On C-commuting graph of matrix algebra

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ON $\mathcal{C}$–COMMUTING GRAPH OF MATRIX ALGEBRA*

P. RAJA† AND S. M. VAEZPOUR*

Abstract. Let $D$ be a division ring, $n \geq 2$ a natural number, and $\mathcal{C} \subseteq M_n(D)$. Two matrices $A$ and $B$ are called $\mathcal{C}$–commuting if there is $C \in \mathcal{C}$ that $AB - BA = C$. In this paper the $\mathcal{C}$–commuting graph of $M_n(D)$ is defined and denoted by $\Gamma_{\mathcal{C}}(M_n(D))$. Conditions are given that guarantee that the $\mathcal{C}$–commuting graph is connected.

Key words. Division ring, Matrix Algebra, Commuting.

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1. Introduction. Given a graph $G$, a path $P$ is a sequence $v_0e_1v_1e_2\ldots e_kv_k$ whose terms are alternately distinct vertices and distinct edges in $G$, such that for any $i$, $1 \leq i \leq k$, the ends of $e_i$ are $v_{i-1}$ and $v_i$. We say $u$ is connected to $v$ in $G$ if there exists a path between $u$ and $v$. The graph $G$ is connected if there exists a path between any two distinct vertices of $G$. For more details see [2].

Let $D$ be a division ring and $M_n(D)$ be the set of all $n \times n$ matrices over $D$. As is defined in [1], for $S \subseteq M_n(D)$ the commuting graph of $S$, denoted by $\Gamma(S)$, is the graph with vertex set $S \setminus Z(S)$ such that distinct vertices $A$ and $B$ are adjacent if and only if $AB = BA$, where $Z(S) = \{ A \mid A \in S, AB = BA \text{ for every } B \in S \}$.

Let $A \in M_n(D)$. If $A^2 = I$, $A$ is called an involution, and $A$ is reducible if it has a non-trivial invariant subspace in $D^n$. It is easily seen that if $A$ is reducible, then there are an invertible matrix $P$ and integers $k$ and $m$ so that $(P^{-1}AP)_{ij} = 0$, for all $i$ and $j$ with $k + 1 \leq i \leq n$ and $1 \leq j \leq m$.

Some properties of commuting graph of $M_n(D)$ were considered in [1]. In particular, we proved the following theorems that are useful in this paper.

THEOREM 1.1. [1, Theorem 1] Let $D$ be a division ring and $n > 2$ a natural number. If $A$ is the set of all non-invertible matrices in $M_n(D)$, then $\Gamma(A)$ is a connected graph.

THEOREM 1.2. [1, Theorem 2] Let $D$ be a division ring with center $F$ and $n > 1$
a natural number. If \( A \in M_n(D) \) is a non-cyclic matrix, then \( A \) is connected to \( E_{11} \) in \( \Gamma(M_n(D)) \).

In the following we extend the definition of commuting graph and define the \( C \)-commuting graph of \( M_n(D) \). We prove the connectivity of \( C \)-commuting graphs for some special cases of \( C \).

**Notation.** For a division ring \( D \) and \( a \in D \), we use \( C_D(a) \) for the centralizer of \( a \) in \( D \). Also the ring of all \( m \times n \) matrices over \( D \) is denoted by \( M_{m \times n}(D) \), and for simplicity we put \( D^n = M_{1 \times n}(D) \). The zero matrix, the identity matrix, the zero matrix of size \( r \), and the identity matrix of size \( r \), are denoted by \( 0 \), \( I \), \( 0_r \) and \( I_r \), respectively, and we use \( X^t \) for the transpose of \( X \), for every \( X \in D^n \).

**2. Main Results.** Throughout this section \( E_{ij} \) denotes the matrix in \( M_n(D) \) whose \((i,j)\)-entry is 1 and other entries are zero, and \( e_i \) denotes the element in \( D^n \) whose \( i \)th entry is 1 and other entries are zero, for \( i \) and \( j \) with \( 1 \leq i, j \leq n \). Also we recall that if \( A \in M_n(D) \) is a cyclic matrix, then the representation of \( A \) in a special basis has the following form

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n
\end{pmatrix}
\]

for some \( a_1, \ldots, a_n \in D \).

**Definition 2.1.** For a division ring \( D \), \( n \in \mathbb{N} \), and \( C \subseteq M_n(D) \), a pair of matrices \( A \) and \( B \) in \( M_n(D) \) is called \( C \)-Commuting if \( AB - BA = C \), for some \( C \in C \).

Thus, if \( A \) and \( B \) commute, then they are \( \{0\} \)-commuting.

**Definition 2.2.** For a division ring \( D \) with center \( F \), \( n \in \mathbb{N} \), and \( C \subseteq M_n(D) \), the \( C \)-Commuting graph of \( M_n(D) \), denoted by \( \Gamma_C(M_n(D)) \), is a graph with vertex set \( M_n(D) \setminus FI \) such that distinct vertices \( A \) and \( B \) are adjacent if and only if they are \( C \)-Commuting, where \( FI = \{ \alpha I \mid \alpha \in F \} \).

Note that the \( \{0\} \)-Commuting graph of \( M_n(D) \) is the commuting graph of \( M_n(D) \) that was defined in [1].

Now, we are going to establish basic properties of this graph.

**Theorem 2.3.** Let \( D \) be a division ring with center \( F \) and \( n \geq 3 \) a natural number. Then the following hold:
(i) If $D$ is non-commutative and $C_1$ is the set of all matrices in $M_n(D)$ such that their ranks are at most 1, then $\Gamma_{C_1}(M_n(D))$ is a connected graph.

(ii) If $D$ is commutative and $C_2$ is the set of all matrices in $M_n(D)$ such that their ranks are at most 2, then $\Gamma_{C_2}(M_n(D))$ is a connected graph.

Proof. Since the zero matrix is in $C_1$ and $C_2$, then by Theorem 1.1, each pair of non-invertible matrices are joined by a path in $\Gamma_{C_1}(M_n(D))$ and $\Gamma_{C_2}(M_n(D))$. So to prove the theorem it suffices to show that for every non-scalar invertible matrix $A \in M_n(D)$, $A$ is joined to a non-zero, non-invertible matrix in $\Gamma_{C_1}(M_n(D))$ and $\Gamma_{C_2}(M_n(D))$. By Theorem 1.2, we may assume that $A$ is a cyclic matrix. So there is an invertible matrix $P$ such that

$$B = P^{-1}AP = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n
\end{pmatrix},$$

where $a_i \in D$, for $i, 1 \leq i \leq n$. To prove (i), let $D$ be a non-commutative division ring and $\alpha \in C_D(a_n) \setminus F$. Then we have the path $B - \alpha I = E_{11}$ in $\Gamma_{C_1}(M_n(D))$, and so is $A - P(\alpha I)P^{-1} = PE_{11}P^{-1}$. To prove (ii), assume $D$ is commutative and put

$$C = \begin{pmatrix}1 & a_1^{-1}a_2 \\ 0 & 0\end{pmatrix} \oplus I_{n-4} \oplus I_2.$$

Then $C$ is a non-zero, non-invertible matrix and it is easily seen that

$$BC - CB = \begin{pmatrix}0 & -1 & -a_1^{-1}a_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix} \oplus 0_{n-3} \in C_2.$$

So rank $(BC - CB) = 2$ and also rank $(A(PCP^{-1}) - (PCP^{-1})A) = 2$, and the proof is complete. $\Box$

Remark 2.4. Note that in the proof of Theorem 2.3, we have

$$E = B(\alpha I) - (\alpha I)B = \begin{pmatrix}0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 0 \\
b_1 & b_2 & \cdots & b_{n-1} & 0\end{pmatrix},$$

where $b_i \in D$, for $i, 1 \leq i \leq n$. To prove (i), let $D$ be a non-commutative division ring and $\alpha \in C_D(a_n) \setminus F$. Then we have the path $B - \alpha I = E_{11}$ in $\Gamma_{C_1}(M_n(D))$, and so is $A - P(\alpha I)P^{-1} = PE_{11}P^{-1}$. To prove (ii), assume $D$ is commutative and put

$$C = \begin{pmatrix}1 & a_1^{-1}a_2 \\ 0 & 0\end{pmatrix} \oplus I_{n-4} \oplus I_2.$$

Then $C$ is a non-zero, non-invertible matrix and it is easily seen that

$$BC - CB = \begin{pmatrix}0 & -1 & -a_1^{-1}a_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix} \oplus 0_{n-3} \in C_2.$$
where \( b_i \in D \), for \( i, 1 \leq i \leq n - 1 \), and also

\[
G = BC - CB = \begin{pmatrix}
0 & -1 & -a_1^{-1}a_2 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \oplus 0_{n-3}.
\]

So \( E, G \) and consequently \( PEP^{-1}, PGP^{-1} \) are non-invertible, triangularizable, reducible, and also nilpotent matrices. Therefore we have the following corollaries.

**Corollary 2.5.** Let \( D \) be a division ring and \( n \geq 3 \) a natural number. If \( A_n \) is the set of all non-invertible matrices in \( M_n(D) \), then \( \Gamma_{A_n}(M_n(D)) \) is a connected graph.

**Corollary 2.6.** Let \( D \) be a division ring and \( n \geq 3 \) a natural number. If \( T_n \) is the set of all triangularizable matrices in \( M_n(D) \), then \( \Gamma_{T_n}(M_n(D)) \) is a connected graph.

**Corollary 2.7.** Let \( D \) be a division ring and \( n \geq 3 \) a natural number. If \( R_n \) is the set of all reducible matrices in \( M_n(D) \), then \( \Gamma_{R_n}(M_n(D)) \) is a connected graph.

**Corollary 2.8.** Let \( D \) be a division ring and \( n \geq 3 \) a natural number. If \( N_n \) is the set of all nilpotent matrices in \( M_n(D) \), then \( \Gamma_{N_n}(M_n(D)) \) is a connected graph.

**Theorem 2.9.** Let \( F \) be a field with \( \text{char} F = 0 \) and \( n \geq 3 \) a natural number. If \( D_n \) is the set of all diagonalizable matrices in \( M_n(F) \), then \( \Gamma_{D_n}(M_n(F)) \) is a connected graph.

**Proof.** Since the zero matrix is diagonalizable, by Theorem 1.1, we have each pair of non-invertible matrices is joined by a path in \( \Gamma_{D_n}(M_n(F)) \). So to prove the theorem it suffices to show that for every non-scalar invertible matrix \( A \in M_n(F) \), \( A \) is joined to a non-zero, non-invertible matrix in \( \Gamma_{D_n}(M_n(F)) \) if \( D_n \) is the set of all nilpotent matrices in \( M_n(D) \). By Theorem 1.2, we may assume that \( A \) is a cyclic matrix. Hence there is an invertible matrix \( P \) such that

\[
P^{-1}AP = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
0 & \cdots & \cdots & a_{n-1} & a_n
\end{pmatrix},
\]

where \( a_i \in F \), for \( i, 1 \leq i \leq n \). First suppose \( n \) is not a multiple of 4. Let \( C = \sum_{i=2}^{n}(-1)^{i}(i-1)E_{i(i-1)} \). We show that \( BC - CB \) is a lower triangular matrix that has distinct diagonal entries. For \( i, j, 1 \leq i \leq n - 1, 1 \leq j \leq n, \) and \( i < j \), we have \((BC - CB)_{ij} = (B)_{i(i+1)}(C)_{(i+1)j} - (C)_{i(i-1)}(B)_{(i-1)j}\). By the definition of \( B \) and \( C \),
\[(C)_{(i+1)j} = (B)_{(i-1)j} = 0. \] So \(BC - CB\) is a lower triangular matrix. Now, assume \(2 \leq i \leq n - 1\). Then

\[
(BC - CB)_{ii} = (B)_{i(i+1)}(C)_{(i+1)i} - (C)_{i(i-1)}(B)_{(i-1)i} \\
= (-1)^{i+1}i - (-1)^{i}(i - 1) = (-1)^{i+1}(2i - 1). 
\]

Also

\[
(BC - CB)_{11} = (B)_{12}(C)_{21} - 0 = -1, 
\]

\[
(BC - CB)_{nn} = 0 - (C)_{n(n-1)} = (-1)^{n+1}(n - 1). 
\]

It is easily seen that for \(i, j, 1 \leq i, j \leq n, i \neq j\), we have

\[
(BC - CB)_{ii} \neq (BC - CB)_{jj}. 
\]

Hence by [3, Theorem 6, p. 204], \(BC - CB\) is a diagonalizable matrix. Next, assume \(n = 4k\) for some positive integer \(k\). Let

\[
C = \sum_{i=2}^{n-1}(-1)^{i}(i - 1)E_{i(i-1)} - (n - 1)E_{n(n-1)}. 
\]

Similarly to the previous case, it can be shown that \(BC - CB\) is a lower triangular matrix. Now, for \(2 \leq i \leq n - 2\),

\[
(BC - CB)_{ii} = (A)_{i(i+1)}(C)_{(i+1)i} - (C)_{i(i-1)}(B)_{(i-1)i} \\
= (-1)^{i+1}i - (-1)^{i}(i - 1) = (-1)^{i+1}(2i - 1). 
\]

Also \((BC - CB)_{11} = (B)_{12}(C)_{21} - 0 = 1\) and

\[
(BC - CB)_{(n-1)(n-1)} = (B)_{(n-1)n}(C)_{n(n-1)} - (C)_{n(n-1)}(B)_{(n-1)n} \\
= -(n - 1) + (n - 1) = 0. 
\]

It is easily checked that by [3, Theorem 6, p. 204], \(BC - CB\) is diagonalizable and the proof is complete. \(\square\)

**Theorem 2.10.** Let \(D\) be a division ring, \(n \geq 3\) a natural number, and \(C\) is a set that includes the zero matrix and all involutions in \(M_n(D)\). Then we have the following:
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(i) If \( n \) is an odd number and \( D \) is commutative, then \( \Gamma_{\mathcal{C}}(M_n(D)) \) is a connected graph if and only if \( \Gamma(M_n(D)) \) is a connected graph.

(ii) If \( n \) is an even number, then \( \Gamma_{\mathcal{C}}(M_n(D)) \) is a connected graph.

Proof. First, suppose that \( n \) is odd and \( D \) is commutative. If \( \text{char} \, F \neq 2 \), then it is easily check that a matrix \( G \in M_n(D) \) is an idempotent if and only if \( 2G - I \) is an involution. So if \( C \) is an involution, then \( H = 2^{-1}(C + I) \) is an idempotent. Hence there is an invertible matrix \( P \) such that \( P^{-1}HP = I_t \oplus (0_{n-t}) \), for some \( 0 \leq t \leq n \). Therefore \( P^{-1}CP = I_t \oplus (-I)_{n-t} \). Now, one can easily seen that the trace of commutators is equals to zero, for every \( M, N \in M_n(D) \). So if \( C \) has the form \( MN - NM \), for some \( M, N \in M_n(D) \), then \( t = n - t \); i.e. \( n = 2t \). So in this case the \( \mathcal{C} \)-commuting elements are 0-commuting matrices. Now suppose that \( \text{char} \, F = 2 \) and \( C \) is an involution. So \( C^2 = I \) and therefore the minimal polynomial of \( C \) is equal to \( x^2 + 1 = x^2 - 1 = (x - 1)(x + 1) \). By [3, Theorem 5, p. 203], \( C \) is a triangularizable matrix that has only 1 as its eigenvalue. Since only the trace of commutators is equal to 0, then the \( \mathcal{C} \)-commuting elements are 0-commuting matrices, and the result follows. Next, suppose that \( n \) is an even number. Since the zero matrix is in \( \mathcal{C} \), by Theorem 1.1, we have each pair of non-invertible matrices are joined by a path in \( \Gamma_{\mathcal{C}}(M_n(D)) \). So to prove the theorem it suffices to show that each non-scalar invertible matrix \( A \in M_n(D) \), is joined to a non-zero, non-invertible matrix in \( \Gamma_{\mathcal{C}}(M_n(D)) \). By Theorem 1.2, we may assume that \( A \) is a cyclic matrix. Hence there is an invertible matrix \( P \) such that

\[
B = P^{-1}AP = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n
\end{pmatrix},
\]

where \( a_i \in D \), for \( i, 1 \leq i \leq n \). Now, let \( C = \sum_{k=1}^{\frac{n}{2}} E_{(2k)(2k-1)} \). Since \( C \) is non-invertible, then it suffices to show that \( BC - CB \in \mathcal{C} \). It is easily checked that

\[
BC = \sum_{k=1}^{\frac{n}{2}} E_{(2k-1)(2k-1)} + \sum_{k=1}^{\frac{n}{2}} a_{2k} E_{n(2k-1)},
\]

and

\[
CB = \sum_{k=1}^{\frac{n}{2}} E_{(2k)(2k)}.
\]

So

\[
BC - CB = \sum_{k=1}^{n} (-1)^k E_{kk} + \sum_{k=1}^{\frac{n}{2}} a_{2k} E_{n(2k-1)}.
\]
To complete the proof, it suffices to show that the last row of \((BC - CB)^2\) equals 
\((0, \ldots, 0, 1)\). For \(k, 1 \leq k \leq \frac{n}{2} - 2\), \((BC - CB)^2_{n(2k)} = 0\) and \((BC - CB)^2_{n(2k-1)} = a_{2k} - a_{2k} = 0\), and \((BC - CB)^2_{nn} = 1\). This completes the proof. \(\square\)

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