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ON THE CHARACTERIZATION OF GRAPHS WITH PENDENT VERTICES AND GIVEN NULLITY*  

BOLIAN LIU†, YUFEI HUANG†, AND SIYUAN CHEN† 

Abstract. Let $G$ be a graph with $n$ vertices. The nullity of $G$, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in its spectrum. In this paper, we characterize the graphs (resp. bipartite graphs) with pendent vertices and nullity $\eta$, where $0 < \eta \leq n$. Moreover, the minimum (resp. maximum) number of edges for all (connected) graphs with pendent vertices and nullity $\eta$ are determined, and the extremal graphs are characterized.

Key words. Eigenvalue, Nullity, Pendent vertex.

AMS subject classifications. 05C50.

1. Introduction. Let $G$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, the degree and neighborhood of $v$ are denoted by $d(v)$ and $N(v)$, respectively. If $W$ is a nonempty subset of $V(G)$, then the subgraph induced by $W$ is the subgraph of $G$ obtained by taking the vertices in $W$ and joining those pairs of vertices in $W$ which are joined in $G$. We write $G - \{v_1, v_2, \ldots, v_k\}$ for the graph obtained from $G$ by removing the vertices $v_1, v_2, \ldots, v_k$ and all edges incident to any of them.

The disjoint union of two graphs $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$. The disjoint union of $k$ copies of $G$ is often written by $kG$. The null graph of order $n$ is the graph with $n$ vertices and no edges. As usual, the complete graph, the cycle, the path, and the star of order $n$ are denoted by $K_n$, $C_n$, $P_n$ and $S_n$, respectively. An isolated vertex is sometimes denoted by $K_1$.

Let $t$ ($\geq 2$) be an integer. A graph $G$ is called $t$-partite if $V(G)$ admits a partition into $t$ classes $X_1, X_2, \ldots, X_t$ such that every edge has its ends in different classes; vertices in the same partition must not be adjacent. Such a partition $(X_1, X_2, \ldots, X_t)$ is called a $t$-partition of $G$. A complete $t$-partite graph is a simple $t$-partite graph with partition $(X_1, X_2, \ldots, X_t)$ in which each vertex of $X_i$ is joined to each vertex of $G - X_i$ ($1 \leq i \leq t$). If $|X_i| = n_i$ ($1 \leq i \leq t$), such a graph is denoted by $K_{n_1, n_2, \ldots, n_t}$.

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Instead of “2-partite” (resp. “3-partite”) one usually says bipartite (resp. tripartite).

The adjacency matrix $A(G)$ of a graph $G$ of order $n$, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, is an $n \times n$ symmetric matrix $[a_{ij}]$, such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and 0, otherwise. A graph is said to be singular (resp. nonsingular) if its adjacency matrix is a singular (resp. nonsingular) matrix. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of $G$, and to form the spectrum of this graph. The number of zero eigenvalues in the spectrum of a graph $G$ is called its nullity and is denoted by $\eta(G)$. Let $r(A(G))$ be the rank of $A(G)$. Obviously, $\eta(G) = n - r(A(G))$. The rank of a graph $G$ is the rank of its adjacency matrix $A(G)$, denoted by $r(G)$. Then $\eta(G) = n - r(G)$. Clearly, if $G$ is a simple connected graph, then $0 \leq r(G) \leq |V(G)| \leq |E(G)| + 1$.

The problem of characterizing all graphs with $\eta(G) > 0$ was posed in [1] and [10]. This problem is relevant in many disciplines of science (see [2, 3]), and is very difficult. At present, only some particular cases are known (see [3-9,11-12]). On the other hand, this problem is of great interest in chemistry, because, for a bipartite graph $G$ (corresponding to an alternant hydrocarbon), if $\eta(G) > 0$, then it indicates that the molecule which such a graph represents is unstable (see [8]). The nullity of a graph $G$ is also meaningful in linear algebra, since it is related to the singularity and the rank of $A(G)$.

It is known that $0 \leq \eta(G) \leq n - 2$ if $G$ is a simple graph on $n$ vertices and $G$ is not isomorphic to $nK_1$. In [4], B. Cheng and B. Liu characterized the extremal graphs attaining the upper bound $n - 2$ and the second upper bound $n - 3$.

**Lemma 1.1.** ([4]) Suppose that $G$ is a simple graph of order $n$. Then

1. $\eta(G) = n - 2$ if and only if $G$ is isomorphic to $K_{n_1, n_2} \cup kK_1$, where $n_1 + n_2 + k = n \geq 2$ and $n_1, n_2 > 0, k \geq 0$.

2. $\eta(G) = n - 3$ if and only if $G$ is isomorphic to $K_{n_1, n_2, n_3} \cup kK_1$, where $n_1 + n_2 + n_3 + k = n \geq 3$ and $n_1, n_2, n_3 > 0, k \geq 0$.

As a continuation, S. Li ([9]) determined the extremal graphs with pendant vertices which achieve the third upper bound $n - 4$ and fourth upper bound $n - 5$, respectively. Recently, Y. Fan and K. Qian ([6]) characterized all bipartite graphs of order $n$ with nullity $n - 4$.

**Definition 1.2.** ([6]) Let $P_n = v_1v_2 \cdots v_n$ ($n \geq 2$) be a path. Replacing each vertex $v_i$ by an empty graph $O_{m_i}$ of order $m_i$, for $i = 1, 2, \ldots, n$ and joining edges between each vertex of $O_i$ and each vertex of $O_{i+1}$ for $i = 1, 2, \ldots, n - 1$, we get a graph $G$ of order $(m_1 + m_2 + \cdots + m_n)$, denoted by $O_{m_1}O_{m_2}\cdots O_{m_n}$. Such graph is called an expanded path of length $n$, and the empty graph $O_{m_i}$ is called an expanded
vertex of order $m_i$ for $i = 1, 2, \ldots, n$.

**Lemma 1.3.** ([6]) Let $G$ be a bipartite graph of order $n \geq 4$. Then $\eta(G) = n - 4$ if and only if $G$ is isomorphic to a graph $H$ possibly adding some isolated vertices, where $H$ is one of the following graphs: a union of two disjoint expanded paths both of length 2, an expanded path of length 4 or 5.

In Section 2 of this paper, we give a characterization of the graphs (resp. connected graphs) with pendant vertices and nullity $\eta$ ($0 < \eta \leq n$). As corollaries of this characterization, some results in [9] can be obtained immediately. Moreover, all bipartite graphs (resp. bipartite connected graphs) with pendant vertices and nullity $\eta = n - 2k$ are characterized. (It is known from [6] that the nullity set of all bipartite graphs of order $n$ is $\{n - 2k \mid k = 0, 1, \ldots, \lfloor n/2 \rfloor\}$.)

Let $\Gamma(n, e)$ be the set of all simple graphs with $n$ vertices and $e$ edges. In [4], the maximum nullity number of graphs with $n$ vertices and $e$ edges, $M(n, e) = \max \{ \eta(A) \mid A \in \Gamma(n, e) \}$, was studied, where $n \geq 1$ and $0 \leq e \leq \binom{n}{2}$. Conversely, we shall study the number of edges for the graphs with pendant vertices and nullity $\eta$ ($0 < \eta \leq n$). Let $e_{\min}(\eta)$ and $e_{\max}(\eta)$ ($e_{\min}$ and $e_{\max}$) denote the minimum and maximum number of edges for all (connected) graphs with pendant vertices and nullity $\eta$. Let $G_{\min}(\eta)$ (resp. $\tilde{G}_{\min}(\eta)$) denote the graphs (resp. connected graphs) ofnullity $\eta$ with pendant vertices and $e_{\min}(\eta)$ (resp. $e_{\min}$) edges. We call $G_{\min}(\eta)$ (resp. $\tilde{G}_{\min}(\eta)$) the minimum graphs (resp. connected graphs) with pendant vertices and nullity $\eta$. Similarly, we can define $G_{\max}(\eta)$ (resp. $\tilde{G}_{\max}(\eta)$), the maximum graphs (resp. connected graphs) with pendant vertices and nullity $\eta$. In Section 3, we determine the number $e_{\min}(\eta)$, $e_{\max}(\eta)$, $\tilde{e}_{\min}(\eta)$, $\tilde{e}_{\max}(\eta)$ and characterize the graphs $G_{\min}(\eta)$, $G_{\max}(\eta)$, $\tilde{G}_{\min}(\eta)$, $\tilde{G}_{\max}(\eta)$, respectively. Now we list some known results needed in this paper.

**Lemma 1.4.** ([12]) Let $G$ be a simple graph of order $n$. Then

1. $\eta(G) = n$ if and only if $G$ is a null graph.
2. If $G = G_1 \cup G_2 \cup \cdots \cup G_t$, where $G_1$, $G_2$, $\ldots$, $G_t$ are the connected components of $G$, then $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.

**Lemma 1.5.** ([9]) Let $v$ be a pendant vertex of a graph $G$ and $u$ be the vertex in $G$ adjacent to $v$. Then $\eta(G) = \eta(G - \{u, v\})$.

**Lemma 1.6.** ([4])

\[
r(P_n) = \begin{cases} n - 1, & n \text{ is odd;} \\ n, & \text{otherwise.} \end{cases}
\]

\[
r(C_n) = \begin{cases} n - 2, & n \equiv 0 \pmod{4}; \\ n, & \text{otherwise.} \end{cases}
\]

2. The graphs with pendant vertices and nullity $\eta$. Let $\eta$ be an integer with $0 < \eta \leq n$. Now the graphs with pendant vertices and nullity $\eta$ are characterized
as follows, where \( n - 3 \leq \eta \leq n \).

**Lemma 2.1.** Let \( G \) be a simple graph of order \( n \) with pendent vertices. Then

1. There exists no such graph \( G \) with nullity \( \eta(G) = n, n - 1 \) or \( n - 3 \);
2. \( \eta(G) = n - 2 \) if and only if \( G \) is isomorphic to \( S_{n-k} \cup kK_1 \) \((0 \leq k \leq n - 2)\).

**Proof.** (1) Obviously, there exists no such graph \( G \) with nullity \( \eta(G) = n - 1 \).
Moreover, by Lemmas 1.1 and 1.4, the graph \( G \) of nullity \( \eta(G) = n \) (resp. \( n - 3 \)) contains no pendent vertices. This leads to the desired results.

(2) Since the graph \( G \) has pendent vertices, combining this with Lemma 1.1, \( \eta(G) = n - 2 \) if and only if \( G \) is isomorphic to \( K_1, n_2 \cup kK_1 \), where \( 1 + n_2 + k = n \) and \( n_2 > 0, k \geq 0 \).
This completes the proof. \( \square \)

Now we give a characterization of the graphs with pendent vertices and nullity \( \eta \) for \( 0 < \eta \leq n - 4 \). Let \( \widetilde{\Xi}_{n}^{(\eta)} \) be the set of all connected graphs of order \( n \) with nullity \( \eta \) \((0 \leq \eta \leq n)\). Then it follows from Lemmas 1.1 and 1.4 that \( \widetilde{\Xi}_{n}^{(n)} = \widetilde{\Xi}_{n}^{(n-1)} = \emptyset \), \( \widetilde{\Xi}_{n}^{(n-2)} = \{K_{n_1, n_2} \mid n_1 + n_2 = n, \text{ and } n_1, n_2 > 0\} \), \( \widetilde{\Xi}_{n}^{(n-3)} = \{K_{n_1, n_2, n_3} \mid n_1 + n_2 + n_3 = n, \text{ and } n_1, n_2, n_3 > 0\} \).

Let \( n, k, t \) be positive integers with \( 4 \leq k < n \) and \( 1 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \), and let \( p, n_j, p_j \) \((1 \leq j \leq t)\) be integers with \( n_j \geq p_j > 1 \) \((1 \leq j \leq t)\), \( \sum_{j=1}^{t} p_j + 2 = k \), \( \sum_{j=1}^{t} n_j + p + 2 = n \). Let \( H_n, k \) be any graph of order \( n \) created from \( H_j \in \widetilde{\Xi}_{n_j}^{(n_j-p_j)} \) \((j = 1, 2, \ldots, t)\), \( pK_1 \) and \( K_2 \) (suppose \( V(K_2) = \{u, v\} \)) by connecting \( v \) to all vertices of \( pK_1 \) and \( H_j \) \((j = 1, 2, \ldots, t)\) (see Figure 1.). Suppose that \( E^* \) is a subset of \( E(G) \). Let \( \Gamma\{E^*\} \) (resp. \( \tilde{\Gamma}\{E^*\} \)) denote the (resp. connected) spanning subgraph of \( G \) which contains the edges in \( E^* \).

![Figure 1. \( H_n, k \) and \( B_n, k \)](http://math.technion.ac.il/iic/ela)
THEOREM 2.2. Let \( G \) be a graph (resp. connected graph) of order \( n \) with pendent vertices. Then \( \eta(G) = n - k \) (4 \( \leq k < n \)) if and only if \( G \) is isomorphic to \( H_n, k \{ E^* \} \) (resp. \( \tilde{H}_n, k \{ E^* \} \)), where \( E^* = \bigcup_{j=1}^t E(H_j) \cup \{ uv \} \).

**Proof.** To begin with, we need to check that \( \eta(H_n, k \{ E^* \}) = \eta(\tilde{H}_n, k \{ E^* \}) = n - k \) (4 \( \leq k < n \)). Note that \( u \) is a pendent vertex of \( H_n, k \{ E^* \} \) (resp. \( \tilde{H}_n, k \{ E^* \} \)) and \( N(u) = \{ v \} \). Delete \( u, v \) from \( H_n, k \{ E^* \} \) (resp. \( \tilde{H}_n, k \{ E^* \} \)), then the resultant graph is \((\bigcup_{j=1}^t H_j) \cup pK_1\). Since \( H_j \in \tilde{\Upsilon}_{n, j}^{(n_j - p_j)} \), we have \( \eta(H_j) = n_j - p_j \) \((j = 1, 2, \ldots, t)\). Hence by Lemmas 1.4 and 1.5,
\[
\eta(\tilde{H}_n, k \{ E^* \}) = \eta(\tilde{H}_n, k \{ E^* \}) = \eta((\bigcup_{j=1}^t H_j) \cup pK_1) = \sum_{j=1}^t \eta(H_j) + p \cdot \eta(K_1) = \sum_{j=1}^t (n_j - p_j) + p = (\sum_{j=1}^t n_j + p + 2) - (\sum_{j=1}^t p_j + 2) = n - k.
\]

On the other hand, assume that \( \eta(G) = n - k \). Choose a pendent vertex, say \( x \), in \( G \). Let \( N(x) = \{ y \} \). Delete \( x, y \) from \( G \), and let the resultant graph be \( G_1 = G_{11} \cup G_{12} \cup \cdots \cup G_{1q} \), where \( G_{11}, G_{12}, \ldots, G_{1q} \) are connected components of \( G_1 \). Some of these components may be trivial, i.e. \( K_1 \). We conclude that there exist \( t \) nontrivial connected components, where \( 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1 \). Without loss of generality, assume that \( G_{11}, G_{12}, \ldots, G_{1k} \) are nontrivial. By contradiction, suppose that \( t = 0 \) or \( t \geq \lfloor \frac{n}{2} \rfloor \).

**Case 1.** \( t = 0 \). Then all the connected components are trivial, adding \( x, y \) to \( G_1 \) gives a star with some isolated vertices, which contradicts to Lemma 2.1.

**Case 2.** \( t \geq \lfloor \frac{n}{2} \rfloor \). By Lemmas 1.1, 1.4 and 1.5, \( \eta(G) = \sum_{j=1}^t \eta(G_{1j}) + z \eta(K_1) \leq \sum_{j=1}^t \lfloor (|V(G_{1j})| - 2) \rfloor + z \), where \( z \) is the number of isolated vertices in \( G_1 \). The above equality holds if \( G_{11}, \ldots, G_{1t} \) are all complete bipartite graphs.

Therefore, \( \eta(G) \leq \sum_{j=1}^t |V(G_{1j})| - 2t + z = n - 2z - 2t + z = n - 2t - 2 < n - k \) for \( t \geq \lfloor \frac{n}{2} \rfloor \), contradicting that \( \eta(G) = n - k \).

Hence \( 1 \leq t \leq \lfloor \frac{n}{2} \rfloor \). Let \( |V(G_{1j})| = n_j \) \((j = 1, 2, \ldots, t)\). Then \( G_1 = (\bigcup_{j=1}^t G_{1j}) \cup (n - \sum_{j=1}^t n_j - 2)K_1 \). It follows from Lemmas 1.4 and 1.5 that
\[
n - k = \eta(G) = \eta(G_1) = \eta((\bigcup_{j=1}^t G_{1j})) + \eta((n - \sum_{j=1}^t n_j - 2)K_1).
\]

Since \( G_{1j} \) \((j = 1, 2, \ldots, t)\) are nontrivial connected components, suppose that \( \eta(G_{1j}) = n_j - p_j \), where \( 1 \leq p_j \leq n_j \) \((j = 1, 2, \ldots, t)\). Thus we have
\[
n - k = \sum_{j=1}^t (n_j - p_j) + (n - \sum_{j=1}^t n_j - 2).
\]

Hence \( \sum_{j=1}^t p_j + 2 = k \) and \( G_{1j} \in \tilde{\Upsilon}_{n_j}^{(n_j - p_j)} \) \((j = 1, 2, \ldots, t)\).

Let \( p = n - \sum_{j=1}^t n_j - 2 \). In order to recover \( G \), to add \( x, y \) to \( G_1 \), we need
to insert edges from $y$ to $x$ and to some (maybe partial or all) vertices of $pK_1$ and $G_{1j}$ ($j = 1, 2, \ldots, t$). Thus the graph (resp. connected graph) $G$ is isomorphic to $H_{n, k}\{E^*\}$ (resp. $H_{n, k}\{E^*\}$), where $E^* = \cup_{j=1}^{t} E(H_j) \cup \{uv\}$. \hfill $\square$

Now we have the following corollaries of this characterization.

![Figure 2. $Q_1$ and $Q_2$](http://math.technion.ac.il/iic/ela)

Let $Q_1$ be a graph of order $n$ created from $K_{n_1, n_2, pK_1}$ and $K_2$ (suppose $V(K_2) = \{u, v\}$ with $n_1 + n_2 + p + 2 = n$ and $n_1, n_2 > 0$, $p \geq 0$ by connecting $v$ to all vertices of $pK_1$ and $K_{n_1, n_2}$. Let $Q_2$ be a graph of order $n$ created from $K_{n_1, n_2, n_3, pK_1}$ and $K_2 (V(K_2) = \{u, v\})$ with $n_1 + n_2 + n_3 + p + 2 = n$ and $n_1, n_2, n_3 > 0$, $p \geq 0$ by connecting $v$ to all vertices of $pK_1$ and $K_{n_1, n_2, n_3}$ (see Figure 2.).

**Corollary 2.3.** Let $G$ be a graph (resp. connected graph) of order $n$ with pendent vertices. Then

1. $\eta(G) = n - 4$ if and only if $G$ is isomorphic to $Q_1\{E^*\}$ (resp. $\tilde{Q}_1\{E^*\}$), where $E^* = E(K_{n_1, n_2}) \cup \{uv\}$.

2. $\eta(G) = n - 5$ if and only if $G$ is isomorphic to $Q_2\{E^*\}$ (resp. $\tilde{Q}_2\{E^*\}$), where $E^* = E(K_{n_1, n_2, n_3}) \cup \{uv\}$.

**Proof.** By Theorem 2.2, $\eta(G) = n - k = n - 4$ implies $t = 1$, $p_1 = 2$, while $\eta(G) = n - k = n - 5$ implies $t = 1$, $p_1 = 3$. Besides, $\tilde{Y}_n^{(n-2)} = \{K_{n_1, n_2} | n_1 + n_2 = n, \text{ and } n_1, n_2 > 0\}$, $\tilde{Y}_n^{(n-3)} = \{K_{n_1, n_2, n_3} | n_1 + n_2 + n_3 = n, \text{ and } n_1, n_2, n_3 > 0\}$. Then we obtain the results as desired. \hfill $\square$

**Remark.** If $G$ is connected, the results of Corollary 2.3 are that in [9].

Now we shall determine all bipartite graphs with pendent vertices and nullity $\eta = n - 2k$ ($k = 0, 1, \ldots, \lfloor n/2 \rfloor$). Since $S_{n-k} \cup kK_1$ ($0 \leq k \leq n - 2$) is a bipartite graph, combining Lemma 2.1, the following corollary is obvious.

**Corollary 2.4.** Let $G$ be a bipartite graph of order $n$ with pendent vertices. Then
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(1) There exists no such graphs \( G \) with nullity \( \eta(G) = n \);

(2) \( \eta(G) = n-2 \) if and only if \( G \) is isomorphic to \( S_{n-k} \cup kK_1 \) \((0 \leq k \leq n-2)\).

Let \( \tilde{\Phi}^n_{(n)} \) be the set of all connected bipartite graphs of order \( n \) with nullity \( \eta = n - 2k \) \((k = 0, 1, \ldots, \lfloor n/2 \rfloor)\). It is easy to see that \( \tilde{\Phi}^n_{(n)} = \emptyset, \tilde{\Phi}^n_{(n-2)} = \{K_{n_1, n_2} \mid n_1 + n_2 = n, n_1, n_2 > 0\}\). Let \( n, k, t \) be positive integers such that \( k \) is even, \( 4 \leq k < n \), and \( 1 \leq t \leq \frac{k}{2} - 1 \). Let \( p, n_j, p_j \) \((1 \leq j \leq t)\) be integers such that \( p_j \) is even, \( n_j \geq p_j > 1 \) \((1 \leq j \leq t)\), \( \sum_{j=1}^{t} p_j + 2 = k \), \( \sum_{j=1}^{t} n_j + p + 2 = n \).

Let \( B_n, k \) be a graph of order \( n \) created from \( B_j \in \tilde{\Phi}^{(n_j, p_j)} \) \((j = 1, 2, \ldots, t)\), \( pK_1 \) and \( K_2 \) \((\text{suppose } V(K_2) = \{u, v\})\) by connecting \( v \) to all vertices of \( pK_1 \) and to all vertices in one partite set of \( B_j \) \((j = 1, 2, \ldots, t)\) (also see Figure 1.)

**Theorem 2.5.** Let \( G \) be a bipartite graph \((\text{resp. connected graph})\) of order \( n \) with pendent vertices. Then \( \eta(G) = n - k \) \((k \text{ is even and } 4 \leq k < n)\) if and only if \( G \) is isomorphic to \( B_n, k\{E^*\} \) \((\text{resp. } \tilde{B}_n, k\{E^*\})\), where \( E^* = \bigcup_{j=1}^{t} E(B_j) \cup \{uv\}\).

**Proof.** Note that \( B_n, k\{E^*\} \) \((\text{resp. } \tilde{B}_n, k\{E^*\})\) is a bipartite graph. The proof is now analogous to that of Theorem 2.2. \( \square \)

Let \( Q_4 \) be a graph of order \( n \) created from \( K_{n_1, n_2}, pK_1 \) and \( K_2 \) \((\text{suppose } V(K_2) = \{u, v\})\) with \( n_1 + n_2 + p + 2 = n \) and \( n_1, n_2 > 0 \), \( p \geq 0 \) by connecting \( v \) to all vertices of \( pK_1 \) and all vertices in one partite set of \( K_{n_1, n_2} \). Let \( Q_4 \) be a graph of order \( n \) created from \( O_{m_1}O_{m_2}, O_{m_3}O_{m_4}, pK_1 \) and \( K_2 \) \((V(K_2) = \{u, v\})\) with \( m_i > 0 \) \((i = 1, \ldots, 4)\), \( p \geq 0 \) and \( \sum_{i=1}^{4} m_i + p + 2 = n \) by connecting \( v \) to all vertices of \( O_{m_1} \) \((\text{or } O_{m_2})\), \( O_{m_3} \) \((\text{or } O_{m_4})\) and \( pK_1 \). Let \( Q_5 \) be a graph of order \( n \) created from \( O_{m_1}O_{m_2}O_{m_3}O_{m_4}, pK_1 \) and \( K_2 \) \((V(K_2) = \{u, v\})\) with \( m_i > 0 \) \((i = 1, \ldots, 4)\), \( p \geq 0 \) and \( \sum_{i=1}^{4} m_i + p + 2 = n \) by connecting \( v \) to all vertices of \( pK_1, O_{m_3}, O_{m_4} \) \((\text{or } pK_1, O_{m_2}, O_{m_4})\). Let \( Q_6 \) be a graph of order \( n \) created from \( O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}, pK_1 \) and \( K_2 \) \((V(K_2) = \{u, v\})\) with \( m_i > 0 \) \((i = 1, \ldots, 5)\), \( p \geq 0 \) and \( \sum_{i=1}^{5} m_i + p + 2 = n \) by connecting \( v \) to all vertices of \( pK_1, O_{m_3}, O_{m_4} \) \((\text{or } pK_1, O_{m_2}, O_{m_4})\) (see Figure 3.).
Corollary 2.6. Let $G$ be a bipartite graph (resp. connected graph) of order $n$ with pendent vertices. Then

(1) $\eta(G) = n - 4$ if and only if $G$ is isomorphic to $Q_3\{E^*\}$ (resp. $Q_3\{E^*\}$), where $E^* = E(K_{n_1}, n_2) \cup \{uv\}$. \\

(2) $\eta(G) = n - 6$ if and only if $G$ is isomorphic to $Q_4\{E_1^*\}$, $Q_5\{E_2^*\}$ or $Q_6\{E_3^*\}$ (resp. $Q_4\{E_1^*\}$, $Q_5\{E_2^*\}$ or $Q_6\{E_3^*\}$), where $E_1^* = E(O_{m_1}O_{m_2}) \cup E(O_{m_3}O_{m_4}) \cup \{uv\}$, $E_2^* = E(O_{m_1}O_{m_2}O_{m_3}O_{m_4}) \cup \{uv\}$, $E_3^* = E(O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}) \cup \{uv\}$. \\

Proof. (1) Note that $\eta(G) = n - 4$ implies $t = 1$, $p_1 = 2$. Since $\Phi_n^{(n-2)} = \{K_{n_1}, n_2 \mid n_1 + n_2 = n$, and $n_1, n_2 > 0\}$, by Theorem 2.5, the result follows.

(2) Notice that $\eta(G) = n - 6$ implies the following two cases: Case 1. $t = 1$, $p_1 = 4$; Case 2. $t = 2$, $p_1 = 2$, $p_2 = 2$. By Lemma 1.3, we have $\Phi_n^{(n-4)} = \{O_{m_1}O_{m_2}O_{m_3}O_{m_4}, O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}\}$, $\Phi_n^{(n-2)} = \{O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}\}$. Thus the results are obtained by applying Theorem 2.5 to Cases 1 and 2. \[\Box\]

3. The minimum and maximum (connected) graphs with pendent vertices and nullity $\eta$. In this section, we shall determine the number $\tilde{e}_{\min}^{(\eta)}$, $\tilde{e}_{\max}^{(\eta)}$, $\tilde{e}_{\min}^{(\eta)}$, $\tilde{e}_{\max}^{(\eta)}$ and characterize $\tilde{G}_{\min}^{(\eta)}$, $\tilde{G}_{\max}^{(\eta)}$. $\tilde{G}_{\min}^{(\eta)}$, $\tilde{G}_{\max}^{(\eta)}$ for $0 < \eta \leq n$.

Note that there exists no graph $G$ of order $n$ with pendent vertices and nullity $\eta(G) = n$, $n - 1$, $n - 3$ by Lemma 2.1, so we exclude these three cases.

Theorem 3.1. $G_{\min}^{(n-2k)} \cong kK_2 \cup (n-2k)K_1$, $\tilde{e}_{\min}^{(n-2k)} = k$, where $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$.

Proof. Suppose $|E(G_{\min}^{(n-2k)})| = i$ and there are $j$ nontrivial connected components $G_{11}, G_{12}, \ldots, G_{1j}$ of $G_{\min}^{(n-2k)}$. Then $j \leq i$.

Claim 1. $|E(G_{1t})| = k$. By contradiction, suppose $i < k - 1$.

Note that $|V(G_{1t})| \leq |E(G_{1t})| + 1$ ($t = 1, 2, \ldots, j$). It follows that

$$r(G_{\min}^{(n-2k)}) = \sum_{t=1}^{j} r(G_{1t}) \leq \sum_{t=1}^{j} |V(G_{1t})| \leq \sum_{t=1}^{j} |E(G_{1t})| + j = i + j \leq 2k - 2.$$
Hence \( \eta(G^{(n-2k)}_{\min}) = n - r(G^{(n-2k)}_{\min}) \geq n - 2k + 2 \), a contradiction.

Hence \( i \geq k \). Note that \( \eta(kK_2 \cup (n-2k)K_1) = n - 2k \), and \( |E(kK_2 \cup (n-2k)K_1)| = k \), then we have \( |E(G^{(n-2k)}_{\min})| = k \).

**Claim 2.** There are \( k \) nontrivial connected components of \( G^{(n-2k)}_{\min} \).

Since \( |E(G^{(n-2k)}_{\min})| = k \), we have \( j \leq k \). Assume that \( j \leq k - 1 \).

Notice that \( |V(G_{1t})| \leq |E(G_{1t})| + 1 \) \( (t = 1, 2, \ldots, j) \), hence

\[
r(G^{(n-2k)}_{\min}) = \sum_{t=1}^{j} r(G_{1t}) \leq \sum_{t=1}^{j} |E(G_{1t})| + j = k + j \leq 2k - 1.
\]

It is a contradiction that \( n - 2k = \eta(G^{(n-2k)}_{\min}) = n - r(G^{(n-2k)}_{\min}) \geq n - 2k + 1 \).

Hence \( j = k \). Combining Claims 1 and 2, \( G^{(n-2k)}_{\min} \) is isomorphic to a graph with \( k \) edges and \( k \) nontrivial connected components. Clearly, \( G^{(n-2k)}_{\min} \cong kK_2 \cup (n-2k)K_1 \), and \( e^{(n-2k)}_{\min} = |E(G^{(n-2k)}_{\min})| = k \), where \( k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \).

**Theorem 3.2.** \( G^{(n-2k-1)}_{\min} \cong K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1 \), and \( e^{(n-2k-1)}_{\min} = k + 2 \), where \( k = 2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \).

**Proof.** Suppose that \( |E(G^{(n-2k-1)}_{\min})| = i \) and there are \( j \) nontrivial connected components \( G_{11}, G_{12}, \ldots, G_{1j} \) of \( G^{(n-2k-1)}_{\min} \).

**Claim 1.** There are at most \( k \) nontrivial connected components of \( G^{(n-2k-1)}_{\min} \).

By contradiction, suppose \( j \geq k + 1 \). By Lemma 1.4, \( \eta(G_{1t}) \leq |V(G_{1t})| - 2 \) \( (t = 1, 2, \ldots, j) \) and \( \eta(G^{(n-2k-1)}_{\min}) = \sum_{t=1}^{j} \eta(G_{1t}) + z \), where \( z \) is the number of isolated vertices of \( G^{(n-2k-1)}_{\min} \). Hence \( n - 2k - 1 = \eta(G^{(n-2k-1)}_{\min}) = \sum_{t=1}^{j} \eta(G_{1t}) + z \leq \sum_{t=1}^{j} (|V(G_{1t})| - 2) + z \leq n - 2j \leq n - 2k - 2 \), a contradiction.

**Claim 2.** \( |E(G^{(n-2k-1)}_{\min})| = k + 2 \).

Note that \( |V(G_{11})| \leq |E(G_{1t})| + 1 \) \( (t = 1, 2, \ldots, j) \). Thus

\[
r(G^{(n-2k-1)}_{\min}) = \sum_{t=1}^{j} r(G_{1t}) \leq \sum_{t=1}^{j} |V(G_{1t})| \leq \sum_{t=1}^{j} |E(G_{1t})| + j = i + j.
\]

It follows that

\[
n - 2k - 1 = \eta(G^{(n-2k-1)}_{\min}) = n - r(G^{(n-2k-1)}_{\min}) \geq n - i - j.
\]

Hence \( i + j \geq 2k + 1 \). Since \( j \leq k \) by Claim 1, we have \( i \geq k + 1 \).

If \( i = k + 1 \), then \( j = k \). Thus \( G^{(n-2k-1)}_{\min} \cong K_1, 2 \cup (k-1)K_2 \cup (n-2k-1)K_1 \).

However, \( \eta(K_1, 2 \cup (k-1)K_2 \cup (n-2k-1)K_1) = n - 2k \neq n - 2k - 1 \).
Thus \( i \geq k + 2 \). Note that \( \eta(K_3 \cup (k - 1)K_2 \cup (n - 2k - 1)K_1) = n - 2k - 1 \), and \( |E(K_3 \cup (k - 1)K_2 \cup (n - 2k - 1)K_1)| = k + 2 \). Then \( |E(G_{\text{min}}^{(n-2k-1)})| = k + 2 \).

By Claim 2, \( |E(G_{\text{min}}^{(n-2k-1)})| = i = k + 2 \), and it follows that \( i + j = (k + 2) + j \geq 2k + 1 \). Combining this with Claim 1, we have \( j = k - 1 \) or \( k \).

**Case 1.** \( j = k - 1 \). First we show that there is no nontrivial connected components which are isomorphic to \( P_3 \). Suppose to the contrary that \( G_{11} \cong P_3 \).

Note that \( r(P_3) = 2 \) by Lemma 1.6 and \( \sum_{t=2}^{j} |E(G_{11})| = k \). Hence

\[
r(G_{\text{min}}^{(n-2k-1)}) = r(P_3) + \sum_{t=2}^{j} r(G_{11})
\leq r(P_3) + \sum_{t=2}^{j} |V(G_{11})| \leq r(P_3) + \sum_{t=2}^{j} |E(G_{11})| + (j - 1) = 2k.
\]

Thus \( n - 2k - 1 = \eta(G_{\text{min}}^{(n-2k-1)}) = n - r(G_{\text{min}}^{(n-2k-1)}) \geq n - 2k \), a contradiction.

Therefore, \( G_{\text{min}}^{(n-2k-1)} \) may be isomorphic to one of the following:

1. \( T_1 = C_4 \cup (k - 2)K_2 \cup (n - 2k)K_1 \);
2. \( T_2 = P_4 \cup (k - 2)K_2 \cup (n - 2k - 1)K_1 \);
3. \( T_3 = T^* \cup (k - 2)K_2 \cup (n - 2k)K_1 \), where \( T^* \) is a graph of order 4 created from \( C_3 \) and \( K_2 \) by identifying a vertex of \( C_3 \) with a vertex of \( K_2 \);
4. \( T_4 = T^{**} \cup (k - 2)K_2 \cup (n - 2k - 1)K_1 \), where \( T^{**} \) is a graph of order 5 created from \( K_2 \) and \( S_3 \) by connecting the center of \( S_3 \) to a vertex of \( K_2 \);
5. \( T_5 = S_5 \cup (k - 2)K_2 \cup (n - 2k - 1)K_1 \).

By Lemmas 1.4 and 1.6, we get \( \eta(T_1) = \eta(T_5) = n - 2k + 2 \neq n - 2k - 1 \), \( \eta(T_2) = \eta(T_3) = \eta(T_4) = n - 2k \neq n - 2k - 1 \). Hence \( j \neq k - 1 \).

**Case 2.** \( j = k \). \( G_{\text{min}}^{(n-2k-1)} \) may be isomorphic to one of the following:

1. \( U_1 = K_3 \cup (k - 1)K_2 \cup (n - 2k - 1)K_1 \);
2. \( U_2 = K_1 \cup (k - 1)K_2 \cup (n - 2k - 2)K_1 \);
3. \( U_3 = P_4 \cup (k - 1)K_2 \cup (n - 2k - 2)K_1 \);
4. \( U_4 = 2K_1 \cup (k - 2)K_2 \cup (n - 2k - 2)K_1 \).

It is not difficult to check that \( \eta(U_1) = n - 2k - 1 \), \( \eta(U_2) = \eta(U_4) = n - 2k \neq n - 2k - 1 \), \( \eta(U_3) = n - 2k - 2 \neq n - 2k - 1 \).
Graphs with Pendent Vertices and Given Nullity

All in all, $G_{\min}^{(n-2k-1)} \cong U_1 = K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1$, and $e_{\min}^{(n-2k-1)} = k + 2$, where $k = 2, 3, \ldots, \lfloor \frac{n-1}{2} \rfloor$.

Let $S_{n_j}$ be a star of order $n_j$ where $j = 1, 2, \ldots, k$ and $\sum_{j=1}^{k} n_j = n$. Let $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$ denote a tree of order $n$ created from $S_{n_j}$ $(j = 1, 2, \ldots, k)$ by adding $k-1$ edges to connect these stars, but the connection of two non-center vertices (not the center of a star) is not permitted. It is easy to see that $(2 \leq p \leq k)$ can be constructed recurrently by connecting the center of $S_{n_p}$ to one vertex of $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{p-1}}$.

Now $\tilde{G}_{\min}^{(n-2k)}$ can be characterized for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ as follows.

**Theorem 3.3.** $\tilde{G}_{\min}^{(n-2k)} \cong S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$, $\tilde{e}_{\min}^{(n-2k)} = n - 1$, where $\sum_{j=1}^{k} n_j = n$ and $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

**Proof.** On one hand, by the definition of $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$, there is a pendent vertex $u_{n_k}$ which is adjacent to the center of $S_{n_k}$. Then

$$\eta(S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}) = \eta(S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-1}}) + \eta((n_k - 2)K_1)$$

$$= \eta(S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-1}}) + (n_k - 2)$$

$$= \cdots = \eta(S_{n_1}) + \sum_{i=2}^{k} (n_i - 2) = n - 2k.$$ 

On the other hand we prove that $\tilde{G}_{\min}^{(n-2k)}$ is isomorphic to $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$ by induction on $k$, where $\sum_{j=1}^{k} n_j = n$ and $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

For $k = 1$, by Lemma 2.1, $\tilde{G}_{\min}^{(n-2)} \cong S_n$. Thus, the statement holds in this case. Suppose the statement holds for $k \leq p - 1$. Now we consider the case of $k = p$, where $2 \leq p \leq \lfloor \frac{n}{2} \rfloor$.

**Claim 1.** It’s obvious that for any connected graph of order $n$, the minimum connected graph is a tree which has $n - 1$ edges.

**Claim 2.** If $T$ is a tree of order $n$ with $\eta(T) = n - l$, then $l$ is even.

Note that a tree $T$ could be decomposed into $t$ (with possibly $t = 0$) isolated vertices by deleting a pendent vertex and its adjacent vertex from $T$ (and its resultant graph, suppose $s$ times) recurrently. Hence $r(T) = r(tK_1) + 2s = 2s$, and then $\eta(T) = n - r(T) = n - 2s$. Therefore, $l = 2s$ is even.

Notice that $\tilde{G}_{\min}^{(n-2p)}$ has pendent vertices and $\eta(\tilde{G}_{\min}^{(n-2p)}) = n - 2p$. Choose a pendent vertex, say $x$, in $\tilde{G}_{\min}^{(n-2p)}$. Let $N(x) = \{y\}$. Delete $x$, $y$ from $\tilde{G}_{\min}^{(n-2p)}$, and...
let the resultant graph be $\tilde{G}_1 = \tilde{G}_{11} \cup \tilde{G}_{12} \cup \cdots \cup \tilde{G}_{1q} \cup zK_1$, where $\tilde{G}_{1j}$ are nontrivial connected components of order $n_j^* \ (j = 1, 2, \ldots, q)$, and $\sum_{j=1}^q n_j^* + z + 2 = n$.

By the definition of $\tilde{G}^{(n-2p)}$ and Claim 1, each nontrivial connected component $\tilde{G}_{1j}$ should be a tree with $n_j^* - 1$ edges $(j = 1, 2, \ldots, q)$. Moreover, it follows from Claim 2 that we suppose $\eta(\tilde{G}_{1j}) = n_j^* - p_j$, where $p_j$ is even and $0 < p_j \leq n_j^*$ $(1 \leq j \leq q)$. By Theorem 2.2, we have $\sum_{j=1}^q p_j + 2 = 2p$.

Let $p_j = 2k_j$, and then $k_j = \frac{p_j}{2} \leq p - 1 \ (j = 1, 2, \ldots, q)$. According to the inductive assumption, $\eta(\tilde{G}_{1j}) = n_j^* - 2k_j$, each $\tilde{G}_{1j}$ is isomorphic to $S^*_n \oplus S^*_{j_2} \oplus \cdots \oplus S^*_{j_h}$, where $\sum_{i=1}^h n_{j_i}^* = n_j^* \ (1 \leq j \leq q)$.

In order to recover the connected graph $\tilde{G}^{(n-2p)}$, to add $x$, $y$ to $\tilde{G}_1$, we need to insert edges from $y$ to each of $z$ isolated vertices of $\tilde{G}_1$ and $x$. This gives a star $K_1$, $z+1 = S_{z+2}$. Moreover, we shall connect the vertex $y$ (namely, the center of $S_{z+2}$) to one vertex of each $\tilde{G}_{1j}$ $(j = 1, 2, \ldots, q)$. So $\tilde{G}^{(n-2p)}$ is a tree of order $n$ created from $S_{n_j}^*$ $(i = 1, 2, \ldots, k_j; j = 1, 2, \ldots, p)$ and $S_{z+2}$ by adding $\sum_{j=1}^q k_j = p - 1$ edges to connect these stars, and any two non-center vertices are not connected since $y$ is the center of $S_{z+2}$.

All in all, it follows from the induction that $\tilde{G}^{(n-2p)}_{\min} \cong S^*_n \oplus S^*_{n_2} \oplus \cdots \oplus S^*_{n_k}$, and then $\bar{e}^{(n-2k)}_{\min} = n - 1$, where $\sum_{j=1}^k n_j = n$ and $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$.

Let $C_{2h+1}$ be a $(2h+1)$-cycle and let $S_{n_j}$ be a star of order $n_j$, where $1 \leq h < k$, $1 \leq j \leq k - h$ and $(2h+1) + \sum_{j=1}^{k-h} n_j = n$. Let $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$ denote a unicyclic connected graph of order $n$ created from $C_{2h+1}$ $(1 \leq h < k)$ and $S_{n_j}$ $(j = 1, 2, \cdots, k-h)$ by adding $k-h$ edges to connect them, but the connection of two non-center vertices is not permitted. It is easy to see that $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_p}$ $(1 \leq p \leq k - h)$ can be constructed recurrently by connecting the center of $S_{n_p}$ to one vertex of $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{p-1}}$.

**Theorem 3.4.** $\tilde{G}^{(n-2k-1)}_{\min} \cong C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$, $\bar{e}^{(n-2k-1)}_{\min} = n$, where $1 \leq h < k$, $(2h+1) + \sum_{j=1}^{k-h} n_j = n$ and $k = 2, 3, \ldots, \left[\frac{n-1}{2}\right]$.

Proof. By the definition of $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$,

$\eta(C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h}}) = \eta(C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h-1}}) + \eta((n_{k-h} - 2)K_1) = \cdots = \eta(C_{2h+1}) + \sum_{i=1}^{k-h} (n_i - 2) = 0 + (\sum_{i=1}^{k-h} n_i - 2k + 2h) = n - 2k - 1$.

On the other hand, we show that $\tilde{G}^{(n-2k-1)}_{\min}$ is isomorphic to $C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h}}$ by induction on $k$, where $1 \leq h < k$ and $(2h+1) + \sum_{j=1}^{k-h} n_j = n$.

For $k = 2$, we have $h = 1$, and it follows from Corollary 2.3 (2) that $\tilde{G}^{(n-5)}_{\min} \cong$
$C_3 \oplus S_{n-3}$. Therefore, the statement holds in this case. Suppose the statement holds for $k \leq p - 1$. We consider the case of $k = p$, where $3 \leq p \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Note that $\tilde{G}^{(n-2p-1)}$ has pendant vertices and $\eta(\tilde{G}^{(n-2p-1)}) = n-2p-1$. Choose a pendant vertex, say $x$, in $\tilde{G}^{(n-2p-1)}_{\text{min}}$. Let $N(x) = \{y\}$. Delete $x$, $y$ from $\tilde{G}^{(n-2p-1)}_{\text{min}}$, and let the resultant graph be $G_1 = \tilde{G}_{11} \cup \cdots \cup \tilde{G}_{1q} \cup zK_1$, where $G_{1j}$ are nontrivial connected components of order $n^*_j$ (for $j = 1, 2, \ldots, q$), and $\sum_{j=1}^q n^*_j + z + 2 = n$.

Assume that $\eta(\tilde{G}_{11}) = n^*_1 - l^*_1$ (for $j = 1, 2, \ldots, q$) is even by Theorem 3.3 Claim 2, and

$$2p + 1 = n - \eta(\tilde{G}^{(n-2p-1)}_{\text{min}}) = n - \left(\sum_{j=1}^q \eta(\tilde{G}_{1j}) + z\right) = 2 + \sum_{j=1}^q l^*_j,$$

a contradiction. Since the number of edges for $\tilde{G}^{(n-2p-1)}_{\text{min}}$ should be as least as possible, and $C_{2h+1} \oplus S_{n^*_1} \oplus \cdots \oplus S_{n^*_p \cap h}$ with nullity $n - 2p - 1$ which satisfies this claim, it follows that Claim 1 holds.

**Claim 2.** $l^*_1$ is odd. Otherwise, we get a similar contradiction as Claim 1.

**Claim 3.** Let $l^*_1 = 2t^* + 1$. Then $\tilde{G}_{11} \cong C_{2t^*+1}$ (for $n^*_1 = 2t^* + 1$), or $\tilde{G}_{11} \cong C_{2h_1+1} \oplus S_{n^*_1} \oplus \cdots \oplus S_{n^*_1 \cap h_1}$, where $1 \leq h_1 < t^*$, $(2h_1 + 1) + \sum_{j=2}^q t^*_j - h^*_1 = n^*_1$.

**Case 1.** If $\tilde{G}_{11}$ has pendant vertices, since $t^* = \frac{l^*_1 - 1}{2} \leq p - 1$ (note that $\sum_{j=2}^q t^*_j = 2p - 1$) and $\eta(\tilde{G}_{11}) = n^*_1 - 2t^* - 1$, according to the inductive assumption, $\tilde{G}_{11} \cong C_{2h_1+1} \oplus S_{n^*_1 \cap h_1} \oplus \cdots \oplus S_{n^*_1 \cap h_1}$, where $1 \leq h_1 < t^*$, $(2h_1 + 1) + \sum_{j=1}^q t^*_j = n^*_1$.

**Case 2.** If $\tilde{G}_{11}$ has no pendant vertex, since $\tilde{G}_{11}$ is an unicyclic connected graph, $\tilde{G}_{11}$ is an odd cycle of order $n^*_1$. Hence $\tilde{G}_{11} \cong C_{2t^*+1}$ and $l^*_1 = 2t^* + 1 = n^*_1$.

**Claim 4.** Combining Claim 1 with Theorem 3.3, each $\tilde{G}_{1j}$ (for $2 \leq j \leq q$) is isomorphic to $S_{n^*_1 \cap h_1} \oplus S_{n^*_1 \cap h_2} \oplus \cdots \oplus S_{n^*_1 \cap h_{k_j}}$, where $\sum_{i=1}^{k_j} n^*_j - n^*_1$ and $l^*_j = 2k_j$.

In order to recover the connected graph $\tilde{G}^{(n-2p-1)}_{\text{min}}$, to add $x$, $y$ to $\tilde{G}_1$, we insert edges from $y$ to each of $z$ isolated vertices of $\tilde{G}_1$ and $x$. This gives a star $K_1, z + 1 = S_{z+2}$. Moreover, we shall connect the vertex $y$ (namely, the center of $S_{z+2}$) to one vertex of each $\tilde{G}_{1j}$ (for $j = 1, 2, \ldots, q$). Let $t^* - h_1 = k_1$. Then $\tilde{G}^{(n-2p-1)}_{\text{min}}$ is an unicyclic connected graph of order $n$ created from $C_{2h_1+1}$, $S_{n^*_1 \cap h_1}$, $(i = 1, 2, \ldots, k_j; j = 1, 2, \ldots, p)$ and $S_{z+2}$ by adding $\sum_{j=1}^q k_j + 1 = p - h_1$. 
(1 ≤ h1 < p) edges to connect these graphs, and any two non-center vertices are not connected since y is the center of S_{z+2}.

In conclusion,
\[ \tilde{G}_{\min}^{(n-2k-1)} \cong C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k-h}, \]
and then \( \bar{e}_{\min}^{(n-2k-1)} = n \), where \( 1 \leq h < k \), \( 2h + 1 + \sum_{j=1}^{k-h} n_j = n \) and \( k = 2, 3, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \).

The following lemma describes the relationship between \( G_{\max}^{(n)} \) and \( \tilde{G}_{\max}^{(n)} \).

**Lemma 3.5.** \( G_{\max}^{(n)} \cong \tilde{G}_{\max}^{(n)} \), \( e_{\max}^{(n)} = \bar{e}_{\max}^{(n)} \), where \( 0 < \eta \leq n \).

**Proof.** Since we want to insert edges as many as possible, by Lemma 2.1 and Theorem 2.2, this lemma is proved.

Now \( G_{\max}^{(n)} \) (namely, \( \tilde{G}_{\max}^{(n)} \)) is characterized for \( \eta = n - 2, n - 4, n - 5 \).

**Theorem 3.6.** \( G_{\max}^{(n-2)} \cong \tilde{G}_{\max}^{(n-2)} \cong S_n, e_{\max}^{(n-2)} = \bar{e}_{\max}^{(n-2)} = n - 1 \).

**Proof.** By Lemma 2.1 (2), we obtain the results as desired.

Let \( U_{\max}^{(n-4)} \) be a graph of order \( n \) created from \( K_{\left\lfloor \frac{n}{2} \right\rfloor - 1}, \left\lfloor \frac{n}{2} \right\rfloor - 1 \) and \( K_2 \) by connecting a vertex \( v \) of \( K_2 \) to all vertices of \( K_{\left\lfloor \frac{n}{2} \right\rfloor - 1}, \left\lfloor \frac{n}{2} \right\rfloor - 1 \).

**Theorem 3.7.** \( G_{\max}^{(n-4)} \cong \tilde{G}_{\max}^{(n-4)} \cong U_{\max}^{(n-4)}, e_{\max}^{(n-4)} = \bar{e}_{\max}^{(n-4)} = \left\lfloor \frac{n^2}{4} \right\rfloor \).

**Proof.** By Corollary 2.3 (1), \( G_{\max}^{(n-4)} \) should be a graph \( Q_{\max}^{(n-4)} \) of order \( n \) created from \( K_{n_1}, n_2, pK_1 \) and \( K_2 \) such that \( n_1 + n_2 + p + 2 = n \) and \( n_1, n_2 > 0, p \geq 0 \) by connecting a vertex \( v \) of \( K_2 \) to all vertices of \( pK_1 \) and \( K_{n_1}, n_2 \).

Since \( n_2 = n - n_1 - p - 2 \) and \( n_1, n_2 > 0, p \geq 0 \), we have
\[
|E(Q_{\max}^{(n-4)})| = n_1n_2 + n - 1 = -n_1^2 + (n - p - 2)n_1 + (n - 1) \\
\leq -n_1^2 + (n - 2)n_1 + (n - 1) \\
= -(n_1 - \frac{n}{2} + 1)^2 + \frac{n^2}{4} \\
\leq \begin{cases} 
\frac{n^2}{4}, & \text{if } n \text{ is even;} \\
\frac{n^2-1}{4}, & \text{if } n \text{ is odd.}
\end{cases}
\]
where the first equality holds if and only if \( p = 0 \), and the second equality holds if and only if \( n_1 = \frac{n}{2} - 1 \) (n is even); \( n_1 = \frac{n-1}{2} - 1 \) or \( \frac{n+1}{2} - 1 \) (n is odd), which implies that \( n_2 = \frac{n}{2} - 1 \) (n is even); \( n_2 = \frac{n+1}{2} - 1 \) or \( \frac{n-1}{2} - 1 \) (n is odd).

Combining Lemma 3.5, it follows that \( G_{\max}^{(n-4)} \cong \tilde{G}_{\max}^{(n-4)} \cong U_{\max}^{(n-4)} \).
Moreover, \( e_{\text{max}}^{(n-4)} = \bar{e}_{\text{max}}^{(n-4)} = \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4}, & \text{if } n \text{ is odd.} \end{cases} \)

Let \( U_{\text{max}}^{(n-5)} \) be a graph of order \( n \) created from

\[
U^* = \begin{cases} 
K\frac{n+3}{2}, \frac{n+3}{2}, \frac{n+3}{2}, & n \equiv 2 \pmod{3} \\
K\frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, & n \equiv 0 \pmod{3} \\
K\frac{n-4}{2}, \frac{n-4}{2}, \frac{n-4}{2}, & n \equiv 1 \pmod{3}
\end{cases}
\]

and \( K_2 \) by connecting a vertex \( v \) of \( K_2 \) to all vertices of \( U^* \).

**Theorem 3.8.** \( G_{\text{max}}^{(n-5)} \cong \bar{G}_{\text{max}}^{(n-5)} \cong U_{\text{max}}^{(n-5)} \), \( e_{\text{max}}^{(n-5)} = \bar{e}_{\text{max}}^{(n-5)} = \left\lfloor \frac{n^2-n+1}{3} \right\rfloor \).

**Proof.** By Corollary 2.3 (2), \( G_{\text{max}}^{(n-5)} \) is isomorphic to a graph \( C_{\text{max}} \) of order \( n \) created from \( K_{n_1, n_2, n_3} \) \( pK_1 \) and \( K_2 \) satisfying \( n_1 + n_2 + n_3 + p + 2 = n \) and \( n_1, n_2, n_3 > 0, p \geq 0 \) by connecting a vertex \( v \) of \( K_2 \) to all vertices of \( pK_1 \) and \( K_{n_1, n_2, n_3} \).

Since \( n_3 = n - n_1 - n_2 - p - 2 \) and \( n_1, n_2, n_3 > 0, p \geq 0 \), we have

\[
|E(C_{\text{max}})| = n_1n_2 + n_2n_3 + n_3n_1 + n - 1 \\
= -(n_1 + n_2)^2 + (n - 2 - p)(n_1 + n_2) + (n - 1) + n_1n_2 \\
\leq -(n_1 + n_2)^2 + (n - 2 - p)(n_1 + n_2) + (n - 1) + \frac{(n_1 + n_2)^2}{4} \\
= \frac{3}{4}(n - n_3 - p - 2)^2 + (n - 2 - p)(n - n_3 - p - 2) + (n - 1) \\
= \frac{1}{4}[3n_3^2 + 2(n - p - 2)n_3 + (n - p - 2)^2] + (n - 1) \\
\leq \frac{1}{4}[3n_3^2 + 2(n - p - 2)n_3 + (n - 2)^2] + (n - 1) \\
= \frac{3}{4}(n_3 - \frac{n - 2}{3})^2 + \frac{n^2 - n + 1}{3} \leq \begin{cases} \frac{n^2-n+1}{3}, & n - 2 \equiv 0 \pmod{3}; \\
\frac{n^2-n}{3}, & n - 2 \equiv 1 \pmod{3}; \\
\frac{n^2-n}{3}, & n - 2 \equiv 2 \pmod{3}.
\end{cases}
\]

where the first equality holds if and only if \( n_1 = n_2 \), the second equality holds if and only if \( p = 0 \), and the third equality holds if and only if

\[
n_3 = \begin{cases} 
\frac{n-2}{3}, & n - 2 \equiv 0 \pmod{3}; \\
\frac{n}{3}, & n - 2 \equiv 1 \pmod{3}; \\
\frac{n+1}{3}, & n - 2 \equiv 2 \pmod{3}.
\end{cases}
\]

Thus \( n_1 = n_2 = \begin{cases} 
n_3, & n - 2 \equiv 0 \pmod{3}; \\
n_3, & n - 2 \equiv 1 \pmod{3}; \\
n_3, & n - 2 \equiv 2 \pmod{3}.
\end{cases} \)
Hence $G^{(n-5)} \cong U^{(n-5)}_{max}$ and then $e^{(n-5)}_{max} = \begin{cases} \frac{n^2-n+1}{3}, & n - 2 \equiv 0 \pmod{3}; \\ \frac{n^2-n}{3}, & n - 2 \not\equiv 0 \pmod{3}. \end{cases}$

Combining this with Lemma 3.5 gives the desired results. □

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