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SPECTRALLY ARBITRARY COMPLEX SIGN PATTERN MATRICES

YUBIN GAO†, YANLING SHAO†, AND YIZHENG FAN‡

Abstract. An \(n \times n\) complex sign pattern matrix \(S\) is said to be spectrally arbitrary if for every monic \(n\)th degree polynomial \(f(\lambda)\) with coefficients from \(\mathbb{C}\), there is a complex matrix in the complex sign pattern class of \(S\) such that its characteristic polynomial is \(f(\lambda)\). If \(S\) is a spectrally arbitrary complex sign pattern matrix, and no proper subpattern of \(S\) is spectrally arbitrary, then \(S\) is a minimal spectrally arbitrary complex sign pattern matrix. This paper extends the Nilpotent-Jacobian method for sign pattern matrices to complex sign pattern matrices, establishing a means to show that an irreducible complex sign pattern matrix and all its superpatterns are spectrally arbitrary. This method is then applied to prove that for every \(n \geq 2\) there exists an \(n \times n\) irreducible, spectrally arbitrary complex sign pattern with exactly \(3n\) nonzero entries. In addition, it is shown that every \(n \times n\) irreducible, spectrally arbitrary complex sign pattern matrix has at least \(3n - 1\) nonzero entries.

Key words. Complex sign pattern, Spectrally arbitrary pattern, Nilpotent.

AMS subject classifications. 15A18, 05C15.

1. Introduction. The sign of a real number \(a\), denoted by \(sgn(a)\), is defined to be \(1, -1\) or \(0\), according to \(a > 0, a < 0\) or \(a = 0\). A sign pattern matrix \(A\) is a matrix whose entries are in the set \(\{1, -1, 0\}\). The sign pattern of a real matrix \(B\), denoted by \(sgn(B)\), is the matrix obtained from \(B\) by replacing each entry by its sign.

Associated with each \(n \times n\) sign pattern matrix \(A\) is a class of real matrices, called the sign pattern class of \(A\), defined by

\[Q(A) = \{A \mid A \text{ is an } n \times n \text{ real matrix, and } sgn(A) = A\}.\]

For two \(n \times n\) sign pattern matrices \(A = (a_{kl})\) and \(B = (b_{kl})\), if \(a_{kl} = b_{kl}\) whenever \(b_{kl} \neq 0\), then \(A\) is a superpattern of \(B\), and \(B\) is a subpattern of \(A\). Note that each sign pattern matrix is a superpattern and a subpattern of itself. For a subpattern \(B\) of \(A\), if \(B \neq A\), then \(B\) is a proper subpattern of \(A\).
Let $A$ be a sign pattern matrix of order $n \geq 2$. If for any given real monic polynomial $f(\lambda)$ of degree $n$, there is a real matrix $A \in Q(A)$ having characteristic polynomial $f(\lambda)$, then $A$ is a spectrally arbitrary sign pattern matrix.

The problem of classifying the spectrally arbitrary sign pattern matrices was introduced in [1] by Drew et al. In their article, they developed the Nilpotent-Jacobian method for showing that a sign pattern matrix and all its superpatterns are spectrally arbitrary. Work on spectrally arbitrary sign pattern matrices has continued in several articles including [1–9], where families of spectrally arbitrary sign pattern matrices have been presented. In particular, in [3], Britz et al. showed that every $n \times n$ irreducible, spectrally arbitrary sign pattern matrix must have at least $2n - 1$ nonzero entries and they provided families of sign pattern matrices that have exactly $2n$ nonzero entries. Recently this work has extended to zero-nonzero patterns and ray patterns, respectively ([10, 11]).

Now we introduce some concepts on complex sign pattern matrices.

For $n \times n$ sign pattern matrices $A = (a_{kl})$ and $B = (b_{kl})$, the matrix $S = A + iB$ is called a complex sign pattern matrix of order $n$, where $i^2 = -1$ ([12]). Clearly, the $(k,l)$-entry of $S$ is $a_{kl} + ib_{kl}$ for $k, l = 1, 2, \ldots, n$. Associated with an $n \times n$ complex sign pattern matrix $S = A + iB$ is a class of complex matrices, called the complex sign pattern class of $S$, defined by $Q_c(S) = \{C = A + iB \mid A$ and $B$ are $n \times n$ real matrices, $\text{sgn}(A) = A, \text{sgn}(B) = B\}$.

For two $n \times n$ complex sign pattern matrices $S_1 = A_1 + iB_1$ and $S_2 = A_2 + iB_2$, if $A_1$ is a subpattern of $A_2$, and $B_1$ is a subpattern of $B_2$, then $S_1$ is a subpattern of $S_2$, and $S_2$ is a superpattern of $S_1$. If $S_1$ is a subpattern of $S_2$ and $S_1 \neq S_2$, then $S_1$ is a proper subpattern of $S_2$.

For a complex sign pattern matrix $S = A + iB$ of order $n$, the sign pattern matrices $A$ and $B$ are the real part and complex part of $S$, respectively, and the number of nonzero entries of both $A$ and $B$ is the number of nonzero entries of $S$.

It is clear that complex sign pattern matrix and ray pattern are different generalization of sign pattern matrix. For a complex sign pattern matrix $S = A + iB$, if $B = 0$, then $S = A$ is a sign pattern matrix.

Let $S = A + iB$ be a complex sign pattern matrix of order $n \geq 2$. If there is a complex matrix $C \in Q_c(S)$ having characteristic polynomial $f(\lambda) = \lambda^n$, then $S$ is potentially nilpotent, and $C$ is a nilpotent complex matrix. If for every monic $n$th degree polynomial $f(\lambda)$ with coefficients from $\mathbb{C}$, there is a complex matrix in $Q_c(S)$ such that its characteristic polynomial is $f(\lambda)$, then $S$ is said to be a spectrally arbitrary complex sign pattern matrix. If $S$ is a spectrally arbitrary complex sign
pattern matrix, and no proper subpattern of $S$ is spectrally arbitrary, then $S$ is a \textit{minimal spectrally arbitrary complex sign pattern matrix}.

Let $\mathcal{SA}_n$ represent the set of all $n \times n$ spectrally arbitrary complex sign pattern matrices. Then the following result holds.

**Lemma 1.1.** The set $\mathcal{SA}_n$ is closed under the following operations:

(i) Negation,

(ii) Transposition,

(iii) Permutational similarity,

(iv) Signature similarity, and

(v) Conjugation.

**Proof.** The results are clear for cases (i)–(iv). We only prove the case (v). Note that for any $n \times n$ complex matrix $C$ and its conjugate complex matrix $\overline{C}$, the corresponding coefficients of the characteristic polynomials of $C$ and $\overline{C}$ are conjugate, that is, if the characteristic polynomial of $C$ is

$$|\lambda I - C| = \lambda^n + (f_1 + ig_1)\lambda^{n-1} + \cdots + (f_{n-1} + ig_{n-1})\lambda + (f_n + ig_n),$$

where $f_i, g_i$, $i = 1, 2, \ldots, n$, are real, then the characteristic polynomial of $\overline{C}$ is

$$|\lambda I - \overline{C}| = \lambda^n + (f_1 - ig_1)\lambda^{n-1} + \cdots + (f_{n-1} - ig_{n-1})\lambda + (f_n - ig_n).$$

By the definition of spectrally arbitrary complex sign pattern matrix, the result holds for the case (v). $\Box$

We note that, if a complex sign pattern matrix $S = A + iB$ is spectrally arbitrary, then sign pattern matrices $A$ and $B$ are not necessarily spectrally arbitrary. For example,

$$S_3 = \begin{bmatrix} 1 - i & 1 & 0 \\ 1 + i & 0 & -1 \\ 1 & 0 & -1 + i \end{bmatrix}$$

is a spectrally arbitrary complex sign pattern matrix (This fact will be proved in Section 3), but both sign pattern matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are not spectrally arbitrary. On the other hand, if both $A$ and $B$ are spectrally arbitrary, then the complex sign pattern matrix $S = A + iB$ is not necessarily spectrally arbitrary.
arbitrary. For example, let

\[
A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

From [1], both \(A\) and \(B\) are spectrally arbitrary sign pattern matrices. Consider the complex sign pattern matrix

\[
S = A + iB = \begin{bmatrix} -1 - i & 1 - i \\ -1 + i & 1 + i \end{bmatrix}.
\]

Note that for any

\[
C = \begin{bmatrix} -a_1 - ib_1 & a_2 - ib_2 \\ -a_3 + ib_3 & a_4 + ib_4 \end{bmatrix} \in Q_c(S),
\]

where \(a_j > 0\) and \(b_j > 0\) for \(j = 1, 2, 3, 4\), the characteristic polynomial of \(C\) is

\[
|\lambda I - C| = \lambda^2 + ((a_1 - a_4) + i(b_1 - b_4))\lambda + (a_2a_3 - a_1a_4 - b_2b_3 + b_1b_4)
\]

\[
- i(a_4b_1 + a_3b_2 + a_2b_3 + a_1b_4).
\]

Since \(-(a_4b_1 + a_3b_2 + a_2b_3 + a_1b_4) < 0\), \(S\) is not spectrally arbitrary.

In Section 2 we extend the Nilpotent-Jacobian method for sign pattern matrices to complex sign pattern matrices, establishing a means to show that an irreducible complex sign pattern matrix and all its superpatterns are spectrally arbitrary. In Section 3 we give an \(n \times n\) \((n \geq 2)\) irreducible spectrally arbitrary complex sign pattern matrix \(S_n\) with exactly \(3n\) nonzero entries. In Section 4 we prove that every \(n \times n\) \((n \geq 2)\) irreducible spectrally arbitrary complex sign pattern matrix has at least \(3n - 1\) nonzero entries, and conjecture that for \(n \geq 2\), an \(n \times n\) irreducible spectrally arbitrary complex sign pattern matrix has at least \(3n\) nonzero entries.

2. The Nilpotent-Jacobian method. In this section, we extend the Nilpotent-Jacobian method on sign pattern matrices in [1] to the case of complex sign pattern matrices.

Let \(S = A + iB\) be a complex sign pattern matrix of order \(n \geq 2\) with at least \(2n\) nonzero entries.

- Find a nilpotent complex matrix \(C = A + iB\) in the complex sign pattern class \(Q_c(S)\), where both \(A\) and \(B\) are real matrices, and \(A \in Q(A)\) and \(B \in Q(B)\).
- Change the \(2n\) nonzero entries (denoted \(r_1, r_2, \ldots, r_{2n}\)) in \(A\) and \(B\) to variables \(x_1, x_2, \ldots, x_{2n}\). Denote the resulting matrix by \(X\).
Express the characteristic polynomial of $X$ as:

$$|\lambda I - X| = \lambda^n + (f_1(x_1, x_2, \ldots, x_{2n}) + ig_1(x_1, x_2, \ldots, x_{2n}))\lambda^{n-1} + \cdots$$

$$+(f_{n-1}(x_1, x_2, \ldots, x_{2n}) + ig_{n-1}(x_1, x_2, \ldots, x_{2n}))\lambda$$

$$+(f_n(x_1, x_2, \ldots, x_{2n}) + ig_n(x_1, x_2, \ldots, x_{2n})).$$

Find the Jacobian matrix

$$J = \frac{\partial (f_1, \ldots, f_n, g_1, \ldots, g_n)}{\partial (x_1, x_2, \ldots, x_{2n})}$$

If the determinant of $J$, evaluated at $(x_1, x_2, \ldots, x_{2n}) = (r_1, r_2, \ldots, r_{2n})$ is nonzero, then by continuity of the determinant in the entries of a matrix, there is a neighborhood $U$ of $(r_1, r_2, \ldots, r_{2n})$ such that all the vectors in $U$ are strictly positive and the determinant of $J$ evaluated at any of these vectors is nonzero. Moreover, by the Implicit Function Theorem, there exists a neighborhood $V \subseteq U$ of $(r_1, r_2, \ldots, r_{2n}) \subseteq \mathbb{R}^{2n}$, a neighborhood $W$ of $(0, 0, \ldots, 0) \subseteq \mathbb{R}^{2n}$, and a function $(h_1, \ldots, h_{2n})$ from $W$ into $V$ such that for any $(y_1, \ldots, y_n, z_1, \ldots, z_n) \in W$, there exists a strictly positive vector $(s_1, s_2, \ldots, s_{2n}) = (h_1, \ldots, h_{2n})(y_1, \ldots, y_n, z_1, \ldots, z_n) \in V$ where $f_k(s_1, s_2, \ldots, s_{2n}) = y_k$ and $g_k(s_1, s_2, \ldots, s_{2n}) = z_k$ for $k = 1, 2, \ldots, n$. Taking positive scalar multiples of the corresponding matrices, we see that each monic $n$th degree polynomial over $\mathbb{C}$ is the characteristic polynomial of some matrix in the complex sign pattern class $Q_c(S)$. That is, $S$ is a spectrally arbitrary complex sign pattern matrix.

Next consider a superpattern of the complex sign pattern matrix $S$. Represent the new nonzero entries of $A$ by $p_1, \ldots, p_{m_1}$, and the new nonzero entries of $B$ by $q_1, \ldots, q_{m_2}$. Let $f_k(x_1, x_2, \ldots, x_{2n}, p_1, \ldots, p_{m_1}, q_1, \ldots, q_{m_2})$ and $g_k(x_1, x_2, \ldots, x_{2n}, p_1, \ldots, p_{m_1}, q_1, \ldots, q_{m_2})$ represent the new functions in the characteristic polynomial, and $J = \frac{\partial (f_1, \ldots, f_n, g_1, \ldots, g_n)}{\partial (x_1, x_2, \ldots, x_{2n})}$ the new Jacobian matrix. As above, let $(y_1, \ldots, y_n, z_1, \ldots, z_n) \in W$ and $(s_1, s_2, \ldots, s_{2n}) = (h_1, \ldots, h_{2n})(y_1, \ldots, y_n, z_1, \ldots, z_n)$. Then $y_k = f_k(s_1, s_2, \ldots, s_{2n}) = \tilde{f}_k(s_1, s_2, \ldots, s_{2n}, 0, \ldots, 0)$, $z_k = g_k(s_1, s_2, \ldots, s_{2n}) = \tilde{g}_k(s_1, s_2, \ldots, s_{2n}, 0, \ldots, 0)$, and the determinant of $J$ evaluated at $(x_1, \ldots, x_{2n}, p_1, \ldots, p_{m_1}, q_1, \ldots, q_{m_2}) = (s_1, \ldots, s_{2n}, 0, 0, \ldots, 0)$ is equal to the determinant of $J$ evaluated at $(x_1, x_2, \ldots, x_{2n}) = (s_1, s_2, \ldots, s_{2n})$ and hence is nonzero. By the Implicit Function Theorem, there exists a neighborhood $V \subseteq V$ of $(s_1, s_2, \ldots, s_{2n})$, a neighborhood $T$ of $(0, 0, \ldots, 0) \subseteq \mathbb{R}^{m_1+m_2}$ and a function $(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_{2n})$ from $T$ into $V$ such that for any vector $(d_1, \ldots, d_{m_1+m_2}) \in T$ there exists a strictly positive vector $(e_1, e_2, \ldots, e_{2n}) = (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_{2n})(d_1, \ldots, d_{m_1+m_2}) \in V$ where
\[
\hat{f}_k(e_1, \ldots, e_2n, d_1, \ldots, d_{m_1+m_2}) = y_k \quad \text{and} \quad \hat{g}_k(e_1, \ldots, e_2n, d_1, \ldots, d_{m_1+m_2}) = z_k.
\]
Choosing \((d_1, \ldots, d_{m_1+m_2}) \in T\) strictly positive we see that there are also matrices in the superpattern’s class with every characteristic polynomial corresponding to a vector in \(W\). Taking positive scalar multiples of the corresponding matrices, we see that each monic \(n\)th degree polynomial over \(C\) is the characteristic polynomial of some matrix in this superpattern’s class. Thus each superpattern of \(S\) is a spectrally arbitrary complex sign pattern matrix.

**Theorem 2.1.** Let \(S = A + iB\) be a complex sign pattern matrix of order \(n \geq 2\), and suppose that there exists some nilpotent complex matrix \(C = A + iB \in Q_c(S)\), where \(A \in Q(A), B \in Q(B)\), and \(A\) and \(B\) have at least \(2n\) nonzero entries, say \(a_{1i,j1}, \ldots, a_{in,jn1}, b_{1i+1j1+1}, \ldots, b_{2n,j2n}\). Let \(X\) be the complex matrix obtained by replacing these entries in \(C\) by variables \(x_1, \ldots, x_{2n}\), and let the characteristic polynomial of \(X\) be

\[
|\lambda - X| = \lambda^n + (f_1(x_1, x_2, \ldots, x_{2n}) + ig_1(x_1, x_2, \ldots, x_{2n}))\lambda^{n-1} + \cdots + (f_{n-1}(x_1, x_2, \ldots, x_{2n}) + ig_{n-1}(x_1, x_2, \ldots, x_{2n}))\lambda + (f_n(x_1, x_2, \ldots, x_{2n}) + ig_n(x_1, x_2, \ldots, x_{2n})).
\]

If the Jacobian matrix \(J = \frac{\partial(f_1, \ldots, f_n, g_1, \ldots, g_n)}{\partial(x_1, x_2, \ldots, x_{2n})}\) is nonsingular at \((x_1, \ldots, x_{2n}) = (a_{1i,j1}, \ldots, a_{in,jn1}, b_{1i+1j1+1}, \ldots, b_{2n,j2n})\), then the complex sign pattern matrix \(S\) is spectrally arbitrary, and every superpattern of \(S\) is a spectrally arbitrary complex sign pattern matrix.

### 3. Minimal spectrally arbitrary complex sign pattern matrices.

In this section we first consider the following \(n \times n\) \((n \geq 7)\) complex sign pattern matrix

\[
(3.1) \quad S_n = A_n + iB_n = \begin{bmatrix}
1 + i & 1 & & & & \\
1 - i & 0 & -1 & & & \\
1 + i & 0 & 1 & & & \\
& 1 - i & & \ddots & -1 & \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & 0 & \ddots \\
1 + (-1)^{n-1}i & & & & -i & (-1)^n \\
0 & 0 & 0 & (-1)^{\frac{n+1}{2}} & 0 & \cdots & 0 & -1
\end{bmatrix}
\]

We will prove that \(S_n\) is a minimal spectrally arbitrary complex sign pattern matrix, and every superpattern of \(S_n\) is a spectrally arbitrary complex sign pattern matrix.
Take an \( n \times n \) complex matrix

\[
C = \begin{bmatrix}
  a_1 + ib_1 & 1 & & & \\
  a_2 - ib_2 & 0 & -1 & & \\
  a_3 + ib_3 & 0 & & 1 & \\
  & a_4 - ib_4 & \ddots & \ddots & -1 \\
  & & \ddots & \ddots & 0 \\
  \vdots & & \ddots & \ddots & \ddots \\
  \vdots & & & 0 & \ddots \\
  a_{n-1} + (-1)^n b_{n-1} & 0 & \cdots & 0 & -ib_n \end{bmatrix}
\]

where \( a_k > 0 \) and \( b_k > 0 \) for \( k = 1, 2, \ldots, n \). Then \( C \in Q_c(S_n) \). Denote

\[
|\lambda I - C| = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_k \lambda^{n-k} + \cdots + \alpha_{n-1} \lambda + \alpha_n,
\]

and \( \alpha_k = f_k + ig_k, k = 1, 2, \ldots, n \).

**Lemma 3.1.** Let \( a_0 = 1 \) and \( b_0 = 0 \). Then

\[
f_1 = 1 - a_1,
\]

\[
f_k = (-1)^{\frac{k(k+1)}{2}} a_k + (-1)^{\frac{k+1}{2}} a_{k-1} + (-1)^{\frac{k-1}{2}} b_{k-1} b_n + (-1)^{\frac{k-1}{2}} b_{k-2} b_n,
\]

\( k = 2, 3, \ldots, n - 4 \),

\[
f_{n-3} = (-1)^n a_n + (-1)^{\frac{n-1}{2}} a_{n-3} + (-1)^{\frac{n-3}{2}} a_{n-4} + (-1)^{\frac{n-3}{2}} b_{n-4} b_n + (-1)^{\frac{n-3}{2}} b_{n-5} b_n,
\]

\[
f_{n-2} = (-1)^{n+1} a_{n-1} + (-1)^{\frac{n-3}{2}} a_{n-2} + (-1)^{\frac{n-5}{2}} a_{n-3} + (-1)^{\frac{n-5}{2}} b_{n-4} b_n + (-1)^{\frac{n-5}{2}} b_{n-3} b_n,
\]

\[
f_{n-1} = (-1)^{n+1} a_{n-2} + (-1)^{\frac{n-5}{2}} a_{n-3} + (-1)^{\frac{n-7}{2}} a_{n-4} + (-1)^{\frac{n-7}{2}} b_{n-4} b_n + (-1)^{\frac{n-7}{2}} b_{n-3} b_n,
\]

\[
f_n = (-1)^n a_3 a_n + (-1)^{\frac{n-3}{2}} a_{n-1} + (-1)^{\frac{n-5}{2}} b_{n-2} b_n,
\]

and

\[
g_1 = -b_1 + b_n,
\]

\[
g_k = (-1)^{\frac{k(k+1)}{2}} b_k + (-1)^{\frac{k+1}{2}} b_{k-1} b_n + (-1)^{\frac{k-1}{2}} a_{k-1} b_n + (-1)^{\frac{k-1}{2}} a_{k-2} b_n,
\]

\( k = 2, 3, \ldots, n - 3 \),

\[
g_{n-2} = (-1)^{n+1} a_{n-1} b_1 + (-1)^{\frac{n+1}{2}} b_{n-2} + (-1)^{\frac{n-1}{2}} b_{n-3} + (-1)^{\frac{n-1}{2}} a_{n-3} b_n + (-1)^{\frac{n-1}{2}} a_{n-4} b_n,
\]

\[
g_{n-1} = (-1)^{n+2} a_{n-2} b_1 + (-1)^{\frac{n+1}{2}} b_{n-2} + (-1)^{\frac{n+1}{2}} b_{n-3} + (-1)^{\frac{n+1}{2}} a_{n-3} b_n + (-1)^{\frac{n+1}{2}} a_{n-4} b_n,
\]

\[
g_n = (-1)^n a_3 b_3 + (-1)^{\frac{n+3}{2}} b_{n-2} + (-1)^{\frac{n+3}{2}} a_{n-3} b_n.
\]
Proof. Denote $c_0 = 1$, and $c_k = a_k + (-1)^{k+1}ib_k$ for $k = 1, 2, \ldots, n-1$. Then

$$|\lambda I - C| = \begin{vmatrix}
\lambda - c_1 & -1 \\
-c_2 & \lambda & 1 \\
\vdots & \ddots & \ddots & \ddots \\
-c_{n-1} & \lambda & \cdots & -1 \\
0 & 0 & 0 & (\lambda + ib_n)
\end{vmatrix} (\lambda + 1)$$

$$= (\lambda + 1)^n a_n (\lambda^3 - c_1 \lambda^2 - c_2 \lambda + c_3) + (\lambda + 1)(-1)^{n-1} c_{n-1} + (\lambda + 1)(\lambda + ib_n) \Delta_{n-2},$$

where

$$\Delta_{n-2} = \begin{vmatrix}
\lambda - c_1 & -1 & 0 \\
-c_2 & \lambda & 1 \\
\vdots & \ddots & \ddots & \ddots \\
-c_{n-2} & \lambda & \cdots & (1)^{n-3} \\
-c_{n-1} & \lambda & \cdots & (1)^{n-2} \\
\end{vmatrix}_{n-2}$$

$$= (-1)^{\frac{3n-5}{2}} c_{n-2} + \lambda \Delta_{n-3}$$

$$= (-1)^{\frac{3n-5}{2}} c_{n-2} + (-1)^{\frac{3n-7}{2}} c_{n-3} \lambda + \lambda^2 \Delta_{n-4}$$

$$= \cdots$$

$$= (-1)^{\frac{3n-5}{2}} c_{n-2} + (-1)^{\frac{3n-5}{2}} c_{n-3} \lambda + (-1)^{\frac{3n-5}{2}} c_{n-4} \lambda^2 + \cdots$$
\[ +(-1)^{\left\lfloor \frac{3(n-k-1)}{2} \right\rfloor} |c_{n-k-2}\lambda^k + \cdots - c_2\lambda^n - c_1\lambda^{n-3} + \lambda^{n-2} = \sum_{k=0}^{n-2} (-1)^{\left\lfloor \frac{3(k+1)}{2} \right\rfloor} c_k\lambda^{n-k-2}. \]

So

\[ |\lambda I - C| = (-1)^n a_n (\lambda^3 - c_1\lambda^2 - c_2\lambda + c_3) + (\lambda + 1)(-1)^{\left\lfloor \frac{3n}{2} \right\rfloor} c_{n-1} + (\lambda^2 + (1 + ib_n)\lambda + ib_n) \sum_{k=0}^{n-2} (-1)^{\left\lfloor \frac{3(k+1)}{2} \right\rfloor} c_k\lambda^{n-k-2}. \]

Thus

\[ \alpha_1 = -c_1 + (1 + ib_n), \]
\[ \alpha_k = (-1)^{\left\lfloor \frac{3(k+1)}{2} \right\rfloor} c_k + (-1)^{\left\lfloor \frac{3n}{2} \right\rfloor} c_{k-1}(1 + ib_n) + (-1)^{\left\lfloor \frac{3(n-k-1)}{2} \right\rfloor} ic_{k-2}b_n, \]
\[ k = 2, 3, \ldots, n - 4, \]
\[ \alpha_{n-3} = (-1)^n a_n + (-1)^{\left\lfloor \frac{3n-6}{2} \right\rfloor} c_{n-3} + (-1)^{\left\lfloor \frac{3n-9}{2} \right\rfloor} c_{n-4}(1 + ib_n) + (-1)^{\left\lfloor \frac{3n-12}{2} \right\rfloor} ic_{n-5}b_n, \]
\[ \alpha_{n-2} = (-1)^{n+1} a_n c_1 + (-1)^{\left\lfloor \frac{3n-9}{2} \right\rfloor} c_{n-2} + (-1)^{\left\lfloor \frac{3n-6}{2} \right\rfloor} c_{n-3}(1 + ib_n) \]
\[ + (-1)^{\left\lfloor \frac{3n-4}{2} \right\rfloor} ic_{n-4}b_n, \]
\[ \alpha_{n-1} = (-1)^{n+1} a_n c_2 + (-1)^{\left\lfloor \frac{3n}{2} \right\rfloor} c_{n-1} + (-1)^{\left\lfloor \frac{3n-3}{2} \right\rfloor} c_{n-2}(1 + ib_n) \]
\[ + (-1)^{\left\lfloor \frac{3n-4}{2} \right\rfloor} ic_{n-3}b_n, \]
\[ \alpha_n = (-1)^n a_n c_3 + (-1)^{\left\lfloor \frac{3n}{2} \right\rfloor} c_{n-1} + (-1)^{\left\lfloor \frac{3n-4}{2} \right\rfloor} ic_{n-2}b_n. \]

Noticing that \( c_k = a_k + (-1)^{k+1} ib_k \) for \( k = 1, 2, \ldots, n-1 \), the lemma holds. \( \Box \)

**Lemma 3.2.** There are unique positive integers \( \hat{a}_k \) and \( \hat{b}_k \), \( k = 1, 2, \ldots, n \), such that when \( a_k = \hat{a}_k \) and \( b_k = \hat{b}_k \) for \( k = 1, 2, \ldots, n \), the complex matrix \( C \) having the form (3.2) is nilpotent. Further, \( \det(\frac{\partial(f_1, \ldots, f_n, 0_1, \ldots, 0_n)}{\partial(a_1, \ldots, a_n, 0_1, \ldots, 0_n)})|_{a_k=\hat{a}_k,b_k=\hat{b}_k,k=1,\ldots,n} = (-1)^{\frac{n+1}{2}} 6. \)

**Proof.** We prove the lemma according to the four cases \( n = 4m \), \( n = 4m + 1 \), \( n = 4m + 2 \), and \( n = 4m + 3 \).
Let $n = 4m$. By Lemma 3.1, we have

\[
\begin{align*}
    f_1 &= 1 - a_1, \\
    f_k &= (-1)^{\frac{k(k-1)}{2}} a_k + (-1)^{\frac{k-3}{2}} a_{k-1} + (-1)^{\frac{k-1}{2}} b_{k-1} b_n + (-1)^{\frac{k-3}{2}} b_{k-2} b_n, \\
    f_{n-3} &= a_n - a_{n-3} + a_{n-4} + b_{n-4} b_n - b_{n-5} b_n, \\
    f_{n-2} &= -a_1 a_n - a_{n-2} - a_{n-3} + b_{n-3} b_n + b_{n-4} b_n, \\
    f_{n-1} &= -a_2 a_n + a_{n-1} - a_{n-2} - b_{n-2} b_n + b_{n-3} b_n, \\
    f_n &= a_3 a_n + a_{n-1} - b_{n-2} b_n, \\
\end{align*}
\]

and

\[
\begin{align*}
    g_1 &= -b_1 + b_n, \\
    g_k &= (-1)^{\frac{k(k-1)}{2}} b_k + (-1)^{\frac{k-3}{2}} b_{k-1} + (-1)^{\frac{k-1}{2}} a_{k-1} b_n + (-1)^{\frac{k-3}{2}} a_{k-2} b_n, \\
    g_{n-2} &= -a_n b_1 + b_{n-2} - b_{n-3} - a_{n-3} b_n + a_{n-4} b_n, \\
    g_{n-1} &= a_n b_2 + b_{n-1} + b_{n-2} - a_{n-2} b_n - a_{n-3} b_n, \\
    g_n &= a_n b_3 + b_{n-1} - a_{n-2} b_n. \\
\end{align*}
\]

Let $f_k = 0$ and $g_k = 0$ for $k = 1, 2, \ldots, n$. Then

\[
\begin{align*}
    a_1 &= 1, \\
    a_{2k} &= a_{2k+1}, \quad k = 1, 2, \ldots, \frac{n}{2} - 3, \\
    a_{n-4} &= a_{n-3} - a_n, \\
    a_{n-2} &= a_{n-1} - 2a_n b_1^2 - a_2 a_n, \\
    a_{n-1} &= b_{n-2} b_n - a_3 a_n, \\
    a_{2k-1} + a_{2k} &= b_{1}^{2k}, \quad k = 1, 2, \ldots, \frac{n}{2} - 2, \\
    a_{n-3} + a_{n-2} &= b_{1}^{n-2} - a_n, \\
\end{align*}
\]

and

\[
\begin{align*}
    b_1 &= b_2 = b_n, \\
    b_{2k+1} &= b_{2k+2}, \quad k = 1, 2, \ldots, \frac{n}{2} - 3, \\
    b_{n-3} &= b_{n-2} - 2a_n b_1, \\
    b_{n-1} &= a_{n-2} b_n - a_n b_3, \\
    b_{2k} + b_{2k+1} &= b_{1}^{2k+1}, \quad k = 1, 2, \ldots, \frac{n}{2} - 2, \\
    b_{n-2} + b_{n-1} &= b_{1}^{n-1} - a_n b_1 - a_n b_2. \\
\end{align*}
\]
We have that
\[
\begin{align*}
  a_1 &= 1, \\
  a_{2k} &= a_{2k+1} = \sum_{j=0}^{k-2} (-1)^{k-j} b_1^{2j}, \quad k = 1, 2, \ldots, \frac{n}{2} - 3, \\
  a_{n-4} &= \sum_{j=0}^{n-2} (-1)^{j} b_1^{2j}, \\
  a_{n-3} &= a_n + \sum_{j=0}^{n-2} (-1)^{j} b_1^{2j}, \\
  a_{n-2} &= -2a_n + \sum_{j=0}^{n-1} (-1)^{j} b_1^{2j}, \\
  a_{n-1} &= 2a_n b_1^2 + a_2 a_n - 2a_n + \sum_{j=0}^{n-1} (-1)^{j} b_1^{2j}, \\
  a_n &= 2a_n b_1^2 - a_3 a_n + \sum_{j=1}^{n-1} (-1)^{j} b_1^{2j},
\end{align*}
\]

and
\[
\begin{align*}
  b_1 &= b_2 = b_n, \\
  b_{2k+1} &= b_{2k+2} = \sum_{j=0}^{k} (-1)^{k-j} b_1^{2j+1}, \quad k = 1, 2, \ldots, \frac{n}{2} - 3, \\
  b_{n-3} &= \sum_{j=0}^{n-2} (-1)^{j} b_1^{2j+1}, \\
  b_k &= b_1 a_{k-1}, \quad k = 3, 4, \ldots, n - 3, \\
  b_{n-2} &= 2a_n b_1 + \sum_{j=0}^{n-1} (-1)^{j} b_1^{2j+1}, \\
  b_{n-1} &= -a_n b_3 - 2a_n b_1 + \sum_{j=0}^{n-1} (-1)^{j} b_1^{2j+1}, \\
  b_{n-1} &= -4a_n b_1 + \sum_{j=0}^{n-1} (-1)^{j} b_1^{2j+1}.
\end{align*}
\]

From the second equation and last two equations in the second set of equations, respectively, we have \( b_3 = -b_1 + b_1^2 \), and \( a_n b_3 + 2a_n b_1 = 4a_n b_1 \), so \( b_1 = \sqrt{3} \). From the second equation and last two equations in the first set of equations, respectively, we have \( a_2 = -1 + b_1^2 \), and \( 2a_2 a_n - 2a_n - 1 = 0 \), so \( a_n = \frac{1}{2b_1 - 4} = \frac{1}{2} \). Thus there is
unique solution for \( f_k = 0 \) and \( g_k = 0, \ k = 1, 2, \ldots, n, \) as follows.

\[
\begin{align*}
\hat{a}_1 &= 1, \quad \hat{a}_n = \frac{1}{k}, \quad \hat{b}_1 = \hat{b}_2 = \hat{b}_n = \sqrt{3}, \\
\hat{a}_{2k} &= \hat{a}_{2k+1} = \sum_{j=0}^{k-1} (-1)^{k-j} \hat{b}^{3j}_1, \quad k = 1, 2, \ldots, \frac{n}{2} - 1, \\
\hat{a}_{n-4} &= \sum_{j=0}^{\frac{n}{2} - 2} (-1)^{\frac{n}{2} - j} \hat{b}^{3j}_1, \\
\hat{a}_{n-3} &= \hat{a}_n + \sum_{j=0}^{\frac{n}{2} - 2} (-1)^{\frac{n}{2} - j} \hat{b}^{3j}_1, \\
\hat{a}_{n-2} &= -2\hat{a}_n + \sum_{j=0}^{\frac{n}{2} - 1} (-1)^{\frac{n}{2} - 1 - j} \hat{b}^{3j}_1, \\
\hat{a}_{n-1} &= 2\hat{a}_n \hat{b}^2_1 + \hat{a}_2 \hat{a}_n - 2\hat{a}_n + \sum_{j=0}^{\frac{n}{2} - 1} (-1)^{\frac{n}{2} - 1 - j} \hat{b}^{3j}_1, \\
\hat{b}_k &= \hat{b}_1 \hat{a}_{k-1}, \quad k = 3, 4, \ldots, n - 3, \\
\hat{b}_{n-2} &= 2\hat{a}_n \hat{b}_1 + \sum_{j=0}^{\frac{n}{2} - 1} (-1)^{\frac{n}{2} - 2 - j} \hat{b}^{3j+1}_1, \\
\hat{b}_{n-1} &= -\hat{a}_n \hat{b}_3 - 2\hat{a}_n \hat{b}_1 + \sum_{j=0}^{\frac{n}{2} - 1} (-1)^{\frac{n}{2} - 1 - j} \hat{b}^{3j+1}_1. 
\end{align*}
\]

Since \( \det(J) = \det \left( \frac{\partial(f_1, \ldots, f_n, g_1, \ldots, g_n)}{\partial(a_1, \ldots, a_n, b_1, \ldots, b_n)} \right) = 0 \)
we have

\[
\begin{vmatrix}
-1 & 0 & 0 & b_n \\
0 & 1 & 0 & -b_n \\
a_n & a_n & a_3 + a_2 - a_1 - 1 & 0 \\
-1 & b_3 - b_2 - b_1 & -a_n & -a_n \\
\end{vmatrix} = - \begin{vmatrix}
-1 & 0 & 0 & b_1 \\
0 & 1 & 0 & -b_2 \\
a_n & a_n & a_3 + a_2 - a_1 - 1 & 0 \\
-1 & b_3 - b_2 - b_1 & -a_n & -a_n \\
\end{vmatrix}.
\]

As for cases \(n = 4m + 1, n = 4m + 2\) and \(n = 4m + 3\), noting that if \(n = 4m + 1\),
then

\[
\begin{align*}
  f_1 &= 1 - a_1, \\
  f_k &= (-1)^{\lfloor \frac{k}{2} \rfloor} a_k + (-1)^{\lfloor \frac{k+3}{2} \rfloor} a_{k-1} + (-1)^{\lfloor \frac{k+2}{2} \rfloor} b_{k-1} b_n + (-1)^{\lfloor \frac{k-3}{2} \rfloor} b_{k-2} b_n, \\
  & \quad k = 2, 3, \ldots, n - 4, \\
  f_{n-3} &= -a_n - a_{n-3} - a_{n-4} + b_{n-4} b_n + b_{n-5} b_n, \\
  f_{n-2} &= a_2 a_n + a_{n-2} - a_{n-3} - b_{n-3} b_n + b_{n-4} b_n, \\
  f_{n-1} &= a_2 a_n + a_{n-1} + a_{n-2} - b_{n-2} b_n - b_{n-3} b_n, \\
  f_n &= -a_3 a_n + a_{n-1} - b_{n-2} b_n,
\end{align*}
\]

and

\[
\begin{align*}
  g_1 &= -b_1 + b_n, \\
  g_k &= (-1)^{\lfloor \frac{k+1}{2} \rfloor} b_k + (-1)^{\lfloor \frac{k}{2} \rfloor} b_{k-1} b_n + (-1)^{\lfloor \frac{k+1}{2} \rfloor} b_{k-2} b_n, \\
  & \quad k = 2, 3, \ldots, n - 3, \\
  g_{n-2} &= a_n b_1 + b_{n-2} + b_{n-3} - a_{n-3} b_n - a_{n-4} b_n, \\
  g_{n-1} &= -a_n b_2 - b_{n-1} + b_{n-2} + a_{n-2} b_n - a_{n-3} b_n, \\
  g_n &= -a_n b_3 - b_{n-1} + a_{n-2} b_n;
\end{align*}
\]

if \( n = 4m + 2 \), then

\[
\begin{align*}
  f_1 &= 1 - a_1, \\
  f_k &= (-1)^{\lfloor \frac{k+3}{2} \rfloor} a_k + (-1)^{\lfloor \frac{k+2}{2} \rfloor} a_{k-1} + (-1)^{\lfloor \frac{k-3}{2} \rfloor} b_{k-1} b_n + (-1)^{\lfloor \frac{k-2}{2} \rfloor} b_{k-2} b_n, \\
  & \quad k = 2, 3, \ldots, n - 4, \\
  f_{n-3} &= a_n + a_{n-3} - a_{n-4} - b_{n-4} b_n + b_{n-5} b_n, \\
  f_{n-2} &= -a_1 a_n + a_{n-2} + a_{n-3} - b_{n-3} b_n - b_{n-4} b_n, \\
  f_{n-1} &= -a_2 a_n - a_{n-1} + a_{n-2} b_n - b_{n-3} b_n, \\
  f_n &= a_3 a_n - b_{n-1} + b_{n-2} b_n,
\end{align*}
\]

and

\[
\begin{align*}
  g_1 &= -b_1 + b_n, \\
  g_k &= (-1)^{\lfloor \frac{k+1}{2} \rfloor} b_k + (-1)^{\lfloor \frac{k}{2} \rfloor} b_{k-1} b_n + (-1)^{\lfloor \frac{k+1}{2} \rfloor} b_{k-2} b_n, \\
  & \quad k = 2, 3, \ldots, n - 3, \\
  g_{n-2} &= -a_n b_1 - b_{n-2} + b_{n-3} + a_{n-3} b_n - a_{n-4} b_n, \\
  g_{n-1} &= a_n b_2 - b_{n-1} - b_{n-2} + a_{n-2} b_n + a_{n-3} b_n, \\
  g_n &= a_n b_3 - b_{n-1} + a_{n-2} b_n;
\end{align*}
\]

if \( n = 4m + 3 \), then

\[
\begin{align*}
  f_1 &= 1 - a_1, \\
  f_k &= (-1)^{\lfloor \frac{k+3}{2} \rfloor} a_k + (-1)^{\lfloor \frac{k+2}{2} \rfloor} a_{k-1} + (-1)^{\lfloor \frac{k-3}{2} \rfloor} b_{k-1} b_n + (-1)^{\lfloor \frac{k-2}{2} \rfloor} b_{k-2} b_n, \\
  & \quad k = 2, 3, \ldots, n - 4, \\
  f_{n-3} &= -a_n + a_{n-3} + a_{n-4} - b_{n-4} b_n - b_{n-5} b_n, \\
  f_{n-2} &= a_1 a_n - a_{n-2} + a_{n-3} + b_{n-3} b_n - b_{n-4} b_n, \\
  f_{n-1} &= a_2 a_n - a_{n-1} - a_{n-2} + b_{n-2} b_n + b_{n-3} b_n, \\
  f_n &= -a_3 a_n + a_{n-1} + b_{n-2} b_n,
\end{align*}
\]
and

\[
\begin{align*}
g_1 &= -b_1 + b_n, \\
g_k &= (-1)^{\left\lfloor \frac{n-k+1}{2} \right\rfloor}b_k + (-1)^{\left\lfloor \frac{n-k-1}{2} \right\rfloor}a_{k-1}b_n + (-1)^{\left\lfloor \frac{n-k-1}{2} \right\rfloor}a_{k-2}b_n, \\
g_{n-2} &= a_nb_1 - b_{n-2} - b_{n-3} + a_n b_n + a_{n-4} b_n, \\
g_{n-1} &= -a_nb_2 + b_{n-1} - b_{n-2} - a_n b_n + a_{n-3} b_n, \\
g_n &= -a_nb_3 + b_{n-1} - a_{n-2} b_n,
\end{align*}
\]

the proof methods are similar to the case \( n = 4m \), and we omit them. \( \square \)

By Theorem 2.1 and Lemma 3.2, the following theorem is immediately.

**Theorem 3.3.** For \( n \geq 7 \), the \( n \times n \) complex sign pattern matrix \( S_n \) having the form (3.1) is spectrally arbitrary, and every superpattern of \( S_n \) is a spectrally arbitrary complex sign pattern matrix.

**Theorem 3.4.** For \( n \geq 7 \), the \( n \times n \) complex sign pattern matrix \( S_n \) having the form (3.1) is a minimal spectrally arbitrary complex sign pattern matrix.

**Proof.** Let \( S_n = (s_{kl}) \), \( T = (t_{kl}) \) be a subpattern of \( S_n \) and \( T \) be spectrally arbitrary.

Firstly, it is easy to see that \( t_{kk} = s_{kk} \) for \( k = 1, n-1, n \).

Secondly, note that if all matrices in \( Q_c(T) \) are singular, or all matrices in \( Q_c(T) \) are nonsingular, then \( T \) is not spectrally arbitrary. Thus \( t_{k,k+1} = s_{k,k+1} \) for \( k = 1, 2, \ldots, n-1 \).

Finally, since \( T \) is spectrally arbitrary, there is a complex matrix \( C \in Q_c(T) \) which is nilpotent. We may assume \( C \) has been scaled so that the \((n,n)\) entry of \( C \) is \(-1\). We can also assume that the \((k,k+1)\) entry of \( C \) is 1 or \(-1\) for \( k = 1, 2, \ldots, n-1 \) (otherwise they can be adjusted to be 1 or \(-1\) by suitable similarities). Thus, without loss of generality, suppose that \( C \) has the form (3.2). From \( f_k = 0 \) and \( g_k = 0 \) for \( k = 1, 2, \ldots, n \), as in Lemma 3.1, we can conclude that \( a_k \neq 0 \) for \( k = 2, \ldots, n \), and \( b_k \neq 0 \) for \( k = 2, \ldots, n-1 \).

Then \( T = S_n \), and so \( S_n \) is a minimal spectrally arbitrary complex sign pattern matrix. \( \square \)

**Lemma 3.5.** Let complex sign pattern matrices

\[
S_2 = \begin{bmatrix} 1-i & 1 \\ i & -1+i \end{bmatrix},
S_3 = \begin{bmatrix} 1-i & 1 & 0 \\ 1+i & 0 & -1 \\ 1 & 0 & -1+i \end{bmatrix},
S_4 = \begin{bmatrix} 1+i & 1 & 0 & 0 \\ 1+i & 0 & -1 & 0 \\ -1 & i & -i & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},
\]

Spectrally Arbitrary Complex Sign Pattern Matrices

\[ S_5 = \begin{bmatrix}
1 + i & 1 & 0 & 0 & 0 \\
1 - i & 0 & -1 & 0 & 0 \\
1 + i & 0 & 0 & 1 & 0 \\
1 - i & 0 & 0 & -i & -1 \\
0 & 0 & 0 & -1 & -1 \\
\end{bmatrix}, \quad S_6 = \begin{bmatrix}
1 + i & 1 & 0 & 0 & 0 & 0 \\
-1 - i & 0 & -1 & 0 & 0 & 0 \\
1 + i & 0 & 0 & 1 & 0 & 0 \\
-1 & -i & 0 & 0 & -1 & 0 \\
-1 & i & 0 & 0 & -i & 1 \\
0 & 0 & 0 & -1 & 0 & -1 \\
\end{bmatrix}. \]

Then \( S_j, j = 2, 3, 4, 5, 6 \) are minimal spectrally arbitrary complex sign pattern matrices.

**Proof.** First, we prove that each \( S_j \) is spectrally arbitrary. For \( S_2 \), we are able to obtain a nilpotent complex matrix

\[
C_2 = \begin{bmatrix}
\frac{a_1 - ib_1}{ia_2} & \frac{1}{-1 + ib_2} \\
\end{bmatrix} \in \mathbb{Q}_c(S_2),
\]

where \( a_2 = 2, a_1 = b_1 = b_2 = 1 \). Replacing the entries \( a_1, b_1, a_2, b_2 \) of \( C_2 \) by variables in using Theorem 2.1, it can be verified that \( S_2 \) is spectrally arbitrary.

For \( S_3 \), we are able to obtain a nilpotent complex matrix

\[
C_3 = \begin{bmatrix}
\frac{a_1 - ib_1}{a_2 + ib_2} & 1 & 0 \\
\frac{1}{a_3} & 0 & -1 + ib_3 \\
\end{bmatrix} \in \mathbb{Q}_c(S_3),
\]

where \( a_1 = 1, a_2 = 2, a_3 = 8, b_1 = b_3 = \sqrt{3}, b_2 = 2\sqrt{3} \). Replacing the entries \( a_1, b_1, a_2, b_2, a_3, b_3 \) of \( C_3 \) by variables in using Theorem 2.1, it can be verified that \( S_3 \) is spectrally arbitrary.

For \( S_4 \), we are able to obtain a nilpotent complex matrix

\[
C_4 = \begin{bmatrix}
\frac{a_1 + ib_1}{a_2 + ib_2} & 1 & 0 & 0 \\
\frac{-a_3}{-ib_3} & 0 & -1 & 0 \\
0 & 0 & a_3 & -1 \\
\end{bmatrix} \in \mathbb{Q}_c(S_4),
\]

where \( a_1 = 1, a_2 = \sqrt{5}, a_3 = 2(7 + 4\sqrt{5}), b_1 = b_2 = b_4 = \sqrt{3} + 2\sqrt{5}, b_3 = 2\sqrt{3} + 2\sqrt{5} \). Replacing the entries \( a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 \) of \( C_4 \) by variables in using Theorem 2.1, it can be verified that \( S_4 \) is spectrally arbitrary.

For \( S_5 \), we are able to obtain a nilpotent complex matrix

\[
C_5 = \begin{bmatrix}
\frac{a_1 + ib_1}{a_2 - ib_2} & 1 & 0 & 0 & 0 \\
\frac{a_3 + ib_3}{a_4 - ib_4} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -ib_5 & -1 \\
\end{bmatrix} \in \mathbb{Q}_c(S_5),
\]
where \( a_1 = 1, a_2 = 1 + \sqrt{2}, a_3 = 2, a_4 = 6\sqrt{2}, a_5 = \sqrt{2} - 1, b_1 = b_2 = b_5 = \sqrt{1 + 2\sqrt{2}}, b_3 = 2\sqrt{1 + 2\sqrt{2}}, b_4 = 2(2\sqrt{1 + 2\sqrt{2}} - \sqrt{2(1 + 2\sqrt{2})}) \). Replacing the entries \( a_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5 \) of \( C_5 \) by variables in using Theorem 2.1, it can be verified that \( S_5 \) is spectrally arbitrary.

For \( S_6 \), we are able to obtain a nilpotent complex matrix

\[
C_6 = \begin{bmatrix}
  a_1 + ib_1 & 1 & 0 & 0 & 0 & 0 \\
  -a_2 - ib_2 & 0 & -1 & 0 & 0 & 0 \\
  a_3 + ib_3 & 0 & 0 & 1 & 0 & 0 \\
  -a_4 & -ib_4 & 0 & 0 & -1 & 0 \\
  -a_5 & ib_5 & 0 & 0 & -ib_6 & 1 \\
  0 & 0 & 0 & -a_6 & 0 & -1
\end{bmatrix} \in Q_c(S_6),
\]

where \( a_1 = 1, a_2 = \frac{4}{3} - \frac{\sqrt{37}}{6}, a_3 = \frac{1}{6}(2\sqrt{37} - 1), a_4 = 2, a_5 = \frac{1}{12}(4 + 19\sqrt{37}), a_6 = \frac{1}{6}(7 + \sqrt{37}), b_1 = b_2 = b_6 = \frac{\sqrt{\sqrt{37} - \frac{1}{2}}}{\sqrt{6}}, b_3 = 2\sqrt{\frac{\sqrt{37} - \frac{1}{2}}{3}}, b_4 = \frac{10}{9}\sqrt{\frac{\sqrt{37} - \frac{1}{2}}{3}} - \frac{1}{9} \sqrt{37(\frac{\sqrt{37}}{6} - \frac{1}{2})}, b_5 = \frac{13}{12}\sqrt{\frac{\sqrt{37}}{6} - \frac{1}{2} + \frac{1}{8}\sqrt{37(\frac{\sqrt{37}}{6} - \frac{1}{2})}}. \]

Replacing the entries \( a_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6 \) of \( C_6 \) by variables in using Theorem 2.1, it can be verified that \( S_6 \) is spectrally arbitrary.

Next, by the same argument as in Theorem 3.4, we see that each \( S_j \) is minimal spectrally arbitrary. \( \Box \)

Theorem 3.4 and Lemma 3.5 immediately yield the following.

**THEOREM 3.6.** For \( n \geq 2 \), there exists an \( n \times n \) minimal, irreducible, spectrally arbitrary complex sign pattern matrix.

4. The minimum number of nonzero entries in a spectrally arbitrary complex sign pattern matrix. Recall that the number of nonzero entries of a complex sign pattern matrix \( S \) is the number of nonzero entries of both the real and imaginary parts of \( S \). In this section we will study the minimum number of nonzero entries in a irreducible spectrally arbitrary complex sign pattern matrix.

Given a sign pattern \( A \), let \( D(A) \) be its associated digraph. For any digraph \( D \), let \( G(D) \) denote the underlying multigraph of \( D \), i.e., the graph obtained from \( D \) by ignoring the direction of each arc.

**LEMMA 4.1.** ([3]) Let \( A \) be an \( n \times n \) sign pattern and let \( A \in Q(A) \). If \( T \) is a subdigraph of \( D(A) \) such that \( G(T) \) is a forest, then \( A \) has a realization that is positive diagonally similar to \( A \) such that each entry corresponding to an arc of \( T \) has magnitude 1. In particular, if \( A \) is irreducible, then \( G(D(A)) \) contains a spanning tree, and \( A \) must therefore have a realization with at least \( n - 1 \) off-diagonal entries in \( \{-1, 1\} \) that is positive diagonally similar to \( A \).
We easily extend Lemma 4.1 to complex sign pattern matrices.

**Lemma 4.2.** Let $\mathcal{S} = A + iB$ be an $n \times n$ irreducible complex sign pattern matrix, and let $C = A + iB \in Q_c(\mathcal{S})$. Then there is a complex matrix $\hat{C} = \hat{A} + i\hat{B} \in Q_c(\mathcal{S})$ (where $\hat{A}$ and $\hat{B}$ are real matrices, $\hat{A} \in Q(A)$ and $\hat{B} \in Q(B)$) such that the following two conditions hold:

1. $\hat{C}$ has at least $n - 1$ off-diagonal entries in which either the real part or complex part of each entry is in $\{-1, 1\}$;

2. $\hat{C}$ is positive diagonally similar to $C$.

Let $\mathbb{Q}[X]$ be the set of polynomials with rational coefficients and finite degree. A set $H \subseteq \mathbb{R}$ is algebraically independent if, for all $h_1, h_2, \ldots, h_n \in H$ and each nonzero polynomial $p(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[X]$, $p(h_1, h_2, \ldots, h_n) \neq 0$ (see [13, p.316] for further details). Let $\mathbb{Q}(H)$ denote the field of rational expressions

$$\left\{ \frac{p(h_1, h_2, \ldots, h_m)}{q(t_1, t_2, \ldots, t_n)} \mid p(x_1, x_2, \ldots, x_m), q(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[X],
\quad h_1, h_2, \ldots, h_m, t_1, t_2, \ldots, t_n \in H \right\},$$

and let the transcendental degree of $H$ be

$$\text{tr.d.} H = \sup\{|T| \mid T \subseteq H, \ T \text{ is algebraically independent}\}.$$

In [3] it was shown that every $n \times n$ irreducible spectrally arbitrary sign pattern matrix contains at least $2n - 1$ nonzero entries. We adapt that proof to the complex sign pattern matrix case to obtain:

**Theorem 4.3.** For $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary complex sign pattern matrix must have at least $3n - 1$ nonzero entries.

**Proof.** Let $\mathcal{S} = A + iB$ be an $n \times n$ irreducible spectrally arbitrary complex sign pattern matrix with $m$ nonzero entries. Choose a set $V = \{f_1, g_1, \ldots, f_n, g_n\} \subseteq \mathbb{R}$ that $\text{tr.d.} V = 2n$. By Lemma 4.2, there is a complex matrix $\hat{C} = \hat{A} + i\hat{B} \in Q_c(\mathcal{S})$ (where $\hat{A}$ and $\hat{B}$ are real matrices, $\hat{A} \in Q(A)$ and $\hat{B} \in Q(B)$) with characteristic polynomial

$$\lambda^n + (f_1 + ig_1)\lambda^{n-1} + \cdots + (f_{n-1} + ig_{n-1})\lambda + (f_n + ig_n)$$

such that $\hat{C}$ satisfies the two conditions in Lemma 4.2.

Denote $\hat{A} = (\hat{a}_{kl})$, $\hat{B} = (\hat{b}_{kl})$, and $H = \{\hat{a}_{kl} \mid 1 \leq k, l \leq n\} \cup \{\hat{b}_{kl} \mid 1 \leq k, l \leq n\}$. Since for each $1 \leq k \leq n$, $f_k$ and $g_k$ are polynomials in the entries of $H$ with rational coefficients, it follows that $\mathbb{Q}(V) \subseteq \mathbb{Q}(H)$. Then

$$2n = \text{tr.d.} \mathbb{Q}(V) \leq \text{tr.d.} \mathbb{Q}(H) \leq m - (n - 1).$$
Thus $m \geq 3n - 1$. □

Note that the spectrally arbitrary complex sign pattern $S_n$ ($n \geq 2$) in Section 3 is irreducible, and has exactly $3n$ nonzero entries. Then for every $n \geq 2$ there exists an $n \times n$ irreducible, spectrally arbitrary complex sign pattern with exactly $3n$ nonzero entries. By Theorem 4.3 the minimum number of nonzero entries in an $n \times n$ irreducible, spectrally arbitrary complex sign pattern must be either $3n$ or $3n - 1$.

A well known conjecture in [3] is that for $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary sign pattern matrix has at least $2n$ nonzero entries. Here, we extend the conjecture to complex sign pattern matrix case.

**Corollary 4.4.** For $n \geq 2$, an $n \times n$ irreducible spectrally arbitrary complex sign pattern matrix has at least $3n$ nonzero entries.

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**REFERENCES**


