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ON $\lambda_1$-EXTREMAL NON-REGULAR GRAPHS

BOLIAN LIU†, YUFEI HUANG†, AND ZHIFU YOU†

Abstract. Let $G$ be a connected non-regular graph with $n$ vertices, maximum degree $\Delta$ and minimum degree $\delta$, and let $\lambda_1$ be the greatest eigenvalue of the adjacency matrix of $G$. In this paper, by studying the Perron vector of $G$, it is shown that type-I-a graphs and type-I-b (resp. type-II-a) graphs with some specified properties are not $\lambda_1$-extremal graphs. Moreover, for each connected non-regular graph some lower bounds on the difference between $\Delta$ and $\lambda_1$ are obtained.

Key words. Spectral radius, Non-regular graph, $\lambda_1$-extremal graph, Perron vector, Degree.

AMS subject classifications. 05C50, 15A48.

1. Introduction. Throughout this paper, let $G = (V, E)$ be a connected, simple and undirected graph with vertex set $V$ and edge set $E$, where $|V| = n$. Let $uv$ denote the edge joining vertices $u$ and $v$. For a vertex $u$, let $N(u)$ be the set of all neighbors of $u$, and let $d(u) = |N(u)|$ be the degree of $u$. The maximum and minimum degree of $G$ are denoted by $\Delta$ and $\delta$ respectively. The sequence $\pi = (d_1, d_2, \ldots, d_n)$ (always with $d_1 \geq d_2 \geq \cdots \geq d_n$) is called the degree sequence of $G$, where $d_1, d_2, \ldots, d_n$ are the degrees of the vertices of $G$. Let $G + uv$ be the graph obtained from $G$ by inserting an edge $uv \notin E(G)$, where $u, v \in V(G)$. Similarly, $G - uv$ is the graph obtained from $G$ by deleting the edge $uv \in E(G)$, and $G - v$ is the graph obtained from $G$ by deleting the vertex $v$ and all edges $uv \in E(G)$ for some $u \in V(G)$.

Let $A(G)$ be the adjacency matrix of $G$. The spectral radius of $G$, denoted by $\lambda_1(G)$, is the largest eigenvalue of $A(G)$. Thus, by the Perron-Frobenius Theorem (see [8]), when $G$ is connected, $\lambda_1(G)$ is simple and there is a corresponding unique positive unit eigenvector. We refer to such eigenvector $f$ as the Perron vector of $G$.

Given a degree sequence $\pi$, let $C_\pi$ denote the set of connected graphs with degree sequence $\pi$. We say that the graph $G$ has the greatest maximum eigenvalue in class $C_\pi$ provided $\lambda_1(G) \geq \lambda_1(G^*)$ for every $G^*$ in $C_\pi$.

Let $G$ be a connected non-regular graph. In [11], $G$ is called $\lambda_1$-extremal if $\lambda_1(G) > \lambda_1(G^*)$ for every other connected non-regular graph $G^*$ with the same num-
ber of vertices and maximum degree as \( G \). Let \( g(n, \Delta) \) denote the set of all connected non-regular graphs with \( n \) vertices and maximum degree \( \Delta \).

Let \( G \) be a \( \lambda_1 \)-extremal graph of \( g(n, \Delta) \). As we know that if \( \Delta = 2 \), then \( G \) is necessarily a path and \( \lambda_1(G) = 2\cos(\frac{\pi}{n+1}) \), while \( G \) is isomorphic to \( K_n - e \) and \( \lambda_1(G) = \frac{n-3+\sqrt{(n+1)^2-8}}{2} \) if \( \Delta = n - 1 \) \( (n \geq 4) \) (see [11]). In the following, we can suppose that \( 2 < \Delta < n - 1 \).

Let \( V_\Delta = \{u \mid d(u) = \Delta\} \) and \( V_{<\Delta} = \{u \mid d(u) < \Delta\} \). It has been shown that a \( \lambda_1 \)-extremal graph of \( g(n, \Delta) \) has the following special properties.

**Lemma 1.1.** ([9]) Suppose \( 2 < \Delta < n - 1 \). If \( G \) is a \( \lambda_1 \)-extremal graph of \( g(n, \Delta) \), then \( G \) must have one of the following properties:

1. \( |V_{<\Delta}| \geq 2 \), and \( V_{<\Delta} \) induces (i.e., \( G[V_{<\Delta}] \)) a complete graph.
2. \( |V_\Delta| = 1 \).
3. \( V_{<\Delta} = \{u, v\} \), \( uv \notin E(G) \) and \( d(u) = d(v) = \Delta - 1 \).

Moreover, \( G \in g(n, \Delta) \) is called a type-I (resp. type-II or type-III) graph if \( G \) has the property (1) (resp. (2) or (3)).

By studying the properties of \( \lambda_1 \)-extremal graphs, B. Liu et al. proved that

**Lemma 1.2.** ([10]) Suppose \( 2 < \Delta < n - 1 \) and \( G \) is a \( \lambda_1 \)-extremal graph of \( g(n, \Delta) \), then \( G \) must be a type-I or type-II graph.

Now we divide type-I (resp. type-II) graphs into two classes as follows.

**Definition 1.3.** (1) Let \( G \in g(n, \Delta) \) be a type-I graph. If there exist \( u_1, v_1 \in V_\Delta \) and \( u_2, v_2 \in V_{<\Delta} \) such that \( u_1v_1, v_1v_2 \in E \) and \( u_1v_2, v_1u_2 \notin E \), then \( G \) is called a type-I-a graph. Otherwise, \( G \) is a type-I-b graph.

(2) Let \( G \in g(n, \Delta) \) be a type-II graph. Then \( G \) is called a type-II-a graph if \( \delta < \Delta - 2 \). Otherwise, \( G \) is called a type-II-b graph.

In this paper, by investigating the Perron vector of \( G \), we show that type-I-a graphs are not \( \lambda_1 \)-extremal graphs, and type-I-b (resp. type-II-a) graphs with some specified properties are also not \( \lambda_1 \)-extremal graphs, which provide more evidence to confirm the following conjecture in [9].

**Conjecture 1.4.** ([9]) Let \( G \in g(n, \Delta) \) with \( 2 < \Delta < n - 1 \). Then \( G \) is a \( \lambda_1 \)-extremal graph if and only if \( G \) is a graph with greatest maximum eigenvalue in the class \( C_\pi \), where \( \pi = (\Delta, \Delta, \cdots, \Delta, \delta) \), and \( \delta = \begin{cases} \Delta - 1, & \text{when } n\Delta \text{ is odd,} \\ \Delta - 2, & \text{when } n\Delta \text{ is even.} \end{cases} \)
Let $G$ be a connected non-regular graph of order $n$. Stevanović first derived a lower bound of $\Delta - \lambda_1$ for $G$ in [13]. Later this bound was improved in [3, 4, 11]. Let $D$ (resp. $\bar{d}$) denote the diameter (resp. the average degree) of $G$. In [3, 11], the authors showed that

$$\Delta - \lambda_1 > \frac{1}{nD} \quad ([3])$$

and

$$D \leq \frac{3n + \Delta - 5}{\Delta + 1} \quad ([11]).$$

Thus combining (1.1) and (1.2), we have

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 5)} \quad ([9, 11]).$$

Applying Lemma 1.2, the authors in [9] made further improvement on Inequality (1.2) and obtained the following inequality which improves (1.3).

$$D \leq \frac{3n + \Delta - 8}{\Delta + 1} \quad \text{and} \quad \Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)} \quad ([9]).$$

Recently, L. Shi [12] established another strong inequality as follows.

$$\Delta - \lambda_1 > [(n - \delta)D + \frac{1}{\Delta + d} - \left(\frac{D}{2}\right)^{-1}]^{-1} \quad ([12]).$$

**Remark 1.5.** For most almost regular graphs of constant degree and large order (graphs where $n\Delta - 2m$ is a constant and $D = o(\sqrt{n})$), Inequality (1.1) is better than (1.5). However, for many non-regular graphs, i.e. graphs with $(\Delta - d)(\frac{D}{2} + D\delta) \geq 1$, L. Shi’s inequality is better. For example, in graphs where the diameter is a constant fraction of the number of vertices (1.5) is better.

In Section 3, we obtain the following inequalities which improve (1.4).

$$D \leq \frac{3n + \Delta - 3\delta - 5}{\Delta + 1},$$

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 3\delta - 5)}$$

and

$$\Delta - \lambda_1 > [-\frac{(3n + \Delta - 3\delta - 5)^2}{2(\Delta + 1)^2} + \frac{(0.5 + n - \delta)(3n + \Delta - 3\delta - 5)}{\Delta + 1} + \frac{1}{\Delta - d}]^{-1}.$$
2. **On \( \lambda_1 \)-extremal graphs.** As is known to all, the Rayleigh quotient of the adjacency matrix \( A(G) \) on vectors \( f \) on \( V \) is the fraction

\[
R_G(f) = \frac{\langle Af, f \rangle}{\langle f, f \rangle} = \frac{2 \sum_{uv \in E} f(u)f(v)}{\sum_{v \in V} f(v)^2}.
\]

(8)]

By the Rayleigh-Ritz Theorem we have the following well known property for the spectral radius of \( G \).

**Proposition 2.1.** (8) Let \( S \) denote the set of unit vectors on \( V \). Then

\[
\lambda_1(G) = \max_{f \in S} R_G(f) = 2 \max_{f \in S} \sum_{uv \in E} f(u)f(v).
\]

If \( R_G(f) = \lambda_1(G) \) for a (positive) function \( f \in S \), then \( f \) is a Perron vector.

The following technical lemma is useful in this paper.

**Lemma 2.2.** (Shifting [1, 2]) Let \( G(V, E) \) be a connected graph with \( uv_1 \in E \) and \( uv_2 \notin E \). Let \( G^* = G + uv_2 - uv_1 \). Suppose \( f \) is a Perron vector of \( G \). If \( f(v_2) \geq f(v_1) \), then \( \lambda_1(G^*) > \lambda_1(G) \).

Analogously, we introduce another technique called **Splitting.**

**Lemma 2.3.** (Splitting) Let \( G(V, E) \) be a connected graph with \( u_1u_2 \in E \) and \( u_1u_1, u_2u_2 \notin E \) (maybe \( w_1 = w_2 \)). Let \( G^* = G + u_1w_1 + u_2w_2 - u_1u_2 \). Suppose \( f \) is a Perron vector of \( G \). If \( f(w_1) + f(w_2) \geq \max \{ f(u_1), f(u_2) \} \), then \( \lambda_1(G^*) > \lambda_1(G) \).

**Proof.** Without loss of generality, suppose \( f(u_1) \geq f(u_2) \). Then

\[
R_{G^*}(f) - R_G(f) = \langle A(G^*)f, f \rangle - \langle A(G)f, f \rangle
\]

\[
= 2 \left( \sum_{xy \in E^* - E} f(x)f(y) - \sum_{uv \in E^* - E} f(u)f(v) \right)
\]

\[
= 2[f(u_1)f(w_1) + f(u_2)f(w_2) - f(u_1)f(u_2)]
\]

\[
\geq 2[f(u_2)[f(w_1) + f(w_2)] - f(u_1)f(u_2)]
\]

\[
= 2[f(u_2)f(w_1) + f(w_2) - f(u_1)] \geq 0.
\]

Hence \( \lambda_1(G^*) \geq R_{G^*}(f) \geq R_G(f) = \lambda_1(G) \) by Proposition 2.1. Assume that \( \lambda_1(G^*) = \lambda_1(G) \), which implies \( f \) would also be a Perron vector of \( G^* \). Then

\[
\lambda_1(G^*)f(w_1) = \sum_{xw_1 \in E} f(x) + \sum_{yw_1 \in E^* - E} f(y) > \sum_{xw_1 \in E} f(x) = \lambda_1(G)f(w_1).
\]

This is a contradiction. Consequently, \( \lambda_1(G^*) > \lambda_1(G) \). □

Now let’s turn to the study of \( \lambda_1 \)-extremal graphs.
Theorem 2.4. Let $G = (V, E) \in g(n, \Delta)$ be a type-I-a graph with $2 < \Delta < n - 1$. Then $G$ is not a $\lambda_1$-extremal graph of $g(n, \Delta)$.

Proof. By contradiction, suppose $G$ is a $\lambda_1$-extremal graph.

Since $G \in g(n, \Delta)$ is a type-I-a graph, there exist $u_1, v_1 \in V_\Delta$ and $u_2, v_2 \in V_{< \Delta}$ such that $u_1u_2, v_1v_2 \in E$ and $u_1v_2, v_1u_2 \notin E$. Let $f$ be the Perron vector of $G$. We consider the next two cases:

Case 1. $f(u_2) \geq f(v_2)$. Let $G^* = G - v_1v_2 + u_1u_2$. Note that $G[V_{< \Delta}]$ is a complete graph and $d(u_2) < \Delta$, thus $G^* \in g(n, \Delta)$. By Lemma 2.2, we have $\lambda_1(G^*) > \lambda_1(G)$, which is a contradiction.

Case 2. $f(u_2) \leq f(v_2)$. Let $G^* = G - u_1u_2 + u_1v_2$. Similarly, since $G[V_{< \Delta}]$ is a complete graph and $d(v_2) < \Delta$, we have $G^* \in g(n, \Delta)$. It follows from Lemma 2.2 that $\lambda_1(G^*) > \lambda_1(G)$, also a contradiction.

Example 2.5. Let $G_0$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_8\}$ and edge set $E = \{v_iv_j \mid i, j = 1, 2, \ldots, 5 \text{ and } i \neq j\} \cup \{v_6v_i \mid i = 4, 7, 8\} \cup \{v_7v_i \mid i = 5, 8\} \setminus \{v_4v_5\}$. Note that $G_0 \in g(8, 4)$ is a type-I-a graph. By Theorem 2.4, $G_0$ is not a $\lambda_1$-extremal graph.

Proposition 2.6. Let $G = (V, E) \in g(n, \Delta)$ be a type-I-b graph with $2 < \Delta < n - 1$ and let $f$ be a Perron vector of $G$. Assume that $\delta \neq \Delta - 1$ when $|V_{< \Delta}| = 2$. If there exist $u_1, u_2 \in V_\Delta$, $w_1, w_2 \in V_{< \Delta}$ such that $u_1u_2 \in E$, $u_1w_2, u_2w_1 \notin E$ and $f(u_1) + f(w_2) \geq \max\{f(u_1), f(w_2)\}$, then $G$ is not a $\lambda_1$-extremal graph of $g(n, \Delta)$.

Proof. Let $G^* = G + u_1w_1 + u_2w_2 - u_1u_2$. Note that either $|V_{< \Delta}| \geq 3$ or $|V_{< \Delta}| = 2 (\delta \neq \Delta - 1)$. Because $w_1, w_2 \in V_{< \Delta}$ and $G[V_{< \Delta}]$ is a complete graph, we have $G^* \in g(n, \Delta)$. Since $f(u_1) + f(w_2) \geq \max\{f(u_1), f(w_2)\}$, by Lemma 2.3, $\lambda_1(G^*) > \lambda_1(G)$, which implies that $G$ is not a $\lambda_1$-extremal graph of $g(n, \Delta)$.

Example 2.7. Let $G_1$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_{11}\}$ and edge set $E = \{v_iv_j \mid i, j = 1, \ldots, 6 \text{ and } i \neq j\} \cup \{v_7v_i \mid i = 4, 5, 9, 10, 11\} \cup \{v_8v_i \mid i = 3, 6, 9, 10, 11\} \cup \{v_9v_1, v_9v_11, v_9v_11\} - \{v_3v_6, v_4v_5\}$. It is not difficult to see that $G_1 \in g(11, 5)$ is a type-I-b graph with $\lambda_1 \approx 4.82843$ and degree sequence $(5, 5, 5, 5, 5, 5, 5, 4, 4, 4)$.

Directly calculating, $f(v_1) \approx 0.36725$ ($i = 1, 2$), $f(v_j) \approx 0.35150$ ($3 \leq j \leq 6$), $f(v_k) \approx 0.25969$ ($k = 7, 8$), and $f(v_l) \approx 0.18363$ ($l = 9, 10, 11$). Note that $v_3v_6 \in E$, $v_5v_9, v_6v_10 \notin E$ and $f(v_9) + f(v_10) \geq \max\{f(v_9), f(v_10)\}$, it follows from Proposition 2.6 that $G_1$ is not a $\lambda_1$-extremal graph.

Proposition 2.8. Let $G = (V, E) \in g(n, \Delta)$ be a type-II-a graph with $2 < \Delta < n - 1$ and let $f$ be a Perron vector of $G$. Suppose $V_{< \Delta} = \{w\}$ and $d(w) = \delta$. If there
exist $u_1, u_2 \in V_G$ such that $u_1u_2 \in E$, $u_1w, u_2w \notin E$ and $2f(w) \geq \max\{f(u_1), f(u_2)\}$, then $G$ is not a $\lambda_1$-extremal graph of $g(n, \Delta)$.

**Proof.** Let $G^* = G + u_1w + u_2w - u_1u_2$. Since $G$ is a type-II-a graph, we have $d(w) = \delta < \Delta - 2$, and then $G^* \notin g(n, \Delta)$. Note that $2f(w) \geq \max\{f(u_1), f(u_2)\}$, by Lemma 2.3, $\lambda_1(G^*) > \lambda_1(G)$. Therefore, $G$ is not a $\lambda_1$-extremal graph. \[\square\]

**Example 2.9.** Let $G_2$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_9\}$ and edge set $E = \{v_iv_j \mid i, j = 1, \ldots, 8, i \neq j\} \cup \{v_9v_1 \mid i = 5, \ldots, 8\} - \{v_5v_7, v_6v_8\}$.

It is easy to see that $G_2 \notin g(9, 7)$ is a type-II-a graph with $\lambda_1 \approx 6.79944$ and degree sequence $(7, 7, 7, 7, 7, 7, 7, 4)$. Directly computing, $f(v_i) \approx 0.35531$ $(1 \leq i \leq 4)$, $f(v_j) \approx 0.33749$ $(5 \leq j \leq 8)$, and $f(v_9) \approx 0.19854$. Note that $v_1v_2 \in E$, $v_1v_9, v_2v_9 \notin E$ and $2f(v_9) \geq \max\{f(v_1), f(v_2)\}$, by Proposition 2.8, we conclude that $G_2$ is not a $\lambda_1$-extremal graph.

**Remark 2.10.** By Theorem 2.4, Propositions 2.6 and 2.8, type-I-a graphs, and type-II-b graphs (resp. type-II-a graphs) with some specified properties are not $\lambda_1$-extremal graphs. In other words, if $G$ is a $\lambda_1$-extremal graph of $g(n, \Delta)$, then $G$ is most likely to be a type-II-b graph ($|V_{\Delta}| = 1$ and $\delta = \Delta - 2$ or $\Delta - 1$). Hence this provides more evidence to confirm Conjecture 1.4.

**Remark 2.11.** Conjecture 1.4 is true for small $n$, where $n \leq 7$. Since $2 < \Delta < n - 1$, it need to be verified for $n = 5, 6, 7$ as follows.

The $\lambda_1$-extremal graph of $g(5, 3)$ is the Graph 1.17 ([7], pp. 273) with the degree sequence $\pi = (3, 3, 3, 3, 2)$, and $\lambda_1 \approx 2.8558$.

The $\lambda_1$-extremal graph of $g(6, 3)$ is the Graph 65 ([5]) with the degree sequence $\pi = (3, 3, 3, 3, 3, 1)$, and $\lambda_1 \approx 2.895$.

The $\lambda_1$-extremal graph of $g(6, 4)$ is the Graph 14 ([5]) with the degree sequence $\pi = (4, 4, 4, 4, 4, 2)$, and $\lambda_1 \approx 3.820$.

The $\lambda_1$-extremal graph of $g(7, 3)$ is the Graph 10-261 ([6], pp. 193) with the degree sequence $\pi = (3, 3, 3, 3, 3, 3, 2)$, and $\lambda_1 \approx 2.9107$.

The $\lambda_1$-extremal graph of $g(7, 4)$ is the Graph 13-643 ([6], pp. 218) with the degree sequence $\pi = (4, 4, 4, 4, 4, 4, 2)$, and $\lambda_1 \approx 3.8558$.

The $\lambda_1$-extremal graph of $g(7, 5)$ is the Graph 17-835 ([6], pp. 231) with the degree sequence $\pi = (5, 5, 5, 5, 5, 5, 4)$, and $\lambda_1 \approx 4.8809$.

3. The largest eigenvalue of non-regular graphs.
On $\lambda_1$-Extremal Non-regular Graphs

**Theorem 3.1.** Let $G \in g(n, \Delta)$ ($2 < \Delta < n - 1$) be a type-I-$b$ graph (or type-II graph) with diameter $D$ and minimum degree $\delta$. Then

$$D \leq \frac{3n + \Delta - 3\delta - 5}{\Delta + 1}.$$

**Proof.** Since $G \in g(n, \Delta)$ with $2 < \Delta < n - 1$ is non-regular, we have $D \geq 2$. Let $u, v$ be vertices at distance $D$ and let $P : u = u_0 \leftrightarrow u_1 \leftrightarrow \cdots \leftrightarrow u_D = v$ be a shortest path connecting $u$ and $v$. We observe $|V_{<\Delta} \cap V(P)| \leq 2$. Otherwise $G$ is a type-I-$b$ graph. Assume \{ur, ru, ur\} $\subseteq V_{<\Delta} \cap V(P)$ with $p < q < r$. Since $G[V_{<\Delta}]$ is a complete graph, $u_pu_r \in E(G)$, contradicting the choice of $P$.

**Case 1.** $V_{<\Delta} \cap V(P) = \{u_p, u_p+1\}.$

**Subcase 1.1.** $D \equiv 2 \pmod{3}$. Define $T = \{i \mid i \equiv 0 \pmod{3} \text{ and } i < p\} \cup \{i \mid i \equiv D \pmod{3} \text{ and } p + 1 < i \leq D\}$. Then $|T| = \frac{D+1}{3}$. Let $d(u_i, u_j)$ denote the distance between $u_i$ and $u_j$. Since $P$ is a shortest path connecting $u$ and $v$, we have $d(u_i, u_j) \geq 3$ and thus $N(u_i) \cap N(u_j) = \emptyset$ for any distinct $i, j \in T$. Note that $|\{p, p+1\} \cap \{0, D\}| \leq 1$ since $D \geq 2$. Thus

$$n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| \geq (D + 1) + [(\Delta - 1) + (|T| - 1)(\Delta - 2)]$$

$$= (D + 1) + [(\Delta - 1) + (\frac{D+1}{3} - 1)(\Delta - 2)] = \frac{D(\Delta + 1) + \Delta + 4}{3}.$$  (3.1)

**Subcase 1.2.** $D \not\equiv 2 \pmod{3}$. Define $T = \{i \mid i \equiv 0 \pmod{3} \text{ and } 0 \leq i \leq D - 3\} \cup \{D\}$. Then $|T| = \frac{D+2}{3}$. Note that there is at most one $j \in T$ such that $\delta \leq d(u_j) < \Delta$. Similarly as Subcase 1.1, we have

$$n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)|$$

$$\geq (D + 1) + [(\Delta - 1) + (\delta - 1) + (|T| - 2)(\Delta - 2)]$$

$$\geq (D + 1) + [(\Delta - 1) + (\delta - 1) + (\frac{D+2}{3} - 2)(\Delta - 2)]$$

$$= \frac{D(\Delta + 1) - \Delta + 3\delta + 5}{3}.$$

**Case 2.** $V_{<\Delta} \cap V(P) = \{u_p\}.$

**Subcase 2.1.** $D \not\equiv 0 \pmod{3}$. Define $T = \{i \mid i \equiv 0 \pmod{3} \text{ and } i < p\} \cup \{i \mid i \equiv D \pmod{3} \text{ and } p + 1 < i \leq D\}$. Then $|T| = \frac{D+1}{3}$. Similarly as Subcase 1.1,

$$n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| \geq \frac{D(\Delta + 1) + \Delta + 4}{3}.$$  (3.7)
Subcase 2.2. $D \equiv 0 \pmod{3}$. Define $T = \{ i \mid i \equiv 0 \pmod{3} \text{ and } 0 \leq i \leq D \}$. Then $|T| = \frac{D+1}{3}$ and $0, D \in T$. Note that there is at most one $j \in T$ such that $\delta \leq d(u_j) < \Delta$. Similarly as Subcase 1.2, we have
\[
n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| \geq (D+1) + [(\Delta - 1) + (\delta - 1) + (|T| - 2)(\Delta - 2)] = \frac{D(\Delta + 1) + 3\delta + 3}{3}.
\]

Case 3. $V_{<\Delta} \cap V(P) = \emptyset$. Define $T = \{ i \mid i \equiv 0 \pmod{3} \text{ and } i \leq D - 3 \} \cup \{ D \}$. Then $|T| = \lfloor \frac{D+1}{3} \rfloor$. Analogously, we obtain that
\[
n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| \geq (D+1) + [2(\Delta - 1) + (|T| - 2)(\Delta - 2)] = (D + 1) + [2(\Delta - 1) + \frac{D+1}{3} - 2(\Delta - 2)] = \frac{D(\Delta + 1) + \Delta + 7}{3}.
\]

By combining the above inequalities (3.1)-(3.13), Theorem 3.1 holds. \[ \square \]

THEOREM 3.2. Let $G \in g(n, \Delta)$ with $2 < \Delta < n - 1$, and minimum degree $\delta$. Then
\[
\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 5 - 3\delta)}.
\]

Proof. We may assume $G \in g(n, \Delta)$ with $2 < \Delta < n - 1$ is a $\lambda_1$-extremal graph. Applying Theorem 3.1 on Inequality (1.1), we obtain the desired result. \[ \square \]

REMARK 3.3. Note that
\[
\lambda_1 < \Delta - \frac{\Delta + 1}{n(3n + \Delta - 5 - 3\delta)} \leq \Delta - \frac{\Delta + 1}{n(3n + \Delta - 8)}
\]
since $\delta \geq 1$, the bound we obtain improves Inequality (1.4) (also see [9]).

The following corollary is a direct consequence of Theorem 3.2.

COROLLARY 3.4. Let $G \in g(n, \Delta)$ with no pendant vertices. Then
\[
\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 11)}.
\]
Theorem 3.5. Let \( G \in g(n, \Delta) \) with \( 2 < \Delta < n - 1 \), minimum degree \( \delta \), and average degree \( \bar{d} \). Then
\[
\Delta - \lambda_1 > \left[ \frac{(3n + \Delta - 3\delta - 5)^2}{2(\Delta + 1)^2} + \frac{(0.5 + n - \delta)(3n + \Delta - 3\delta - 5)}{\Delta + 1} + \frac{1}{\Delta - \bar{d}} \right]^{-1}.
\]

Proof. Let
\[
f(x) = (n - \delta)x + \frac{1}{\Delta - \bar{d}} - \left( \frac{x}{2} \right)^2.
\]
It is easy to see that
\[
f(x) = \frac{x^2}{2} + \left( \frac{1}{2} + n - \delta \right)x + \frac{1}{\Delta - \bar{d}}
= \left( \frac{1}{2} - \left( \frac{1}{2} + n - \delta \right) \right)^2 + \frac{1}{2} \left( \frac{1}{2} + n - \delta \right)^2 + \frac{1}{\Delta - \bar{d}}.
\]
Then for \( x \leq \frac{1}{2} + n - \delta \), the function \( f(x) \) is monotonically increasing in \( x \).

On the other hand, we have
\[
\frac{1}{2} + n - \delta - \frac{3n + \Delta - 3\delta - 5}{\Delta + 1} = (n - \delta - 0.5)(\Delta - 2) + 4.5 \geq 0.
\]
Combining this with Theorem 3.1, it follows that
\[
D \leq \frac{3n + \Delta - 3\delta - 5}{\Delta + 1} \leq \frac{1}{2} + n - \delta.
\]
Hence
\[
f(D) \leq -\frac{(3n + \Delta - 3\delta - 5)^2}{2(\Delta + 1)^2} + \frac{(0.5 + n - \delta)(3n + \Delta - 3\delta - 5)}{\Delta + 1} + \frac{1}{\Delta - \bar{d}}.
\]
By Inequality (1.5), \( \Delta - \lambda_1 > f(D)^{-1} \), and this completes the proof. \( \square \)

Remark 3.6. For the non-regular graphs with
\[
(\Delta - \bar{d})\left( \frac{3n + \Delta - 3\delta - 5}{\Delta + 1} \right) + \frac{\delta(3n + \Delta - 3\delta - 5)}{\Delta + 1} \geq 1,
\]
the bound in Theorem 3.5 is better than that in Theorem 3.2. On the other hand, for most almost regular graphs of constant degree and large order, the bound in Theorem 3.2 is better. We conclude that these two bounds are incomparable.

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REFERENCES


