3^*A_3^3x_3.
\end{align*}
\]

Since \( \eta \in \mathbb{R} \),

\[
\begin{align*}
    \begin{cases}
        \eta &= x_1^*A_1x_1 + \cos \frac{2\pi}{3}x_2^*A_2x_2 + \cos \frac{4\pi}{3}x_3^*A_3x_3, \\
        \sin \frac{2\pi}{3}x_2^*A_2x_2 + \sin \frac{4\pi}{3}x_3^*A_3x_3 &= 0
    \end{cases} \quad \Rightarrow \quad \begin{cases}
        \eta &= x_1^*A_1x_1 - x_2^*A_2x_2, \\
        x_2^*A_2x_2 &= x_3^*A_3x_3
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    \begin{cases}
        \eta^2 &= x_1^*A_1^2x_1 + \cos \frac{4\pi}{3}x_2^*A_2^2x_2 + \cos \frac{2\pi}{3}x_3^*A_3^2x_3, \\
        \sin \frac{4\pi}{3}x_2^*A_2^2x_2 + \sin \frac{2\pi}{3}x_3^*A_3^2x_3 &= 0
    \end{cases} \quad \Rightarrow \quad \begin{cases}
        \eta^2 &= x_1^*A_1^2x_1 - x_2^*A_2^2x_2, \\
        x_2^*A_2^2x_2 &= x_3^*A_3^2x_3
    \end{cases}
\end{align*}
\]

and

\[
\eta^3 = x^*A^3x = x_1^*A_1^3x_1 + x_2^*A_2^3x_2 + x_3^*A_3^3x_3. \quad \square
\]

THEOREM 3.5. Let \( A = A_1 \oplus e^{i\frac{2\pi}{3}} A_2 \) and \( A_1^* = A_1, A_2^* = A_2 \). Then \( V^3(A) = \sigma(A) \).

Proof. By using [2, Lemma 2.3], \( V^2(A) \subseteq R_2 \cup R_4 \) and by Corollary 3.2, \( V^3(A) \subseteq \mathbb{R} \cup e^{i\frac{2\pi}{3}} \mathbb{R} \cup e^{i\frac{4\pi}{3}} \mathbb{R} \). Hence \( V^3(A) \subseteq V^2(A) \cap \left( \mathbb{R} \cup e^{i\frac{2\pi}{3}} \mathbb{R} \right) \). Now, we will show that

\[
V^2(A) \cap \left( \mathbb{R} \cup e^{i\frac{2\pi}{3}} \mathbb{R} \right) \subseteq \sigma(A).
\]
First, we show that $V^2(A) \cap \mathbb{R} \subseteq \sigma(A)$. Suppose that $x = x_1 \oplus x_2$ and $\eta = x^* Ax \in V^2(A) \cap \mathbb{R}$. By the same method as in the proof of Lemma 3.4, we have

$$V^2(A) \cap \mathbb{R} = \left\{ \eta = x_1^* A_1 x_1 : \begin{array}{l} x_1^* x_1 + x_2^* x_2 = 1, \\ \eta^2 = x_1^* A_1^2 x_1 \end{array} \right\}.$$ 

Then, $(x_1^* A_1 x_1)^2 = x_1^* A_1^2 x_1 = \|A_1 x_1\|^2$.

By the Cauchy-Schwarz Inequality, we have $(x_1^* A_1 x_1)^2 \leq \|x_1\|^2 \|A_1 x_1\|^2$. Hence $A_1 x_1 = 0$ or $\|x_1\| = 1$. In both cases $\eta = x_1^* A_1 x_1 \in \sigma(A) \subseteq \sigma(A)$. Since $V^2(e^{i\alpha} A) = e^{i\alpha} V^2(A)$, similarly, $V^2(A) \cap e^{i\frac{2\pi}{3}} \mathbb{R} \subseteq \sigma(e^{i\frac{2\pi}{3}} A_2) \subseteq \sigma(A)$. Therefore, $V^3(A) = \sigma(A)$.

In the following Theorem, we show that if $A_1$, $A_2$ and $A_3$ are positive semi-definite matrices as in (3.1), then $V^3(A) = \sigma(A)$.

**Theorem 3.6.** Let $A$ be as in (3.1). If $A_1$, $A_2$, $A_3$ are positive semi-definite matrices, then $V^3(A) = \sigma(A)$.

**Proof.** By Lemma 3.4,

$$V^3(A) \cap \mathbb{R} \subset \left\{ \eta : \begin{array}{l} x_1^* x_1 + x_2^* x_2 + x_3^* x_3 = 1, \\ \eta = x_1^* A_1 x_1 - x_2^* A_2 x_2, \\ \eta^3 = x_1^* A_1^3 x_1 + x_2^* A_2^3 x_2 + x_3^* A_3^3 x_3 \end{array} \right\}$$

$$= \left\{ \eta : (\eta, \eta^3) \in \text{conv} \left\{ \left( (a, a^3) \right)_{a \in \sigma(A_1)} \cup \left( (-b, b^3) \right)_{b \in \sigma(A_2)} \cup \left( (0, c^3) \right)_{c \in \sigma(A_3)} \right\} \right\}.$$ 

Assume $A_1 = \text{diag}(a_1, \ldots, a_\ell), A_2 = \text{diag}(b_1, \ldots, b_m)$, and $A_3 = \text{diag}(c_1, \ldots, c_n)$, where $0 \leq a_1 \leq \cdots \leq a_\ell$, $0 \leq b_1 \leq \cdots \leq b_m$, and $0 \leq c_1 \leq \cdots \leq c_n$. Let $p_1 = (a_1, a_1^3), q_j = (-b_j, b_j^3), r_k = (0, c_k^3)$. By the following Figure, $V^3(A) \cap \mathbb{R} = \sigma(A_1)$. Similarly, $V^3(A) \cap e^{i\frac{2\pi}{3}} \mathbb{R} \subseteq \sigma(A_2)$ and $V^3(A) \cap e^{i\frac{4\pi}{3}} \mathbb{R} \subseteq \sigma(A_3)$. Hence, $V^3(A) = \sigma(A)$ and the proof is complete. □
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PROPOSITION 3.7. Let $A$ be as in (3.1). Assume $A_1, A_2$ are positive semidefinite matrices and $A_3$ is a negative semi definite matrix. Then $V^3(A) \subseteq \sigma(A) \cup e^{i\frac{\pi}{2}}(0, \infty)$.

Proof. Without loss of generality, we assume that $A_3$ is a negative definite matrix. By [2, Theorem 1.4.], $V^2(A) \cap (\mathbb{R} \cup e^{i\frac{\pi}{2}} \mathbb{R}) \subseteq \sigma(A)$. Hence $V^3(A) \cap (\mathbb{R} \cup e^{i\frac{\pi}{2}} \mathbb{R}) \subseteq \sigma(A)$. By Corollary 3.2, $V^3(A) \subseteq (\mathbb{R} \cup e^{i\frac{\pi}{2}} \mathbb{R}) \cup e^{i\frac{\pi}{2}} \mathbb{R}$. Also, $V^3(A) \subseteq W(A)$, therefore, $V^3(A) \subseteq \sigma(A) \cup e^{i\frac{\pi}{2}}(0, \infty)$. □

In the following example, we show that Theorem 3.6 may not be true if $A_1, A_2$ are positive semi definite matrices and $A_3$ is a negative definite matrix.

EXAMPLE 3.8. Let $A = \text{diag}(0, 2\sqrt{3}, a, 0)$. After a rotation and a translation, by using Theorem 2.5 (a), it is readily seen that $V^3(A) = \sigma(A) \cup \{\sqrt{3}e^{i\frac{\pi}{2}}\}$.

4. $K$th roots of a convex set. In this section we introduce the concept of $k$th roots of a convex set and we show that the concepts “inner cross” and “outer cross” in [2, Section 3] are special cases of this concept.

DEFINITION 4.1. Let $S$ be a convex set and $R := S^{\frac{1}{k}} = \{z \in \mathbb{C} : z^k \in S\}$. Then $R$ is called $k$th root of the convex set $S$.

In the following Lemma, we list some properties of the $k$th roots of a convex set.

LEMMA 4.2. Let $P$ and $Q$ be two convex sets. Then

a) $(P \cap Q)^{\frac{1}{k}} = P^{\frac{1}{k}} \cap Q^{\frac{1}{k}}$.

b) $(P^c)^{\frac{1}{k}} = \left( P^{\frac{1}{k}} \right)^c$.

c) $(e^{ik\theta} P)^{\frac{1}{k}} = e^{i\theta} P^{\frac{1}{k}}$.

The following is a key Theorem in this section:

THEOREM 4.3. Let $A$ be a normal matrix and $S$ be an arbitrary convex set. If $\sigma(A) \subseteq S^{\frac{1}{k}}$, then $V^k(A) \subseteq S^{\frac{1}{k}}$.

Proof. If $\sigma(A) \subseteq S^{\frac{1}{k}}$, then $\sigma(A^k) \subseteq S$. Since $W(A^k) = \text{conv}(\sigma(A^k)) \subseteq S$. Thus, $\{z^k : z \in V^k(A)\} \subseteq S$, and hence $V^k(A) \subseteq S^{\frac{1}{k}}$. □

LEMMA 4.4. The 2-roots of a line is a rectangular hyperbola with center at the origin.

Proof. Suppose that $(a, b) \neq (0, 0)$ and let $S = \{(x, y) : ax + by + c = 0\}$. Therefore

$$R = S^{\frac{1}{2}} = \{(x, y) : a(x^2 - y^2) + b(2xy) + c = 0\}.$$
It is clear that $R$ is an arbitrary rectangular hyperbola with center at the origin. □

**Corollary 4.5.** [2, Theorem 3.1] Let $A \in M_n$ be a normal matrix and $\sigma(A) \subseteq R$, where $R$ is a rectangular hyperbola. Then $V^2(A) \subseteq R$.

**Proof.** Since $V^2(\alpha A + \beta I) = \alpha V^2(A) + \beta$, we assume that the center of $R$ is origin. Now, by Theorem 4.3 and Lemma 4.4 the result holds. □

**Corollary 4.6.** [2, Lemma 3.3] Let $A \in M_n(\mathbb{C})$ be a normal matrix and $\Delta$ be an inner or outer cross. If $\sigma(A) \subseteq \Delta$, then $V^2(A) \subseteq \Delta$.

**Proof.** Without loss of generality we assume that $\Delta = \{x + iy : x^2 - y^2 \leq 1\}$. Then, $\Delta = \{z \in \mathbb{C} : \Re(z^2) \leq 1\}$. Define $S := \{z \in \mathbb{C} : \Re(z) \leq 1\}$. Thus, $\Delta = S^{1/2}$. This means that $\Delta$ is the $2^{nd}$ root of the convex set $S$. By Theorem 4.3, the result holds. □

Let

\[
R_k^e = \bigcup_{t=0}^{k-1} R_k^{2t} \quad \text{and} \quad R_k^o = \bigcup_{t=0}^{k-1} R_k^{2t+1},
\]

where $R_k^e$ be as in (1.1). It is clear that $C = R_k^e \cup R_k^o$ and $R_k^e = e^{\pi R_k}$. The following is a generalization of Theorem 3.1.

**Theorem 4.7.** Let $A$ be a normal matrix and let $z_0 \in \mathbb{C}$ and $\eta \in \mathbb{R}$. If $\sigma(A) \subseteq z_0 + e^{i \eta} R_k^e$, then $V^k(A) \subseteq z_0 + e^{i \eta} R_k^e$.

**Proof.** Let $A := e^{-i \eta} (A - z_0 I)$, then $\sigma(A) \subseteq R_k^e$. Define $S = R_k^e$ (upper half plane), it is easy to show that $S^{1/k} = R_k^e$. Since $\sigma(A) \subseteq S^{1/k} = R_k^e$, by Theorem 4.3 $V^k(A) \subseteq S^{1/k} = R_k^e$. Also, $V^k(A) = e^{-i \eta} (V^k(A) - z_0)$, hence

$V^k(A) \subseteq z_0 + e^{i \eta} R_k^e$. □

**Corollary 4.8.** Let $A$ be a normal matrix of the form

$A = A_1 \oplus e^{\frac{2\pi i}{k}} A_2 \oplus \cdots \oplus e^{\frac{2(k-1)\pi}{k}} A_k$, \quad $A_i^* = A_i$, \quad $i = 1, \ldots, k$.

Then, $V^k(A) \subseteq \mathbb{R} \cup e^{\frac{2\pi i}{k}} \mathbb{R} \cup \cdots \cup e^{\frac{2(k-1)\pi}{k}} \mathbb{R}$.

**Proof.** It is clear that $\sigma(A) \subseteq \mathbb{R} \cup e^{\frac{2\pi i}{k}} \mathbb{R} \cup \cdots \cup e^{\frac{2(k-1)\pi}{k}} \mathbb{R} = R_k^e \cap R_k^o$, where $R_k^e$ and $R_k^o$ be as in (1.1). By Theorem 4.7,

$V^k(A) \subseteq R_k^e \cap R_k^o = \mathbb{R} \cup e^{\frac{2\pi i}{k}} \mathbb{R} \cup \cdots \cup e^{\frac{2(k-1)\pi}{k}} \mathbb{R}$. □
Acknowledgement. This research has been supported by Mahani Mathematical Research Center, Kerman, Iran. Research of the first author was supported by Vali-E-Asr University of Rafsanjan, Rafsanjan, Iran.

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