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MATRICES OF POSITIVE POLYNOMIALS

DAVID HANDELMAN

Abstract. Associated to a square matrix all of whose entries are real Laurent polynomials in several variables with no negative coefficients is an ordered “dimension” module introduced by Tuncel, with additional structure, which acts as an invariant for topological Markov chains, and is also an invariant for actions of tori on AF C*-algebras. In describing this invariant, we are led naturally to eventually positivity questions, which in turn lead to descriptions of the Poisson boundaries (of random walks affiliated with these processes). There is an interplay between the algebraic, dynamical, and probabilistic aspects, for example, if the (suitably defined) endomorphism ring of the dimension module is noetherian, then the boundary is more easily described, the asymptotic behaviour of powers of the matrix is tractible, and the order-theoretic aspects of the dimension module are less difficult to deal with than in general. We also show that under relatively modest conditions, the largest eigenvalue function is a complete invariant for finite equivalence (early results of Marcus and Tuncel showed that it is not a complete invariant in general, but is so if the large eigenvalue is a polynomial).

Keywords. positive polynomial, trace, point evaluation, Newton polyhedron, convex polytope, primitive matrix, random walk, order ideal, (topological) shift equivalence, finite equivalence, Choquet simplex, real analytic function.

AMS subject classifications. 37B10, 46B40, 46L80, 32A38, 60J05, 16W80, 19K14, 15A54.

Introduction.

Let $M$ be a matrix whose entries are polynomials in $d$ real variables with no negative coefficients. Tuncel [Tu] initiated a study of “dimension modules” (certain ordered vector spaces) associated to such matrices, in connection with classification problems for (topological) Markov chains. (See [Tu, Tu2] for details.) In this article, we develop structure and classification theory in somewhat different directions.

Let $A$ be either $\mathbb{Z}[x_i^\pm 1]$ or $\mathbb{R}[x_i^\pm 1]$ (Laurent polynomial rings in $d$ variables), and let $A^+$ denote the corresponding set of Laurent polynomials with no negative coefficients. Then $A$ is a partially ordered ring with positive cone $A^+$, and the space of columnsof size $n$, denoted $A^n$, is an ordered $A$-module. If $M$ is in $\text{M}_n A^+$ (i.e., $M$ is an $n \times n$ matrix with entries from $A^+$), then $M$ induces an order preserving $A$-module map, $M : A^n \rightarrow A^n$. Iterating this map, we obtain the dimension module associated to $M$ as the direct limit,

$$G_M := \lim_{\rightarrow} A^n \overset{M}{\rightarrow} A^n \overset{M}{\rightarrow} A^n \overset{M}{\rightarrow} \ldots .$$

This is an ordered $A$-module, consisting of equivalence classes, $[c, k]$ where $(c, k) \in A^n \times \mathbb{N}$, and $[c, k] = [c', k']$ if there exists $N \geq k, k'$ such that $M^{N-k} c = M^{N-k'} c'$; $[c, k]$ is in the positive cone $G^+_M$ if there exists $N$ such that $M^N c \in (A^+)^n$. The shift map sending $[c, k] \mapsto [c, k+1]$ is an order automorphism, with inverse $\hat{M} : G_M \rightarrow G_M$ defined via $\hat{M}([c, k]) = [Mc, k]$. The structural problems referred to earlier arise from attempts to say something about the positive cone of $G_M$. Explicitly, one of the crucial problems is

(\text{$\mathcal{S}$}) Decide in terms of $M$ (in $\text{M}_n A^+$) and $c$ (in $A^n$) whether there exists an integer $N$ so that the column $M^N c$ has all of its entries in $A^+$.

(Of course, $[c, k]$ belongs to $G^+_M$ if and only if such an $N$ exists.) When $n = 1$, $M$ is just a polynomial $P$ in $A^+$, and $f$ is a Laurent polynomial with integer (or real) coefficients; in this case, (\text{$\mathcal{S}$}) asks to decide

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if the Laurent polynomial \( P^m f \) has no negative coefficients for some integer \( m \). This was solved in [H3, Theorems A and B]. Special cases had been solved by Meissner [Me]. In this situation, the solution leads to connections with a result of Ney and Spitzer [NS], determining the Martin boundary of the random walk on \( \mathbb{Z}^d \) associated with \( P \). There were also connections to the weighted moment map of algebraic geometry [Od, p. 94], and to an affine \( \text{AGL}(d, \mathbb{Z}) \)-invariant for compact lattice polytopes [H5], as well as a real affine \( \text{(AGL}(d, \mathbb{R})) \) invariant for general compact convex polytopes.

The problem \((\mathfrak{S})\) arises in other contexts than topological Markov chains. If \( A = \mathbb{Z}[x_1^\pm 1, x_2^\pm 1] \), the limiting dimension module is a direct limit of partially ordered abelian groups. Let \( R \) be an AF \( C^* \)-algebra [El] whose ordered Grothendieck group is the limit obtained by evaluating all the variables appearing everywhere in (1) at 1—in other words, each \( x_i \mapsto 1 \):

\[
G_M(1) = \lim_{n \to \infty} Z^n \xrightarrow{M(1)} Z^n \xrightarrow{M(1)} Z^n \xrightarrow{M(1)} \ldots .
\]

(Here \( M(1) \) an abbreviation of the real nonnegative matrix \( M(1, 1, \ldots, 1) \). In other words, \( K_0(R) = G_M(1) \), as ordered abelian groups. Then there is an action \( \alpha \) of the \( d \)-torus \( T = T^d \) on \( R \) so that the equivariant Grothendieck group of \( R \), \( K^T_0(R) \) is \( G_M \) when viewed as a partially ordered module over the representation ring of \( T \), which of course is \( A \). Alternatively, \( G_M = K_0(R \times \alpha T) \). This is a special case of more general results in [HR], which deals with classification results for locally finite actions of tori and other compact Lie groups on AF \( C^* \)-algebras.

There is a probabilistic interpretation that is closely related. Begin with the matrix \( M = (M_{ij}) \) with entries \( M_{ij} \) in \( \mathbb{A}^+ \). Form the set of sites, \( S = \mathbb{Z}^d \times \{1, 2, \ldots, n\} \). It is helpful to think of \( S \) as \( n \) disjoint copies of the lattice \( \mathbb{Z}^d \) piled on top of each other, with the second coordinate indexing the level. A particle at point \((z, k) \in S \) (lattice point \( z \), level \( k \)) is constrained to jump to level \( j \) in the next unit of time; its motion in this level is governed by the entry \( M_{jk} \). Write \( M_{jk} = \sum \lambda_w x^w \) where we use multinomial notation \( x^w = x_1^{w_1(1)} \cdots x_d^{w(d)} \), \( \lambda_w \) are all nonnegative real numbers, and we have suppressed the dependence on \( jk \) for expository convenience. Then on level \( j \), the particle moves from \( z \) according to the rule that the probability of displacement by \( w \) is \( \lambda_w / M_{jk}(1) \). Then \((\mathfrak{S})\) amounts to asking which initial finitely supported signed distributions on \( S \) become nonnegative after a finite number of iterations of this process.

In the case that \( n = 1 \), this is related to the Poisson boundary, although knowledge of the latter is insufficient to solve \((\mathfrak{S})\). In this case, let \( \mu \) be the measure on \( \mathbb{Z}^d \) obtained from \( P: \mu(\{w\}) = \lambda_w / P(1) \), where \( P = \sum \lambda_w x^w \). Let \( \nu \) be a finitely supported signed measure, and set \( \nu_0 = \|\nu\| \) (any positive measure with the same support as \( \nu \) will do just as well). The specialization of \((\mathfrak{S})\) asks if there exists \( m \) so that the \( m \)-fold convolution of \( \mu \), convolved (once) with \( \nu \), is positive. The answer is affirmative if and only if for every extremal \([0, \infty] \)-valued harmonic function \( h \) (on \( \mathbb{Z}^d \) finite with respect to \( \nu_0, \sum_{w \in \text{supp} \nu} \nu(w)h(w) > 0 \).

If \( \nu_0 \) is mutually absolutely continuous with respect to \( \mu \) or some convolution power \( \mu^{(k)} = \mu \ast \cdots \ast \mu \) for some \( k \), then the set of these extremal harmonic functions is compact (and in this situation is contractible). However, for general \( \nu_0 \), the set of such functions need not be either compact or connected (if \( d = 1 \), it is compact, but need not be connected).

For \( n > 1 \), there is a choice of initial (row of) measures so that a boundary, analogous to the Poisson boundary, occurs (it is the positive portion of the real maximal ideal space of a partially ordered ring naturally associated to \( M \) and is at least compact, although not generally connected). Other choices lead to very complicated situations that must be disentangled in order to attack \((\mathfrak{S})\).

One can also put the random walk interpretation (actually these admit the same formalism of the “matrix-valued random walks” occurring in [CW]—although the problems considered are almost entirely unrelated) in the context of subshifts of finite type. By symbol splitting, we may assume that all of the entries of \( M \) are either monomials or zero (although the matrix size may change). Form \( S = \mathbb{Z}^d \times \{1, 2, \ldots, n\} \) as we did before, and set

\[
X = S^N \quad X_M = \left\{ x = (w(x,i), k(x,i)) \in S^N \mid M_{k(x,i+1), k(x,i)} = x^w, \ w = w(x,i+1) - w(x,i) \right\}.
\]
That is, if \( x(i) = (w(x, i), k(x, i)) \) where \( w(x, i) \in \mathbb{Z}^d \) and \( k(x, i) \in \{1, 2, \ldots, n\} \), then the sequence \( x = x(i) \) belongs to \( X_M \) provided that for each \( i \in \mathbb{N} \), the transition from \( w(x, i) \) to \( w(x, i+1) \) is permitted—that is, \( w = w(x, i + 1) - w(x, i) \) is the exponent of the monomial appearing in the \( k(x, i + 1), k(x, i) \) entry of \( M \). One can think of this as a shift space on an enormous state space (\( S \)), but with relatively few transitions permitted. In general, \( X_M \) is not compact, but we may find a lot of compact cylinder sets, by selecting a finite set of elements “\( z \)” in \( S \) and times \( i \) in \( \mathbb{N} \), \( U = \{(z_j, i(j))\} \) (\( j \) running over a finite index set), and set \( K_U = \{x \in X | x \text{ there exists } j \text{ such that } x(i(j)) = z_j \} \).

The path \( x \) belongs to \( K_U \) if it passes through at least one of the \( z_j \) at time \( i(j) \). This corresponds to the selection of an initial measure.

We normally assume \( M \) is primitive, i.e., some power of \( M \) has no zero entries; however, at times, we have to face up to the reducible case, which is somewhat more complicated.

The approach used when \( n = 1 \) is elaborated in order to apply to the general case. Associated to the ordered vector space \( G_M \) is the “bounded subring” (defined in section 1), denoted \( E_b(G_M) \). This is a partially ordered ring of endomorphisms of \( G_M \) that are bounded in the appropriate sense. Much of the information concerning positivity of elements of \( G_M \) is reflected, although not terribly clearly, in the structure of \( E_b(G_M) \). For example, the (densely defined) traces on \( G_M \) in principal permit one to decide positivity of a specific element, and these traces factor through (in the appropriate sense) \( E_b(G_M) \). Recall that a trace on an ordered vector space is a nonzero positive linear functional. In this case, “densely defined” means the domain of the trace is an order ideal in \( G_M \). The (global, that is, everywhere defined) traces of \( G_M \) yield some information about positivity (for example, necessary for \([c, k]\) to be positive is that \( \tau([c, k]) > 0 \) for every global trace on \( G_M \), but these are far from sufficient to determine the ordering, even if \( d = 1 \) (in contrast, when \( n = d = 1 \), the global traces do determine the ordering). The pure traces on \( E_b(G_M) \) that are not faithful determine ideals of \( E_b(G_M) \), and part of the structure theory is to describe these order ideals, which turn out to correspond (not bijectively) with faces of the convex polytope associated to \( M \) by Marcus and Tuncel [MT]. (When \( n = 1 \), the association is a bijection and many very nice properties hold [H5].)

In the case that \( n = 1 \), the ordered ring \( E_b(G_M) \) (in the context of \( n = 1 \), called \( R_P \) [H5]) is always a finitely generated commutative ring/algebra (depending on whether we use \( \mathbb{Z} \) or \( \mathbb{R} \) in the definition of \( A \)) with no zero divisors. It is easy to see that \( E_b(G_M) \) may have zero divisors and need not be commutative, but generically the characteristic polynomial is irreducible over \( \mathbb{A} \), and that case, it is a commutative domain. Unfortunately, if \( n \geq 2 \), generically it is not finitely generated; it is not even noetherian; in fact, it does not even satisfy the ascending chain condition on order ideals. This means that the positivity problem is far more complicated than expected. We give necessary conditions in order that this ascending chain condition hold.

Among classification problems is the question of deciding finite equivalence of two such matrices; that is, given square \( M \) and \( M' \) with entries in \( A^+ \), give criteria in order that there be a nonzero rectangular matrix \( X \) with entries in \( A^+ \) such that \( MX = XM' \). A necessary condition is given by equality of the corresponding beta functions; for each \( r = (r_i) \in (\mathbb{R}^d)^{++} (= \{r \in \mathbb{R}^d \mid r_i > 0, i = 1, 2, \ldots, d\}) \), \( M(r) \) (all the entries of \( M \) evaluated at \( r \)) is a real nonnegative matrix, and our initial primitivity assumption means the large eigenvalue, denoted \( \beta_M(r) \) (or simply \( \beta(r) \)) has multiplicity one; it follows that \( r \mapsto \beta_M(r) \) is a real analytic function on \( (\mathbb{R}^d)^{++} \). A necessary condition for finite equivalence is that \( \beta_M = \beta_M' \) on \( (\mathbb{R}^d)^{++} \). This was conjectured to be sufficient as well. However, Marcus and Tuncel [MT] gave two examples to show this was not the case; I showed (independently, but only in the form of a talk at the Adler conference) that the conjecture failed, for one of the pairs that Marcus and Tuncel considered, by completely different methods. However, it had been known already that if \( d = 1 \), the conjecture is true (Parry), and Marcus and Tuncel [MT2] also showed that if \( \beta \) were a polynomial, the conjecture succeeds.

Here we give conditions on \( M \) and \( M' \) that guarantee that \( \beta \) is a complete invariant for finite equivalence. For any (primitive) \( M \), we can find a left eigenvector for \( \beta \) whose entries are polynomials in \( \beta \) with coefficients
from $A$, i.e., in $\mathbb{R}[x_1^{\pm 1}][\beta]$, and each entry is real analytic and strictly positive on $(\mathbb{R}^d)^{++}$. If we can arrange that each entry of the eigenvector is bounded above and below away from zero by a quotient of elements of $A^+$ (this is a simplification of the real definition; the actual definition is more general), then we say $M$ satisfies $(\%)$ (a local criterion, using the facial structure of $M$, can also be given, 8.7). For example, if $\beta$ is polynomial, or if $M$ is a companion matrix or the transpose thereof, or if the Newton polyhedra of all the entries of some power of $M$ are the same, then $(\%)$ holds. We say $M$ satisfies $(\#)$ if the ratio of the second largest absolute value of eigenvalues of $M(r)$ to $\beta_M(r)$ is bounded above by $1 - \epsilon$ as $r$ varies over $(\mathbb{R}^d)^{++}$. This holds if for every face $F$ of the convex set associated to $M_F$, there is a unique irreducible block which hits the spectral radius, and it is primitive (actually, in the statement of the result, the last “primitive” can be deleted), so it is relatively easy to test for.

Imprecisely, we show that among matrices satisfying $(\#)$ and $(\%)$, the beta function is a complete invariant for finite equivalence. It is conceivable that $(\#)$ implies $(\%)$, so the latter condition may be redundant. It is also possible that $(\%)$ alone is sufficient for $\beta$ to be a complete invariant for finite equivalence.

In results leading up to this, we note that analytic and algebraic functions that are positive on $(\mathbb{R}^d)^{++}$ can behave strangely, and this strangeness is reflected in boundary behaviour. Explicitly, if $\rho : (\mathbb{R}^d)^{++} \to \mathbb{R}^{++}$ is algebraic (i.e., satisfies a polynomial equation with coefficients from $A$) and real analytic, and if $X : \mathbb{R}^+ \to (\mathbb{R}^d)^{++}$ is the exponential of a ray in $\mathbb{R}^d$ with directional derivative $v$, then

$$
\lim_{t \to \infty} \frac{\log \rho(X(t))}{\log t}
$$

may depend on the starting point of the ray; if $\rho$ is bounded above and below by integer multiples of a rational function (whose numerator and denominator are in $A^+$), then the limit does not depend on the starting point. This provides a notion of (local) “linkages” between the various blocks of $M_F$ (for a fixed face $F$), which would be part of the data for shift equivalence. In the survey article [Tu2], Tuncel provides a global way of linking blocks.

**Principal results.** Theorem 4.5 describes completely the pure (global) traces on $G_M$ and its order ideals under the mildest of conditions, as well as the faithful pure traces on $E_b(G_M)$. Roughly speaking, they factor through the left Perron eigenvector for $\beta$ evaluated at points of $(\mathbb{R}^d)^{++}$.

The unfaithful pure traces on order ideals of $E_b(G_M)$ and of $G_M$ are described in several situations, completely if the necessary condition for noetherianness holds (sections 5 and 6). Roughly speaking, they factor through quotients corresponding to taking facial matrices. In fact, the unfaithful pure traces could be completely described in this form if an innocuous-looking conjecture were true.

The consequences, and subsequent non-genericity, of the ascending chain condition on order ideals of $G_M$ are discussed in section 7. Some of the relevant conditions can be simply expressed in terms of the weighted graph associated to $M$.

The finite equivalence result discussed above, Theorem 10.7, is proved in section 10.

In section 12, necessary and sufficient conditions on a size 2 matrix $M$ are determined in order that it satisfy the condition $(\%)$ discussed previously. It turns out to be easy to express but difficult to prove: $(\%)$ holds if and only if the discriminant of the matrix divides a polynomial with no negative coefficients, Theorem 12.1. It follows that for size 2 matrices wherein the diagonal entries have no common monomials, $(\%)$ is equivalent to $M + pI$ satisfying $(\#)$ for some $p$ in $A^+$ (this property is called weak $(\#)$, and admits a much less awkward formulation).

Finally, in section 13, we have a weak realization theorem, which says that under suitable circumstances, given suitable functions $\beta$, there exists a primitive matrix $M$ for which $\beta_M = \beta^N$ for some undetermined integer $N$; moreover, $M$ may be selected to satisfy a strong condition, $(\ast\ast)$, which asserts that in some power of $M$, the Newton polyhedra of all the entries are identical.

This paper is a considerable elaboration of results in the preprint, “Eventual positivity and finite equivalence for matrices of polynomials”, versions of which were distributed from 1987 on.
Appendix Nonnegative real matrices

1 The bounded endomorphism ring

In this section, we define and develop the basic properties needed of the ring (algebra) of bounded endomorphisms. Curiously, even in the case of zero variables, the notion is of some interest.

We recall some notation and definitions from the introduction. The ring of Laurent polynomials in the $d$ variables $x_1, x_2, \ldots, x_d$ over either the integers or reals is denoted $A$, and the set of elements of $A$ having no negative coefficients is a positive cone $A^+$ for $A$, making the latter into a partially ordered ring. The ring of $n \times n$ matrices with entries from $A$ will be denoted $M_nA$; of course, $M_nA^+$ will denote the set of matrices with entries from $A^+$. For an integer $n$, $A^n$ will denote the set of columns of size $n$, viewed as an ordered $A$-module (with the direct sum ordering). Fix $n$ and $M$ in $M_nA^+$. The real matrix obtained by evaluating all the variables at 1 will be denoted $M(1)$, an abbreviation for $M(1, 1, \ldots, 1)$. Define the direct limit, as ordered $A$-modules,

\[
G_M = \lim_{\leftarrow} A^n \xrightarrow{M} A^n \xrightarrow{M} A^n \xrightarrow{M} \ldots.
\]

This is the set of equivalence classes,

\[
\{(a, k) \mid a \in A^n, \quad k \in \mathbb{N}\} / \sim,
\]

where $(a, k) \sim (a', k')$ if there exists $m > k, k'$ such that $M^{m-k}a = M^{m-k'}a'$. The $A$-module structure is obvious, and $G_M$ admits an ordering making it into a directed partially ordered $A$-module, by means of the positive cone

\[
G_M^+ = \{ [a, k] \in G_M \mid \text{there exists } (a', k') \sim (a, k) \text{ with } a' \in (A^n)^+ \}.
\]

We observe that $[a, k] = [Ma, k+1]$, and $M$ induces an order isomorphism (that commutes with the action of $A$) by means of $M[a, k] = [Ma, k]$; its inverse is given by a “shift”, $M^{-1}[a, k] = [a, k+1]$. More generally, if $N$ is in $M_nA$ and commutes with $M$, then $N$ induces an $A$-module endomorphism of $G_M$ via $N[a, k] = [Na, k]$. This will be positive (i.e., order preserving) if and only if there exists an integer $m$ so that $NM^k$ belongs to $M_nA^+$. Thus $\sim$ describes an assignment (actually a homomorphism of $A$-modules and of rings), from the centralizer of $M$ in $M_nA$, $\mathcal{C}_A(M)$, to the ring of $A$-module endomorphisms of $G_M$, $\text{End}_A G_M$.

The matrix $M$ is an element of the centre of $\mathcal{C}_A(M)$, and we may formally invert $M$; if the determinant of $M$ is nonzero, this amounts to adjoining $M^{-1}$ from the matrices with entries in the field of fractions of $A$ (the rational functions in the $d$ variables). In general, the construction is given as limit ring,

\[
\mathcal{C}_A(M)[M^{-1}] = \lim_{\leftarrow} \mathcal{C}_A(M) \xrightarrow{\times M} \mathcal{C}_A(M) \xrightarrow{\times M} \mathcal{C}_A(M) \xrightarrow{\times M} \ldots.
\]
We have been a little sloppy in notation; if the determinant of $M$ is zero, then the ideal of $C_A(M)$, 
$\{ N \in C_A(M) \mid M^nN = 0 \}$ will be factored out. So, despite the notation, $C_A(M)$ will not be a subring of $C_A(M)[M^{-1}]$. Note that since $M$ belongs to an $n \times n$ matrix ring over a field (the rational functions in the $d$ variables), for any column $a$ such that $M^na$ is zero, it follows that $M^n a$ is also zero. In any case, $\sim$ induces a map from $C_A(M)[M^{-1}]$ to $\text{End}_A G_M$.

Define $E(G_M) = \{ e \in \text{End}_A G_M \mid e\hat{M} = \hat{M} e \}$; then the range of $\sim : C_A(M)[M^{-1}] \rightarrow \text{End}_A G_M$ is obviously inside $E(G_M)$. In fact the range is onto, and the map is an isomorphism.

Lemma 1.1 The map

$$\sim : C_A(M)[M^{-1}] \rightarrow E(G_M)$$

is an isomorphism of rings and $A$-modules.

Proof: It is routine to verify the map is a well-defined homomorphism of rings and $A$-modules. We first show the map is onto. Select $e$ in $E(G_M)$. Let $E_j (j = 1, 2, \ldots, n)$ denote the standard basis elements for $A^n$. Then we may find elements $a_j$ in $A^n$ together with integers $k(j)$ such that $e[E_j, 1] = [a_j, k(j)]$. We may assume that $k(j)$ are all equal, say to $k$, so that $e[E_j, 1] = [a_j, k]$ for all $j$. Define $N_0$ to be the $n \times n$ matrix with entries from $A$ whose $j$th column is $a_j$. Obviously, $(\hat{M})^{-k(n-1)}\hat{N}_0$ has the same effect as $e$ on the $[E_j, 1]'s$. It is a routine verification that $\sum_j [E_j, 1]A$ is essential (as an $A$-module) in $G_M$, so that $A$-module endomorphisms agree on $\{[E_j, 1]\}$, they are equal. As the map $C_A(M) \rightarrow C_A(M)[M^{-1}]$ is not necessarily one to one, it is not immediate that $N_0$ commutes with $M$. However, it easily follows that there exists an integer $s$ so that $N_0M^s$ commutes with $M$. Set $N = N_0M^s$, and observe that $(\hat{M})^{-k(n-1+s)}\hat{N} = e$. Thus the map is onto.

It is immediate from the definitions that the map is one to one: $\hat{N}\hat{M}^{-t} = 0$ implies $[NE_j, 1] = [0, 1]$, so that $M^sNE_j = 0$ for some $s$ (and all $j$), whence $M^sN = 0$, which means $N$ is in the kernel of the map $C_A(M) \rightarrow C_A(M)[M^{-1}]$.

Remark. 1: The portion of the preceding that does not involve the ordering applies if $A$ is a general commutative unital ring and $M$ is an arbitrary square matrix (see [BoH, p. 125ff] where $E(\cdot)$ is defined for square matrices with entries in a commutative ring).

Now we discuss the positive cone of $E(G_M)$. The natural definition of a positive cone on a ring of endomorphisms is the following:

$$E(G_M)^+ = \{ e \in E(G_M) \mid e(G_M^+) \subseteq G_M^+ \} .$$

Lemma 1.2 Under the map

$$\sim : C_A(M)[M^{-1}] \rightarrow E(G_M),$$

an element of $C_A(M)[M^{-1}]$ is sent to a positive element $\hat{M}^{-k}\hat{N}$ of $E(G_M)$ if and only if there exists a positive integer $m$ such that $M^mN$ belongs to $M_+A^+$.

Proof: Let $e$ denote the endomorphism described by $\hat{M}^{-k}\hat{N}$. If $e \geq 0$, $e[E_j, 1] \geq [0, 1]$ for all $j$. Hence there exists $k$ such that for all $j, e[E_j, 1] = [a_j, k]$ where $a_j$ belongs to $(A^n)^+$. It follows from the definitions that multiplication by some power of $M$ will render all the columns of $N$ positive, which is what we want. The converse is trivial.

Now we observe that every element of $E(G_M)$ is a difference of elements of its positive cone (i.e., as a partially ordered abelian group, $E(G_M)$ is directed).

Finally, we can define the bounded subring of $E(G_M)$; Let $I : G_M \rightarrow G_M$ denote the identity element of $E(G_M)$. Set

$$E_b(G_M) = \{ e \in E(G_M) \mid \text{there exists } k \in \mathbb{N} \text{ such that } -kI \leq e \leq kI \}$$

$$E_b(G_M)^+ = E_b(G_M) \cap E(G_M)^+$$

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We observe that with the relative ordering, \( E_b(G_M) \) is a partially ordered ring; it need not be commutative, although this is generic; it is an order ideal in \( E(G_M) \), and moreover \( I \) is an order unit for \( E_b(G_M) \). Obviously \( G_M \) is an ordered module over \( E_b(G_M) \). However, that \( A \) does not act on \( E_b(G_M) \), because (informally) all of the former’s elements (except the constants) are unboundedin any reasonable sense.

For now, we make the standing hypothesis that the real matrix \( M(1) \) is primitive. This is equivalent to saying that there exists a power of \( M \) so that every entry is not zero. One of the reasons for doing this is to guarantee that if \( H \) is any nonzero order ideal of \( G_M \) then its bounded subring is the same as \( E_b(G_M) \), so that all order ideals of \( G_M \) may be viewed as ordered modules over a common partially ordered ring.

Now let \( H \) be an order ideal of \( G_M \). Recall from the introduction that to describe the ordering on \( G_M \) we must analyze all the pure states of the order ideals (that have order units).

Specifically, suppose that \( G \) is a partially ordered unperforated abelian group and \( a \) is an element thereof. If there exists an order ideal with order unit of \( G \), \( I \), containing \( a \) such that for all (pure) traces \( \tau \) of \( I \), \( \tau(a) > 0 \), then \( a \) is in the positive cone of \( G \). If no such order ideal exists, then \( a \) is not in the positive cone; in particular, there need not be a minimal order ideal containing \( a \), or there could be a trace on a minimal order ideal with order unit among those containing \( a \), at which \( \tau(a) < 0 \).

To analyze the traces on an order ideal \( I \), we examine the bounded subrings \( E_b(I) \); these automatically contain \( I \) as an order ideal, and their pure trace spaces are compact. Moreover, there is a natural map from the pure trace space of the latter to that of \( I \). The pure trace spaces are described in sections 4–6.

We may form \( E(H) = \{ e \in E(G_M) \mid eH \subseteq H \} \), and put the relative order on it. It is an ordered subring. Unfortunately, even if \( H \) admits an order unit, \( E(H) \) need not admit the identity as an order unit, so the latter is not generally equal to its bounded subring (which we shall define shortly) even when \( n = d = 1 \).

Example 1.3 An order ideal \( H \) in \( G_M \) which has an order unit but for which \( E(H) \) does not admit the identity as an order unit.

Here \( n = d = 1 \), so \( M = P \) a polynomial in one variable. Set \( P = 1 + x + x^2 \). As in [H5], \( G_M \) is the ring \( R[x, x^{-1}, P^{-1}] \), and the bounded subring, \( E_b(G_M) \) is just \( R_P = R[x/P, 1/P] \). The positive cone of the latter is generated additively and multiplicatively by \( \{ 1/P, x/P, x^2/P \} \) [H5, I.4]. Let \( H = I \) be the ideal of \( R_P \) generated by \( \{ 1/P, x/P \} \). As in [H5], this is also an order ideal of \( R_P \) (hence of \( G_M \)), and \( (1 + x)/P \) is an order unit for \( I \). Since each of \( x/P^2, x^2/P^2 \), and \( x^3/P^2 \) belong to \( I \), multiplication by \( x^2/P \) is a positive endomorphism of \( I \), which obviously extends to a positive endomorphism of \( G_M \). However, it is not bounded by a multiple of the identity endomorphism (so that the identity is not an order unit for \( E(H) \)).

To see this, we it is sufficient to show that there does not exist an integer \( k \) such that \( (x^2/P) \cdot 1 \leq k \cdot 1 \), i.e., \( x^2/P \leq k \) is impossible. If \( x^2/P \leq k \), then according to the definition of the ordering, there would exist \( K \) so that \( x^2/P^{K-1} \leq kP^K \) with respect to the usual ordering on \( A = R[x\pm 1] \). In other words, every monomial appearing in \( x^2(1 + x + x^3)^{K-1} \) would have to appear in \( P^K \). It is a simple exercise to show that \( x^3K^{-1} \) appears in the former but not in the latter.

In the case that \( n = 1 \), failure of \( E(H) \) to admit an order unit if \( H \) is an order ideal of \( R_P \) forces the latter ring to be not integrally closed in its field of fractions (see [H5, Section III]).

If \( H \) is an order ideal of \( G_M \), we define its bounded subring

\[
E_b(H) = \{ e \in E(H) \mid \text{there exists } k \in \mathbb{N} \text{ such that } -kI \leq e \leq kI \}
\]

\[
E_b(H)^+ = E_b(H) \cap E(H)^+.
\]

We will show that \( E_b(H) = E_b(G_M) \), at least when \( M(1) \) is primitive. Thus \( E_b(H) \) is independent of the choice of (nonzero) \( H \). This requires a series of lemmas.

Lemma 1.4 Suppose that \( M(1) \) is primitive. If \( e \) belongs to \( E(G_M)^+ \) and \( eh = 0 \) for some \( h \) in \( G_M \setminus \{ 0 \} \), then \( e = 0 \).
Corollary 1.5 Suppose that $M(1)$ is primitive and $H$ is an order ideal in $G_M$. If $e$ belongs to $E(G_M)^+$ and $eH = 0$, then $e = 0$.

Proof. The conclusion amounts to $x^w \mathbf{1} \leq \mathbf{K}^w$ for some positive integer $K$, the ordering computed with respect to that on $E(G_M)$. Form the graph of $M$. Select a loop from 1 to 1 that contains every state in \{1, 2, ..., $n$\} at least once (this is of course possible, because $M(1)$ is primitive). Let $k$ be the length of the path; so if $w$ is the sum of the weights along the loop, $x^w$ appears in the loop as a cycle of the form (1, 1) entry of $M^k$. For any particular state in, we can obtain a loop of the same length by starting at the first occurrence of $i$ instead of the first 1. Since the resulting loop is a cyclic permutation of the original, it has the same sum of weights. This means that each diagonal entry of $M^k$ contains $x^w$; it follows immediately that $x^w \mathbf{1} \leq \mathbf{K}^w$, as desired.

Since everything remains the same for our immediate purposes if we replace $M$ by $x^{-w} M^k$, we may even assume that $1 \leq K M$ (that is, $0 \in \text{Log} M_{i,i}$ for all $i$). We will have to be a bit more careful about this when we come consider the invariants $\Delta$ and $\Gamma$ of Krieger (our assumption for now is that they are the same).

Lemma 1.6 If $M(1)$ is primitive, then there exist $w$ in $\mathbb{Z}^d$ and a positive integer $k$ such that $x^w M^{-k}$ belongs to $E_b(G_M)$.

Proof. The order ideal contains to $x^w \mathbf{1} \leq \mathbf{K}^w$ for some positive integer $K$, the ordering computed with respect to that on $E(G_M)$. Form the graph of $M$. Select a loop from 1 to 1 that contains every state in \{1, 2, ..., $n$\} at least once (this is of course possible, because $M(1)$ is primitive). Let $k$ be the length of the path; so if $w$ is the sum of the weights along the loop, $x^w$ appears in the loop as a cycle of the form (1, 1) entry of $M^k$. For any particular state in, we can obtain a loop of the same length by starting at the first occurrence of $i$ instead of the first 1. Since the resulting loop is a cyclic permutation of the original, it has the same sum of weights. This means that each diagonal entry of $M^k$ contains $x^w$; it follows immediately that $x^w \mathbf{1} \leq \mathbf{K}^w$, as desired.

Since everything remains the same for our immediate purposes if we replace $M$ by $x^{-w} M^k$, we may even assume that $1 \leq K M$ (that is, $0 \in \text{Log} M_{i,i}$ for all $i$). We will have to be a bit more careful about this when we come consider the invariants $\Delta$ and $\Gamma$ of Krieger (our assumption for now is that they are the same).

Lemma 1.7 Suppose that $M(1)$ is primitive. If $H$ is a nonzero order ideal of $G_M$, then

$$A[M^{\pm 1}] \cdot H = G_M \quad \text{and} \quad A[M^{\pm 1}]^+ \cdot H^+ = G_M^+.$$

Proof. The order ideal contains an element of the form $[x^w E_j, k]$ for some integers $j$ and $k$, and some lattice point $w$. By multiplying by an element of the form $a x^{-w}$ (for $a$ in $A^+$), we obtain $[aE_j, k]$. By multiplying by a power (positive or negative) of $M$, we obtain anything of this form but with arbitrary $k$. Finally, by primitivity of $M(1)$ and the convexity of $H$, we can always find $w'$ and $k'$ (depending on $j$) so that for any state in, $[x^w E_j, k']$ belongs to $H$. Hence by applying suitable elements of $a$ and powers of $M$, an arbitrary element of $G_M^+$ is attainable; this yields the second assertion and the first follows immediately.

Proposition 1.8 Suppose that $M(1)$ is primitive. Let $H$ be a nonzero order ideal of $G_M$, and let $e$ be an element of $E(G_M)$ such that $eH = 0$. Then $e$ is zero.

Proof. Select $[x^w E_j, k]$ in $H$. Apply $M$ using the identity $[Ma, k + 1] = [a, k]$. From the convexity of $H$ together with the primitivity of $M(1)$, we may find $k'$, together with lattice points $w_j$ such that for all $j = 1, 2, \ldots, n$, $[x^w E_j, k']$ belongs to $H$. Write $e = \hat{M}^{s} N$. As $\hat{M}$ is an automorphism, we must have $N[x^w E_j, k'] = 0$ for all $j$. Thus $x^w M^{\pm 1} N E_j$ are all zero, and this obviously forces $M^w N = 0$. Hence $e$ is zero.

Proposition 1.9 Suppose that $M(1)$ is primitive. If $H$ is an order ideal in $G_M$, then $E_b(G_M) = E_b(H)$.
Proof: We first show that $E_b(H)$ is the order ideal of $E(G_M)$ generated by the identity. By definition, $E_b(H)$ is a subring of $E(G_M)$. Suppose that $e$ belongs to $E(G_M)$ and $e(H^+) \subseteq H^+$; we wish to show that $e(G_M^+) \subseteq G_M^+$. This follows from Lemma 1.7. Thus the relative ordering on $E_b(H)$ agrees with the natural ordering.

Since the identity element of $E_b(H)$ is automatically an order unit for it, $E_b(H)$ is directed. Finally, suppose that $e' \leq e$ where $e$ and $e'$ are positive elements of $E(G_M)$ and $e$ belongs to $E_b(H)$. As $e(H^+) \subseteq H^+$ and $H$ is an order ideal, it easily follows that $e'(H^+) \subseteq H^+$, so that $e(H) \subseteq H$. Hence $e'$ belongs to $E(H)$; but $e' \leq KI_H$ for some $K$ (since the same inequality holds for $e$), so that $e'$ belongs to $E_b(H)$.

Hence $E_b(H)$ is an order ideal of $E(G_M)$ and has the identity as an order unit. However, by definition $E_b(G_M)$ is the order ideal generated by the identity. So the two must be equal.

Now we show that $E_b(G_M)$ is a shift invariant for $M$. Recall that if $\mathcal{R}$ is a (unital) partially ordered commutative ring, and $M$ and $M'$ are square matrices with entries from $\mathcal{R}^+$, then we say that they are (lag $l$) shift equivalent (over $\mathcal{R}^+$) if there exist rectangular matrices $R$ and $S$ with entries from $\mathcal{R}^+$ such that

$$MR = RM' \quad SM = M'S \quad RS = M^l \quad SR = (M')^l.$$ 

If all the matrices involved have entries in $\mathcal{R}$ but not necessarily in the positive cone, then we say $M$ and $M'$ are algebraically shift equivalent (over $\mathcal{R}$).

The reason $G_M$ itself is not a shift invariant is that it depends on the choice of $A$, that is, on the number of variables, and how they are labelled. Let $B$ be a subgroup of $\mathbb{Z}^d$, and suppose that all the entries of $M$ have all of their exponents lying in $B$. Let $A_0$ denote the subring (if $A = \mathbb{Z}[x_{\pm 1}^\pm]$) or subalgebra (if $A = \mathbb{R}[x_{\pm 1}^\pm]$) generated by $B$. We say that $M$ is defined over $A_0$. We may define $G_{M,0}$ by replacing $A^n$ with $A_0^n$ throughout (1). Then it is immediate that $G_M$ is naturally isomorphic (in all possible ways) with the tensor product of ordered $A_0$-modules, $G_{M,0} \otimes_{A_0} A$.

Lemma 1.10 Under these conditions,

$$E_b(G_M) = E_b(G_{M,0}).$$

Proof: Select $b$ in $G_{M,0}^+$, and form $H_0$, the order ideal generated by $b$ therein. We claim that $H$ is also an order ideal in $G_M$. Write $A = F[B_1]$ (giving the name $B_1$ to the standard copy of $\mathbb{Z}^d$) and $A_0 = F[B]$, where $F$ is either the integers or the reals. Let $T$ be a transversal of $B$ in $B_1$; that is, $T$ is a subset of $B_1$ such that $B = \cup_{t \in T}(t + B)$ and for all $t \neq t'$ in $T$, $t - t'$ does not belong to $B$. We may assume that zero belongs to $T$. Obviously, $A = \oplus_{t \in T} x^t A_0$. Since the coefficients of the entries of $M$ all lie in $A_0$, we have that for all $t$, $M((x^t A_0)^n) \subseteq (x^t A_0)^n$. It easily follows that $G_M$ decomposes as $\oplus_{t \in T} x^t G_{M,0}$, as ordered $A_0[\bar{M}^\pm]$-modules. Moreover, each summand is an order ideal in $G_M$, so that $H$ (being an order ideal in the summand corresponding to $t = 0$) is also an order ideal. (Order ideals in order ideals of a dimension group are themselves order ideals in the big dimension group.)

Next, we observe that if $G = \oplus G_i$, a direct sum of partially ordered (directed) abelian groups with the coordinatewise ordering on $G$, and $e : G \to G$ is a positive endomorphism such that $e \leq KI_G$ (as endomorphisms), then for each $i$, $e(G_i) \subseteq G_i$. To see this, select $g$ in $G_i^+$. Then $e(g) \leq Kg$ so that $e(g)$ belongs to the order ideal generated by $g$; this obviously is contained in $G_i$. Since $G_i$ is directed, it follows that $e(G_i) \subseteq G_i$.

Hence if $e$ is in $E_b(G_M)$, for all $t$, $e(x^t G_{M,0}) \subseteq x^t G_{M,0}$. We have an order-preserving ring homomorphism, $\Delta : E_b(G_{M,0}) \to E_b(G_M)$, by permitting $e$ to commute with each $x^t$. By Proposition 1.8, an element of $E_b(G_M)$ is determined by its effect on a nonzero order ideal. Hence the map is onto (since these endomorphisms must commute with elements of $A$).

•
2. A brief look at $E_b(G_M)$ as a shift invariant

Suppose that $M$ and $M'$ are matrices with entries from $A^+$ (a common Laurent polynomial ring) and they are shift equivalent; by enlarging the ring we may assume that the entries in the implementing matrices $R$ and $S$ are also in $A$ (this is not really necessary — a result in [PT] asserts that the equivalence can be implemented by matrices with entries in $A^+$).

Proposition 2.1 If $M$ and $M'$ are defined over a common Laurent polynomial ring $A$, and are shift equivalent (with the shift equivalence implemented over $A$), then the shift equivalence induces an order-isomorphism of ordered $A$-algebras, $E(G_M) \to E(G_{M'})$, and an isomorphism of ordered rings $E_b(G_M) \to E_b(G_{M'})$.

Proof. The equation $MR = RM'$ induces an order-preserving $A$-module homomorphism, $\hat{R} : G_{M'} \to G_M$ by means of $[a, k]_{M'} \mapsto [Ra, k]_M$. It is a triviality to check that $\hat{M} \hat{R} = \hat{R}M'$. Similarly, $S$ induces $\hat{S} : G_M \to G_{M'}$, that also intertwines $\hat{M}$ and $\hat{M}'$. In general, having two such intertwining homomorphisms is not sufficient to obtain a ring homomorphism, either $E(G_M) \to E(G_{M'})$ or between their bounded subrings (examples exist in the case that $d = 0$). However, we also see from $RS = M^1$ and $SR = (M')^1$, that $\hat{R} \hat{S} = \hat{M}^1$ and $\hat{S} \hat{R} = (\hat{M}')^1$. Define $\phi_R : E(G_{M'}) \to E(G_M)$ as follows. Pick $e'$ in $E(G_{M'})$ and define

$$e(h) = (\hat{M})^{-1} \hat{R} e' (\hat{S} h).$$

The intertwining relations guarantee that $e(h) = (\hat{R}) e' \left( (\hat{S})(\hat{M})^{-1} h \right)$, as well. The so-defined $e$ is clearly linear. It commutes with the actions of $A$ and $\hat{M}$ (and thus with $\hat{M}^{-1}$), because of the following identities:

$$e(a \hat{M}^t h) = \hat{M}^{-1} \hat{R} e'(a \hat{M}^t \hat{S} h) = \hat{M}^{-1} \hat{R} e'(\hat{S} \cdot a \cdot \hat{M}^t \hat{S} h) = \hat{M}^{-1} \hat{R} e'(\hat{S} \cdot a \cdot \hat{M}^t \hat{S} h) = a \hat{M}^t e(\hat{S} h)$$

If $e' \geq 0$, then it follows that $\phi_R(e')$ is also positive. So $\phi_R(e')$ is at least well-defined. We check that $\phi_R$ is a ring homomorphism.

$$\phi_R(e'_1 \circ e'_2)(h) = \hat{M}^{-1} \hat{R}(e'_1 \circ e'_2)(\hat{S} h) = \hat{R}(e'_1 \circ e'_2)(\hat{S} \hat{M}^{-1} h) = \phi_R(e'_1)(\hat{R} e'_2(\hat{S} \hat{M}^{-1} h)) = \hat{R} \left[ \hat{S} \hat{M}^{-1} \hat{R} e'_2(\hat{S} \hat{M}^{-1} h) \right] = \hat{R} e'_1(\hat{S} \hat{M}^{-1} h) = \phi_R(e'_1 \circ e'_2)(h).$$

Hence under the assumptions $MR \geq RM'$, $SM = M' S$, and $SR = (M')^1$, we deduce that $\phi_R$ is an order-preserving $A$-module ring homomorphism that intertwines the action of the original pair of matrices. One defines $\phi_S$ in a similar fashion, and a five-line computation reveals that they are mutually inverse.

It is routine to verify that the bounded subrings are mapped isomorphically to each other under these isomorphisms.
Beginning with shift equivalent $M$ and $M'$, find a Laurent polynomial ring over which they and the implementing matrices are all defined. We have just obtained a pair of natural inverse ring isomorphisms between the rings of endomorphisms (after enlarging the polynomial rings). These isomorphisms induce an isomorphism on the bounded endomorphism rings; however, the latter are independent of the choice of coefficient ring. So the bounded subring is a shift invariant. (In fact, a little more is true—if the first three equations hold, we obtain a map induced by $\phi_R$ from the bounded subring corresponding to $M'$ to that corresponding to $M$.)

Example 2.2 Zero variables.

Suppose that $d = 0$, so that $M = M(1) = M_0$ is a matrix of constants with entries from either $\mathbb{Z}$ or $\mathbb{R}$. Assume the former, and that $M_0$ is primitive. Then $G_{M_0}$ is simply Krieger’s dimension group, and it is simple (as a dimension group) with a unique trace. Let $v$ in $\mathbb{R}^{1 \times n}$ be the unique (up to positive scalar multiple) strictly positive left eigenvector of $M_0$, with corresponding eigenvalue $\rho$, which is of course the spectral radius. The unique trace $V : G_M \to \mathbb{R}$ is determined via $V[c, k] = v \cdot c / \rho^k$. Then $G_{M_0}^+ \setminus \{0\} = V^{-1}((0, \infty))$. If $N$ is a matrix commuting with $(M_0)$, then $V\hat{N} = \lambda(N)V$, where $\lambda(N)$ is the eigenvalue of $N$ whose eigenvector is $v$. It follows easily that $E(G_{M_0}) = E_b(G_{M_0})$. (This equality can only occur if $d = 0$, or other equally degenerate situations.)

Thus $E_b(G_{M_0}) = C_{\mathbb{Z}}(M_0)[M_0^{-1}]$. This is a finer shift invariant (over $\mathbb{Z}$) than $\mathbb{Z}[1/\rho]$ studied in [BMT], but not so fine as the complete invariant (when $d = 0$) which is $G_{M_0}$ as a module over $E_b(G_{M_0})$.

For example, let $a$ and $b$ be positive integers. Set $M_0 = \begin{bmatrix} a & 8b \\ b & a \end{bmatrix} \quad M_0' = \begin{bmatrix} a & 4b \\ 2b & a \end{bmatrix}.$

Both matrices have $\rho = a + 2b\sqrt{2}$. The centralizer of $M_0$ is the centralizer of $\begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix}$ (subtract $a$ times the identity and divide by $b$); this is the companion matrix for the polynomial $x^2 - 8$. Hence $C_{\mathbb{Z}}(M_0) \cong \mathbb{Z}[2\sqrt{2}]$, and thus $E_b(G_{M_0}) \cong \mathbb{Z}[2\sqrt{2}][(a + 2b\sqrt{2})^{-1}]$ (the isomorphisms are given by the effect on the left eigenvector, or the function $\lambda$ defined earlier). On the other hand, the centralizer of $M_0'$ is that of $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

Thus $E_b(G_{M_0'}) \cong \mathbb{Z}[\sqrt{2}][(a + 2b\sqrt{2})^{-1}]$. It is straightforward to check that the two bounded endomorphism rings are isomorphic if and only if

\[ \mathbb{Z}[2\sqrt{2}][(a + 2b\sqrt{2})^{-1}] = \mathbb{Z}[\sqrt{2}][(a + 2b\sqrt{2})^{-1}] \]

(note the equality). This will occur if and only if there exists a positive integer $k$ so that $(a + 2b\sqrt{2})^k\sqrt{2} = c + 2d\sqrt{2}$ for some integers $c$ and $d$. On taking norms, this would imply $2(a^2 - 8b^2)^k = c^2 - 8d^2$. Hence 2 divides $c$, so 4 divides the right side. Thus 2 divides $a^2$. Conversely, if $a$ is even, $\rho^2 = 2(a - 2a^2 - 4b^2)$, so that invertibility of $\rho$ entails invertibility of 2 and thus (*) holds.

We have deduced that (*) holds if and only if $a$ is even. However, $\mathbb{Z}[\sqrt{2}]$ is a principal ideal domain, so $\mathbb{Z}[\sqrt{2}][(a + 2b\sqrt{2})^{-1}]$ is as well. Hence, with very little effort, we obtain that $M_0$ is shift equivalent to $M'_0$ if and only if $a$ is even.

On the other hand, if $M_0 = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} \quad M_0' = \begin{bmatrix} 4 & 15 \\ 1 & 4 \end{bmatrix},$

we have that the bounded endomorphism rings of both matrices are $\mathbb{Z}[\sqrt{15}]$ (as $4 + \sqrt{15}$ is a unit). The matrices correspond to the two ideal classes of $\mathbb{Z}[\sqrt{15}]$, and so are not shift equivalent (this is easy to see directly—as the matrices are invertible, shift equivalence is the same as conjugacy via $GL(2, \mathbb{Z})$).

If we add the identity to both matrices, the bounded endomorphism ring is then $\mathbb{Z}[\sqrt{15}][(5 + \sqrt{15})^{-1}]$. This is a principal ideal domain (as the ideal in $\mathbb{Z}[\sqrt{15}]$ that contains 5 and $\sqrt{15}$ represents the non-trivial ideal class). Hence in this case the matrices are shift equivalent.
Example 2.3 FOG examples (two variables).

Now let \( d = 2 \); instead of \( x_1, x_2 \), we write \( x, y \). We consider a pair of matrices suggested by Jack Wagoner and Mike Boyle in connection with FOG (“finite order generation”) and other murky conjectures, most of which have been resolved:

\[
M = \begin{pmatrix} x & 1 & 0 \\ 0 & y & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad M' = \begin{pmatrix} y & 1 & 0 \\ 0 & x & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\]

The second matrix is obtained from the first by interchanging \( x \) and \( y \). These matrices arise from the graphs:

(The unlabelled arrows have weight \( 1 = x^0 y^0 \).) The automorphism of \( A = \mathbb{Z}[x^\pm 1, y^\pm 1] \) given by interchanging \( x \) and \( y \) induces an order isomorphism \( G_M \to G_{M'} \), and correspondingly between the rings of endomorphisms, and the rings of bounded endomorphisms. These do not implement module isomorphisms, because they do not (in the first two cases) commute with the action of \( A \). Although the matrices are conjugate via an elementary matrix of \( \text{GL}(3,A) \), they are not shift equivalent; in fact they are not even finitely equivalent, a fact that was discovered independently by Marcus and Tuncel [MT2] using quite general techniques and by me (using completely ad hoc methods).

By examining the left eigenvector for the large eigenvalue function of \( M \), it is routine to check that in fact \( M \) is conjugate (i.e., using \( \text{GL}(3,A) \)) to the companion matrix for its characteristic polynomial. Thus \( C_M(A) = A[M] \), the polynomial algebra (over \( A \)) in \( M \). So \( E(G_M) = A[M^\pm 1] \)—but the bounded subring, \( E_b(G_M) \) is more difficult to compute.

We shall have more to say about this example in section 7.

Now we prove a result which is related to the converse of Proposition 2.1. This is already known.

Proposition 2.4 If \( M \) and \( M' \) are defined over a common Laurent polynomial ring (algebra) \( A \), then any order isomorphism between \( G_M \) and \( G_{M'} \) as \( A[\hat{M}]-\text{modules} \) is induced by a shift equivalence between \( M \) and \( M' \), up to a power of \( \hat{M} \).

Proof. Suppose \( \phi : G_M \to G_{M'} \) is a positive \( A \)-module homomorphism satisfying \( \phi \hat{M} = \hat{M}' \phi \). Then there exists an integer \( k \) together with positive columns \( a_j \) in \( A^{n'} \) such that for all \( j = 1, 2, \ldots, n \),

\[
\phi \left( [E_j, 1] \right) = [a_j, k].
\]

Define the \( n' \times n \) matrix \( R_0 \) by setting its \( j \)th column to be \( a_j \). All the entries of \( R_0 \) lie in \( A^+ \), and we observe that as \( \phi \) intertwines the shift(s) and is an \( A \)-module homomorphism, \( \phi \) has the same effect on \( G_M \) as applying \( R_0 \) to the equivalence classes and following this with \( \hat{M} \). In particular, \( \hat{R}_0 \) is well-defined and satisfies \( \hat{R}_0 \hat{M} = \hat{M}' \hat{R}_0 \). It follows immediately that \((R_0 M - M' R_0)M^n = 0\). If we replace \( R_0 \) by \( R_1 = R_0 M^n \), then we see that \( R_1 \) has all its entries in \( A^+ \), intertwines \( M \) and \( M' \), and satisfies \( \hat{R} = \phi \hat{M}^{n-k+1} \).

Similarly, if \( \psi : G_{M'} \to G_M \) is positive and intertwines \( \hat{M} \) and \( \hat{M}' \), there exists a rectangular matrix \( S \) with entries from \( A^+ \) that intertwines \( M \) and \( M' \) and implements \( \psi \), up to a power of \( \hat{M} \) or \( \hat{M}' \). By
multiplying on or the other of \( R_1 \) or \( S \) by a power of \( M \) we may assume that the power of \( \hat{M} \) that indicates the perturbation from \( \phi \) and \( \psi \) is the same; say \( \tilde{R}_1 = \phi \tilde{M}^t \) and \( \tilde{S} = \tilde{M}^t \psi \).

Now suppose that \( \phi \tilde{M}^t \tilde{M}^t \) for some integer \( t \). Then \( \tilde{S} \tilde{R}_1 = \tilde{M}^{t+2t} \). As before, \( (S \tilde{R}_1 - \tilde{M}^{t+2t})M^n = 0 \). Replacing \( \tilde{R}_1 \) by \( R = \tilde{R}_1 \tilde{M}^n \), we obtain that \( SR = \tilde{M}^{t+2t+n} \), and \( R \) is positive and still intertwines \( M \) and \( M' \). By incorporating another power of \( M' \) into \( S \), we obtain that if \( \phi \psi = (\tilde{M}^t)^t \), then \( RS \) is also a power of \( M' \). In particular, this yields a shift equivalence. 

Remark 1. If we let \( A \) be an arbitrary commutative unital ring and \( M \) an arbitrary square matrix (see Remark 1), then the unordered versions of Proposition 2.1 and Proposition 2.4 still hold, where we now use algebraic shift equivalence.

In view of Proposition 2.1 and Proposition 2.4, what is the point of discussing \( E_b(G_M) \) (especially since it generally is more difficult to compute than \( G_M \))? For determining the pure traces on the order ideals, it is essential (the trace space of \( E(G_M) \) is inadequate for this purpose). Moreover, the preceding depends on the choice of \( A \). There is a close relationship between \( E_b(G_M) \) and \( E(G_M) \) if \( M \) is defined over the minimal possible \( A \).

Krieger [Kr] has defined two invariants for shift equivalence, which amount to the following (where \( \text{tr} \) denotes trace). If \( f = \sum \lambda_w x^w \) is a polynomial in \( d \) variables expressed in monomial notation, then \( \log f = \{ w \in \mathbb{Z}^d \mid \lambda_w \neq 0 \} \).

\[
\Gamma(M) = \left\{ \log \text{tr} M^j \right\}_{j=1}^{\infty}
\]

\[
\Delta(M) = \bigcup_{j=1}^{\infty} (\log \text{tr} M^j - \log \text{tr} M^{j+1}).
\]

The angle brackets \( \langle \rangle \) indicate the group generated by the contents. If \( S \) and \( T \) are subsets of \( \mathbb{Z}^d \), then \( S - T = \{ s - t \mid s \in S, t \in T \} \). Note that \( \Delta(M) \) is already a group.

Lemma 2.5 If \( M(1) \) is primitive, then

\[
\Gamma(M) = \{ w \in \mathbb{Z}^d \mid x^w M^{-1} \in E_b(G_M) \text{ for some } l \in \mathbb{N} \}\]

\[
\Delta(M) = \{ w - w_0 \mid x^w M^{-1} \in E_b(G_M) \text{ for some } l \in \mathbb{N} \}.
\]

Proof: Select \( z \) in \( \log \text{tr} M^i \). This means that \( z \) is the total weight of a loop of length \( l \) whose initial and terminal state\(^1\) is \( j \) for some \( j \). There exists a loop of length \( k \) that passes through every state (as \( M(1) \) is primitive). Let its total weight be denoted \( w \). In this latter loop, insert the original loop immediately adjacent to an occurrence of \( j \). This creates a loop of length \( k + l \) containing every state, of weight \( z \).

As the loop of weight \( w \) contains every state, \( w \) belongs to \( \log (M^k)^{i_{ij}} \) for all \( i = 1, 2, \ldots, n \), \( n \) and thus \( x^w I \leq M^k \) for some positive integer \( K \) (as endomorphisms of \( G_M \)). Hence \( x^w M^{-k} \) belongs to \( E_b(G_M) \).

Similarly, \( x^{z+w} M^{-(k+l)} \) belongs to \( E_b(G_M) \). Hence \( z = z + w - w \) belongs to the right side.

Conversely, if \( x^w M^{-l} \) is a member of \( E_b(G_M) \), then \( x^w I [E_j, 1] \leq K M^l [E_j, 1] \) for some \( K \) and all \( j \). Hence there exists an integer \( N \) so that \( M^N (K M^l - x^w I) E_j \) has all of its entries in \( A^+ \). This simply means that all the entries of the matrix \( K M^{l-N} x^w M^N \) belong to \( A^+ \). Taking a diagonal entry, we see that if \( u \) belongs to \( \log (M^N)^{i_{ii}} \) for some \( i \), then \( u + w \) belongs to \( \log (M^{l+N})^{i_{ii}} \). Hence \( w = u + w - u \) expresses \( w \) as the difference of two elements of \( \Gamma(M) \).

The argument for \( \Delta(M) \) is parallel.

\(^1\) We use ‘vertex’ in another context.
Parry and Schmidt [PS] (viz. [MT, 1.9]) had shown that if $M(1)$ is primitive, there exists a diagonal matrix $D$ with monomial entries so that $DMD^{-1}$ has all of its entries in $\mathbb{Z}[\Gamma(M)]$. Since this implements a very strong form of shift equivalence, we may assume that $M$ already has support in $\Gamma(M)$, i.e. $\text{Log} M \subset \Gamma(M)$. For many purposes (including $(\mathcal{G})$), we can replace $M$ by $x^wM$ (for any lattice point $w$), and so even assume that $\text{Log} M \subset \Delta(M)$. The result in [PS] can be deduced from the following, which is interesting in itself.

Proposition 2.6 Suppose that for some $i$, no entry in the $i$th row and column of $M(1)$ zero. Then

$$\text{Log} trM \cup \text{Log} trM^2 \cup \text{Log} trM^3$$

generates $\Gamma(M)$ as an abelian group, and there exists a diagonal matrix $D$ with monomial entries such that $\text{Log} DMD^{-1} \subset \Gamma(M)$.

Proof. Let $W$ denote the subgroup of $\Gamma(M)$ generated by the three sets. For each $(k, l)$, select any exponent $w_{k,l}$ appearing in $\text{Log} M_{k,l}$, unless $M_{k,l}$ is zero, in which case we avoid that choice of $(k, l)$ for now. Whenever it makes sense, $w_{k,l} + w_{l,k} \in W$, and similarly, for any three indices, $w_{i,j} + w_{j,k} + w_{k,l} \in W$. We deduce that if $w_{k,l}$ is defined, then it is congruent modulo $W$ to $w_{l,k} - w_{i,j}$ (which is always defined). Moreover, if $w_{k,l}'$ is another choice replacing $w_{k,l}$, then their difference belongs to $W$ (since we may use the same $w_{l,k}$ and $w_{i,j}$ for $w_{k,l}'$). Set $D = \text{diag}(1, x^{w_{1,2}}, \ldots, x^{w_{1,n}})$. Then $(DMD^{-1})_{kk} = M_{kk}$, while for $k \neq l$, $(DMD^{-1})_{kl}$ is either zero or $x^{-w_{k,l}'}x^{w_{k,l}}M_{kl}$ for some $w_{k,l}$ in $W$. Thus $\text{Log} (DMD^{-1})_{kl} \subset W$ (where $\text{Log} 0$ is the empty set).

Lemma 2.7 Suppose that $M(1)$ is primitive and $\text{Log} M \subset \Delta(M)$. If $H$ and $H'$ are nonzero order ideals of $G_M$, then $H \cap H'$ is not empty. (In other words, all nonzero order ideals of $G_M$ are essential.)

Proof. We may find (nonzero positive) elements of the form $[x^wE_j, k]$ and $[x^wE_j', k]$ in $H$ and $H'$ respectively. By replacing these by $[x^wM^tE_j, k + t]$ and $[x^wM^tE_j', k + t]$ for suitable $t$, and using the convexity property of order ideals (and the fact that $M(1)$ is primitive), we may even assume that originally $j = j'$. There exists an integer $l$ so that $x^{w_{k,l}^0}E_j$ is an element of $E_b(G_M)$. In particular, if it is applied to any element of an order ideal, the outcome still belongs to that order ideal. Applying it to $[x^wE_j, k]$, the outcome is not zero, positive and lies in $H$; on the other hand, the image is $[x^{w_{k,l}^0}E_j, k + l]$. Under the current assumption that $I \preceq KM$, this element is dominated by an integer multiple of $[x^wE_j, k]$, so belongs to $H'$.

Let $A$ be one of the ordered rings $\mathbb{Z}[x_i^\pm 1]$ or $\mathbb{R}(x_i^\pm 1)$ (Laurent polynomial rings in $d$ variables), equipped with positive cone consisting of the Laurent polynomials with no negative coefficients. For two rectangular matrices of the same dimensions with entries from $A$, we write $S \preceq S'$ if $S'$ has all of its entries in $A^+$ and $\text{Log} S_{ij} \subseteq \text{Log} S'_{ij}$ for all coordinates.

Theorem 2.8 If $M(1)$ is primitive and $\Delta(M) = \mathbb{Z}^d$, then there exists $e_0 = x^{v_0}M^{-1}$ in the centre of $E_b(G_M)$ such that $E(G_M) = E_b(G_M)[e_0^{-1}]$. Additionally, for $h$ in $G_M$, if $e_0h \succeq 0$ then $h \succeq 0$.

Proof. If $f$ is an element of $E(G_M)$, we may replace it by $f$ multiplied by any power of $M$. Hence we may assume that $f = \tilde{N}$ where $N$ is in $C_A(M)$.

First assume that $\text{tr} M \neq 0$ and 0 belongs to $\text{Log} \text{tr} M$. As $M(1)$ is irreducible, there exists $k$ ($k = n!$ will do) together with a weight $w$ so that there is a loop $l$ of length $k$ containing every state from 1 to $n$ with total weight $w$. Set $W_l = \{ y \in \mathbb{Z}^d \mid x^yM^{-l+k} \in E_b(G_M) \}$.

Claim: $W_l \supseteq \bigcup_{0 \leq i \leq \ell} (w + \text{Log} trM^i)$. Select $v$ appearing in the $i$, $i$ position of $M^i$. This corresponds to a path $l'$ from $i$ to $i$ with total weight $v$. Insert $l'$ into $l$ so that the initial state $i$ appears adjacent to an occurrence of $i$ in $l$. This creates a loop $l$ of
length \( k + s \) with total weight \( w + v \). Set \( m = t - s \). As \( 0 \) belongs to \( \text{Log} \, \text{tr} \, M \), there exists a state \( j' \) so that \( 0 \) lies in \( \text{Log} \, M_{i'j'} \) yielding a loop of length \( 1 \) from \( j' \) to \( j' \). Insert the \( m \)-fold concatenation of this loop in \( l \cup l' \), adjacent to an occurrence of \( j' \). The weight is unchanged, but of course the length is increased to \( k + t \). As every state appears in the resulting loop, we have that \( x^{w+v} M^{-(k+t)} \) belongs to \( E_b(G_M) \). This proves the claimed statement.

Let \( s(1), s(2), \ldots \) be positive integers, and let \( t \geq \sum s(j) \).

**Claim:** \( w + \sum \text{Log} \, \text{tr} \, M^{s(j)} \subset W_t \).

Select \( v(j) \) in \( \text{Log} \, \text{tr} \, M^{s(j)} \); let \( l(j) \) be a corresponding loop with initial state \((i(j))\) and total weight \( v(j) \). If \( m = t - \sum s(j) \), set \( l_0 \) to be the loop consisting of the constant path at \( j' \) of length \( m \). Insert each \( l(j) \) adjacent to an occurrence of \( i(j) \) in \( l \) and \( l_0 \) adjacent to an occurrence of \( j' \). The resulting loop has total weight \( w + \sum \text{Log} \, \text{tr} \, M^{s(j)} \) and is of length \( t \). This establishes the claim.

The cone in \( \mathbb{Z}^d \) with vertex \( w \) generated by all of the \( \text{Log} \, \text{tr} \, M^{s} \) generates \( \mathbb{Z}^d \) as an abelian group, by hypothesis (\( \Delta(M) = \mathbb{Z}^d \)). Hence it contain an element \( v_0 \) such that all of \( w + v_0 \pm e_i \) \((i = 1, 2, \ldots, n)\) belong to it as well (here \( e_i \) run over the standard basis of \( \mathbb{Z}^d \)). By the last claim, there exists \( t_0 \) such that \( \{w + v_0 \pm e_i\} \subset W_{t_0} \) in fact, \( x^{w+v_0 \pm e_i} \ll M^{t_0} \).

Choose \( z \) in \( \mathbb{Z}^d \), and write \( z' = z - w \). Now decompose

\[
z' = \sum I a_i e_i + \sum I' b_{i'} e_{i'}
\]

where \( I \cup I' \) is a partition of a subset of \( \{1, 2, \ldots, n\} \), \( a_i \) are positive integers and \( b_{i'} \) are negative. Set \( m = \sum a_i + \sum |b_{i'}| \). We note that for all \( t \geq mt_0 \),

\[
w + \sum a_i (v_0 + e_i) + \sum (-b_{i'}) (v_0 - e_{i'}) \in W_t.
\]

Thus \( w + mv_0 + z \) lies in \( W_t \), whence \( x^w (v_0)^m x^z 1 \ll M^{mt_0+k} \). By the preceding, \( x^v (v_0)^m x^z 1 \ll M^{mt_0+k} \), and so \( x^z 1 \ll (x^{v_0} M^{t_0+k})^m \). Moreover, this holds with \( m \) replaced by any integer exceeding it.

Let \( E_{ij} \) denote the matrix with a 1 in the \( i \), \( j \) position and zeroes elsewhere. We shall show that for any \( z \) in \( \mathbb{Z}^d \) and choice of \( i, j \),

\[
x^z E_{ij} \ll (x^{v_0} M^{t_0+k})^m
\]

(in other words, \( z \in \text{Log} \, (x^{v_0} M^{t_0+k})^m \) \((i,j)\) for all sufficiently large \( m \)).

Choose \( v \) in \( \text{Log} \, (x^{v_0} M^{t_0+k})^m \) \((i,j)\) (note that \( t_0 + k > n! \), so every entry of the matrix is nonzero). This corresponds to a path of length \( t_0 + k \) from \( i \) to \( j \) with total weight \( v \). In particular, \( x^v E_{ij} \ll x^{v_0} M^{t_0+k} \). By the preceding, for all sufficiently large \( m \), \( x^z x^z 1 \ll (x^{v_0} M^{t_0+k})^m \), so the result obtains on multiplication.

Now let \( N \) be an element of \( C_A(M) \). Since \( \text{Log} \, N \) is finite, for all sufficiently large \( m \), for all \( i \) and \( j \),

\[
\text{Log} \, N_{ij} \subset \text{Log} \, (x^{v_0} M^{t_0+k})^m \).
\]

Hence there exists an integer \( K \) so that \(-K x^{v_0} M^{t_0+k} \ll N \ll K x^{v_0} M^{t_0+k} \). In particular, \( \hat{N} \cdot (x^{v_0} M^{-(t_0+k)})^m_1 \) belongs to \( E_b(G_M) \) for all sufficiently large \( m_1 \). Applying the argument with \( z' = -w \), we deduce that for some \( m_0 \), the element defined as \( e_0 = (x^{v_0} M^{-(t_0+k)})^{m_0} \) belongs to \( E_b(G_M) \), and is actually in the centre. Thus if \( m_1 \) is chosen to be a multiple of \( m_0 \),

\[
\hat{N} = (\hat{N} \cdot e_0) e_0^{-1}
\]

where \( \hat{N} e_0 \) belongs to \( E_b(G_M) \) and \( e_0 \) is in the centre of \( E_b(G_M) \). Moreover, \( e_0 \) is not a zero divisor in \( E(G_M) \), and thus is not a zero divisor in \( E_b(G_M) \).
If \( \text{tr} M \neq 0 \) but 0 is not in \( \log \text{tr} M \), choose \( w_0 \) in \( \log \text{tr} M \), and replace \( M \) by \( x^{-w_0} M \).

Finally, if \( \text{tr} M = 0 \) we observe that \( \text{tr} M^{s} \neq 0 \) for all \( s \geq n! \) (and in fact \( s \geq n^2 \) is sufficient). If g. c. d. \((s, n!) = 1 \) and \( NM^s = M^s N \), then \( (M^n N) M = M(M^n N) \) (by Lemma A1.1). Thus \( M^n N \) belongs to \( C_A(M) \), so \( M^n N M^{-n} \) is in \( E(G_M) \) and implements \( N \). In other words, \( C_A(M^s)[M^{-s}] = C_A(M)[M] \). So we can replace \( M \) by \( M^s \) in the preceding.

The methods used here suggest the following shift invariant. For \( M \in M_n A^+ \), define the subset of \( \mathbf{Z}^d \times \mathbf{N} \),
\[
T(M) = \{(w, t) \in \mathbf{Z}^d \times \mathbf{N} \mid \text{there exists } s \text{ such that } x^w M^s < M^{s+t}\}
\]

Our first assertion is that \( T(M) \) is a shift invariant for \( M \). This follows immediately from the observation (practically a tautology) that \( (w, t) \) belongs to \( T(M) \) if and only if \( x^w \tilde{M}^{-t} \) is an element of the bounded endomorphism ring \( E_b(G_M) \). There is a natural map from \( T(M) \) to \( \text{WPS}(M) := \cup(\log \text{tr} M^k)/k \), given by \( \tau : (w, t) \mapsto w/t \); again verification is obvious. The image is in general a proper subset, and of course its image is also a shift invariant, but there is some information lost in going from \( T(M) \) to its image. This will be seen as a consequence of the result to be proved below, that if \( d = 1 \), \( \text{cvx} \tau T(M) = \text{cvx} \text{WPS}(M) \).

(For the formal definition of \( \text{WPS} \), introduced by Marcus & Tuncel [MT], will be given in section 3.)

First, we present an example of what can happen when \( d = 2 \). Suppose
\[
M = \begin{bmatrix}
x & 0 & 1 \\
1 & y & 0 \\
0 & 1 & 1
\end{bmatrix},
\]

our familiar FOG example (2.3). Then it is fairly easy to see that
\[
T(M) = \{(a, b, t) \in \mathbf{Z}^2 \times \mathbf{N} \mid a \geq 0, b \geq 0, \text{ and } t \geq a + b + 3\}
\]

Thus although \((0, 0)\) (one of the vertices of \( \text{cvx} \text{WPS}(M) \), the standard triangle in \( \mathbf{R}^2 \)) belongs to \( \tau T(M) \) (via the element \((0, 0, 3) - 3\) can be replaced by any larger integer, but no smaller one), neither \((1, 0)\) nor \((0, 1)\) does. We do obtain that both of these points as limits of elements of \( \tau T(M) \), e.g., \( \tau((k, 0), k + 3)\), but this is a general phenomenon. In this example, \( \tau \) causes a loss of information in the sense that although \((0, 0)\) is in its image, we cannot tell that \((0, 0, 1)\) is not in \( T(M) \). This example \( E_b(G_M) \) will be investigated more deeply in the next section.

Valerio de Angelis asked whether the following is true, in the form given in the third sentence.

Proposition 2.9 Let \( M \) be a primitive square matrix of size \( n \) with entries from \( A^+ \), where \( d = 1 \). Then \( \text{cvx} \tau T(M) = \text{cvx} \text{WPS}(M) \). In particular, if \( 0 \) is the left endpoint of \( \text{cvx} \text{WPS}(M) \), there exist positive integers \( s \) and \( t \) such that \( M^s < M^{s+t} \).

Proof. The conclusion of the second sentence is that \((0, t)\) belongs to \( T(M) \). It is clearly enough to prove this, since we may replace \( x \) (the lone variable) by its inverse and multiply by a suitable power of \( x \) to obtain the other endpoint. The following proof yields extravagantly large choices for both \( s \) and \( t \), but it may be that this is necessary (particularly when \( n \) is large).

We assume that \( 0 \) is the left endpoint of \( \text{cvx} \text{WPS}(M) \); in particular, \( 0 \) itself belongs to \( \text{WPS}(M) \). The first step is to find a state \( s \in \{1, 2, \ldots, n\} \) and a positive integer \( k \) such that the set \( S = \text{Log} (M^k)_{ss} \) satisfies:
(i) \( 0 \in S \)
(ii) \( \gcd S = \gcd (\cup \text{Log} \text{tr} M^e) \).
(iii) There exists an integer \( m > 2n \) in \( S \) such that for all integers \( l \geq m \) lying in the right side of (ii), \( l \) can be expressed as a nonnegative integer combination of elements of \( S \).
Note that (ii) makes sense—$S$ cannot contain any negative integers, as $\text{WPS}(M)$ contains no negative rationals (negative exponents may appear in entries in all powers of $M$, but not on the diagonal). Of course (ii) implies there exists an integer $m$ with the properties ascribed to it in (iii), but there we also insist that $m$ belong to $S$ (this is easy to arrange).

Since 0 is in $\text{WPS}(M)$, there exists a loop of length $k_1$ (for some $k \leq n$) whose weight is zero; let $s$ be one of the states it passes through (i.e., 0 belongs to $\text{Log} M^{k_1}$). Now replace $M$ by $M^{k_1}$ (which we can do throughout). Look at all the loops on the state $s$. Since the matrix is primitive, we may assume the greatest common divisor of their weights, $c$, is that of the right side of (ii). To see this, let $u$ be a state, and suppose there is a loop $u \rightarrow u$ (of some unspecified length) with weight $v$ such that $c$ does not divide $v$. There exist paths $s \rightarrow u$ and $u \rightarrow s$ of weights $e, f$ respectively (by the primitivity of the matrix), so there exists a loop $s \rightarrow u \rightarrow u \rightarrow s$ on $s$ with weight $e + f$. Then $c$ divides $e + f$. Inserting the loop of weight $v$ on $u$ into this loop, we create a loop $s \rightarrow u \rightarrow u \rightarrow s$ of weight $v + e + f$. As $c$ must divide this, it divides $v$, a contradiction. Replace the current power of $M$ by a larger one, by raising it to the power $\prod l_i$ where $l_i$ run over the lengths of loops required to implement a set of weights whose greatest common divisor is $c$. By replacing this power by a still higher one (so that some $\text{m}^n$ belongs, which we can arrange to exceed $2n$), we obtain condition (iii). The $k$ in this construction is enormous.

Replace the original $M$ by $M^{n!}$, to absorb all minimal loops. Now select a path of length $N$ ($N$ to be determined later, but much larger than $n$) from state $i$ to state $j$, having weight $w$ (this can be negative); i.e., $x^w$ appears in $M^N_{ij}$.

If there exists a loop $i_1 \rightarrow i_2 \cdots \rightarrow i_t = i_1$ appearing in this path (permitting permutations) of weight zero, there exists a loop of length 1, $i_h \rightarrow i_h$ with weight zero, where $i_h$ is one of the states appearing in the loop. Since we may repeat $i_h \rightarrow i_h$ as often as we wish in the original path, increasing the length by 1 each time but not increasing the weight, we deduce that for all $y > 0$, $x^w$ appears in $M^{N+y}_{ij}$.

If no loop of weight zero appears in the path, we perform surgery on it. Suppose there exist $F$ loops used in the path (appearing with various multiplicities); we can ensure that $F \geq (N + n - 1)/n$, as there exists one state that appears at least $N/n$ times in the path. Since none of these loops can have weight zero, they all have nonzero weight, hence $w \geq Fc \geq c(N/n - 1)$. There exists an edge from $i$ to $s$ (since we have replaced $M$ by a large power of itself) of some weight $K$ say, and an edge in the reverse direction of $L$. Then $K + L \geq 0$. Let $p$ from $i$ to $j$ be an edge of least weight, say $e$ (which can be negative). Form a new path from $i$ to $j$ via $i \rightarrow s \rightarrow s \rightarrow \ldots s \rightarrow i \rightarrow p$ as follows. The $i$ to $s$ and $s \rightarrow i$ edges have total weight $K + L$. Let $u = w - K - L - e$ (we will adjust $N$ later so the following construction will work). Write $u = \sum_{s_i \in \mathcal{S}} a_i s_i + am$ where $a$ and the $a_i$ are positive integers; we can do this if $u$ is sufficiently large and we arrange that $a$ is defined so that $2m > u - am \geq m$. Use the loop of length 1 on $s$ of weight $m, a$ times; use the loops of length 1 and weight $s_i, a_i$ times. The upshot is a path of length $3 + a + \sum a_i$ with total weight $w$. Moreover, we can throw in any number of loops of length 1 and weight zero, since the path hits $s$.

If we guarantee that $F \geq m + K + L + e$, e.g., if $N \geq n(m + K + L + e + 1)$ and $N \geq 3 + a + \sum a_i$, then we have that $x^w$ appears in $\text{Log}(M^{N+y})_{ij}$ for all $y > 0$. Now $\sum a_i \leq 2m$ and $a \leq N/m$, so we only require $N(1 - 1/m) \geq m + 3$, i.e., $N \geq (m - 1)(m + 3)/m$ for the latter constraint. Now $m, K, L, e$ depend only on $i$ and $j$, so we take $N \geq n \max \{m(i, j) + K(i, j) + L(i, j) + 1\}$. We have shown that for all $i$ and $j$, and for all $y > 0$, $\text{Log}(M^N)_{ij} \subseteq \text{Log}(M^{N+y})_{ij}$ for all $y > 0$. Since we have made a replacement of $M$ by $M^{n!}$, we deduce $M^{n!N} \prec M^{(n+1)N}$, so that $(0, n!)$ belongs to $T(M)$.

3. The many faces of $M$

As usual, $M$ will denote an $n \times n$ matrix with entries in $A^+$, the set of $d$-variable Laurent polynomials having no negative coefficients. Define a convex polytope in $\mathbb{R}^d$ associated to $M$,

$$K(M) = \frac{1}{n!} \text{cvx} \text{Log} \text{tr} M^{n!}.$$ 

This has vertices with coefficients in $1/n! \mathbb{Z}$, and agrees with the weight per symbol polytope of Marcus and Tuncel [MT, Section 3], which is denoted $\text{WPS}_M$. If $n = 1$, this is just the Newton polyhedron (see [H5]).
Marcus and Tuncel [op. cit.] considered the facial structure of \( K(M) \). To every face \( F \), they associated a matrix \( M_F \) also with entries in \( A^+ \). We will show that there are natural maps \( E_b(G_M) \to E_b(G_{M_F}) \). In general, there are no maps \( G_M \to G_{M_F} \) nor \( E(G_M) \to E(G_{M_F}) \); the bounded subring is a shift invariant that detects facial matrices. We also use the facial matrices to determine the pure traces on \( E_b(G_M) \) and on order ideals in \( G_M \). In order to proceed, we have to see how the facial matrices behave on replacing \( M \) by powers of itself.

Define \( \beta_M : (\mathbb{R}^d)^{++} \to \mathbb{R} \) by setting \( \beta_M(r) \) to be the spectral radius of \( M(r) \). If \( M(1) \) is primitive, then \( \beta \) is a real analytic function (by Hartogs’ theorem, we may assume that there is only one variable; since \( \beta \) is a solution to the characteristic equation of \( M \) and the multiplicity of \( \beta(r) \) is one for every \( r \), analyticity follows).

Let \( A_Q \) denote the algebra generated by the monomials \( \{x^w\} \) where \( w \) is allowed to range over rational \( d \)-tuples. Obviously, \( A^+_Q \) will denote the positive cone consisting of positive linear combinations of the monomials. If \( S \) is any convex subset of \( \mathbb{R}^d \), we denote by \( \partial_s S \), the set of vertices (extreme points) of \( S \). If \( S = \text{cvx Log } P \), we sometimes abbreviate \( \partial_s \text{cvx Log } P \) to \( \partial_s \text{Log } P \).

Recall from [A, p. 290–295] that if \( p \) is a (real) analytic function on \((0, \infty)\) that satisfies an equation with coefficients from \( \mathbb{R}[t^{\pm 1}] \), then its behaviour at 0 is of fractional power order, that is, for some integers \( a \) and \( b \), \( \lim_{t \to 0^+} p(t)/t^{a/b} \) exists and is not zero, and similarly, \( \lim_{t \to \infty} p(t)/t^{a/b} \) is not zero.

**Lemma 3.1** Let \( p \) be a nonconstant function of one real variable \( t \) that is continuous on an interval \((a, \infty)\) and satisfies the equation

\[
p^n + \sum_{i=0}^{n-1} s_i t^i = 0
\]

where the \( s_i \) are rational functions. Suppose that for all sufficiently large \( t \), \( p(t) \) is nonnegative.

(a) If for all sufficiently large \( t \), \( p(t) \) is larger than the absolute value of all the other roots of the equation \( \lambda^n + \sum s_i(t)\lambda^i = 0 \), then either \( \lim_{t \to \infty} p(t) \) exists or \( \lim_{t \to \infty} p(t) = \infty \).

(b) If all of the \( s_i \) are bounded on \([a, \infty)\), then \( \lim_{t \to \infty} p(t) \) exists.

**Proof.** Suppose that \( a < a_1 < a_2 < \ldots \), is a sequence with \( \sup a_j = \infty \) such that \( p(a_j) \) converges to a limit, \( r \). As \( t \) increases, \( p(t)/t^{c/b} \) is bounded on a real neighbourhood of \( \infty \) for some integers \( b \) and \( c \), with \( 0 \leq b \leq n \). Obviously \( c > 0 \) is impossible; if \( c < 0 \), \( r = 0 \) and the limit exists. This leaves the situation that \( c = 0 \), that is, that \( p \) is bounded. This is also a consequence of (b).

Assume \( p \) is bounded; as the complement in its Stone–Čech compactification of \([a, \infty)\) is connected, if \( \beta \) has more than one limit point at infinity, it has a whole interval of them. We observe that the roots at each point are uniformly bounded, and so in case (a) holds, the symmetric functions \( s_i \) are themselves uniformly bounded. As they are rational, we may apply l’Hôpital’s rule; thus \( \lim_{t \to \infty} s_i(t) = r_i \) exists. Hence any limit point at infinity, \( r \), of \( \{p(t)\} \) must satisfy the monic equation \( r^n + \sum r_i t^i = 0 \). This equation has only finitely many roots, so \( \{p(t)\} \) admits at most finitely many limit points at infinity. Since the limit points form an interval, there can only be one, so that \( \lim p(t) \) exists.

**Lemma 3.2** Suppose that \( M(1) \) is primitive. For \( n! \) dividing \( s \), define the Laurent polynomial \( P_s = \text{tr } M^s \). Then \( \beta_M/P_s \) and \( P_s/\beta_M \) are bounded as functions on \((\mathbb{R}^d)^{++}\). Moreover, for all \( r \) in \((\mathbb{R}^d)^{++}\),

\[
\lim_{s \to \infty} \frac{P_s}{\beta_M^s}(r) = 1.
\]

**Proof.** Abbreviate \( \beta_M \) to \( \beta \). Let \( \lambda^n + \sum \sigma_i \lambda^{n-i} \) be the characteristic polynomial of \( M^s \) (so \( \sigma_1 = -\text{tr } M^s \), \( \sigma_n = (-1)^n \det M^s \), etc.). Set \( g = \beta^s/P_s \); this is a positive real analytic function on \((\mathbb{R}^d)^{++}\) such that

\[
g^n + \sum \frac{\sigma_i}{P_s} g^{n-i} = 0.
\]
We observe that as functions on \((\mathbb{R}^d)^{++}\), \(\left| \sigma_i \right| \leq O\left( \text{tr } M^i \right)\): for any matrix \(N\) with entries in \(A^+\), \(\text{tr } N^j \cdot \text{tr } N^k \leq \text{tr } N^{j+k}\), and now Newton’s identities apply. Next, we claim that \(\text{tr } M^{is} = O\left( (\text{tr } M^s)^i \right)\). It is obviously sufficient to show that every lattice point \(w\) appearing in \(\text{Log } \text{tr } M^{is}\) is a convex combination of lattice points appearing in \(\text{Log } (\text{tr } M^s)^i\). By definition, there is a loop of length \(is\) with initial state \(j\) of total weight \(w\). All minimal loops must be of length \(n\) or less, as there are only \(n\) states. Thus \(w\) can be decomposed as a sum of weights of minimal loops, \(l_1, l_2, \ldots, l_k\) with a common initial state which can be concatenated to form the original loop (we may have to perform some surgery, e.g.,

\[1 \rightarrow 2 \rightarrow 2 \rightarrow 3 \rightarrow 3 \rightarrow 2 \rightarrow 1\]

can be re-formed into \(2 \rightarrow 3 \rightarrow 3 \rightarrow 2 \) and \(2 \rightarrow 1 \rightarrow 2\) by cyclically permuting the original). Since \(n!\) divides \(s\), the concatenation of \(l_i\) occurs a suitable number of times in the graph associated to \(M^s\). If the weight of \(l_i\) is \(w(t)\) and the length of \(l_i\) is \(L(t)\), we have \(\sum L(t) = is\) and \(\sum w(t) = w\). The weight of \(l_t\) concatenated \(is/L(t)\) times is \(w(t) \cdot is/L(t)\), and so

\[
\sum_t \left( \frac{is}{L(t)} \right) w(t) \cdot \frac{L(t)}{is} = \sum w(t) = w,
\]

expressing \(w\) as a convex combination of lattice points appearing in \(\text{Log } (\text{tr } M^s)^i\).

So there exist constants \(u\) and \(U\) such that \(\left| \sigma_i \right| \leq u \text{tr } M^{is} \leq U P^s\) as functions on \((\mathbb{R}^d)^{++}\). Thus \(\sigma_i/P^s\) is bounded. Hence \(g\) is bounded; in other words, \(\beta^s = O\left( P^s\right)\). On the other hand, \(\text{tr } M^k \leq n^\beta^k\), so \(P^s = O\left( \beta^s\right)\).

The limit result is a consequence of the simple fact that for a primitive matrix \(N\) with spectral radius \(\rho\), \(\text{tr } N^s/\rho^s \longrightarrow 1\).

**Lemma 3.3** If \(Q\) and \(Q'\) belong to \(A_Q^+\) and \(Q/Q'\) is bounded on \((\mathbb{R}^d)^{++}\), then \(\text{Log } Q \subseteq \text{cvx } \text{Log } Q'\). If both \(Q/Q'\) and \(Q'/Q\) are bounded, then \(\text{cvx } \text{Log } Q = \text{cvx } \text{Log } Q'\).

**Proof.** This can be deduced from results in [H5]; we recall the method of proof. By redefining the lattice, we may replace \(A_Q^+\) by \(A^+\), that is, all the supporting monomials are exponentials of lattice points. If the first inclusion fails, then there exists \(z\) in \(\partial_0 \text{cvx } \text{Log } Q \setminus \text{cvx } \text{Log } Q'\); since \(\partial_0 \text{cvx } \text{Log } Q \subseteq \text{Log } Q\), \(z\) belongs to \(\text{Log } Q\). There exists a linear functional \(\alpha : \mathbb{R}^d \rightarrow \mathbb{R}\) such that \(a = \alpha(z) > b = \sup \left\{ \alpha \cdot w \mid w \in \text{cvx } \text{Log } Q' \right\}\); we may assume that \(\alpha\) is defined over the rationals, and thus over the integers. Define the path \(X(t) : \mathbb{R}^+ \rightarrow (\mathbb{R}^d)^{++}\) as \(\exp \alpha(t)\) (that is, write \(\alpha\) as given by the row of integers \((\alpha_1, \ldots, \alpha_d)\), and set \(X(t) = (t^{\alpha_1}, \ldots, t^{\alpha_d})\). For a monomial, \(w^w(X(t)) = t^{w^w}.\) Since all the coefficients are nonnegative, the growth of \(Q(X(t))\) (as a function of \(t\)) is at least as large as that of \(t^a\), while the growth of \(Q'(X(t))\) is at most that of \(t^b\). Hence \(Q/Q'\) is unbounded on \((\mathbb{R}^d)^{++}\), a contradiction. The second part is an immediate consequence.

**Corollary 3.4** (a) The convex polytope \(K(M)\) depends only on \(\beta\), in the sense that if \(Q\) is any element of \(A_Q^+\) such that both \((\beta^m/Q)^{\pm 1}\) are bounded on \((\mathbb{R}^d)^{++}\) for some \(m\), then \(\text{cvx } \text{Log } Q = mK(M)\).

(b) For all integers \(k\), \(K(M^k) = kK(M)\).

**Proof.** Let \(Q\) be as described. Setting \(P = \text{tr } M^n\), we have that both \((\beta^{kn}/P^k)^{\pm 1}\) are bounded. By Lemma 3.3, \(n! \text{cvx } \text{Log } Q = k \text{cvx } \text{Log } P = kn! K(M)\). Hence \(K(M)\) is determined entirely by the choice of \(\beta\), proving (a). Part (b) follows from this and Lemma 3.2.

Now let \(F\) be a face of \(K(M)\). Marcus and Tuncel [MT] have defined a matrix associated to \(M\) and \(F\). For each loop \(l\) of weight \(w\) and length \(t\) in the graph associated to \(M\) define its weight per symbol, \(wps_M(l) = \frac{1}{t} w \in Q^d\). Define

\[\text{WPS } (M) = \{ wps_M(l) \mid l \text{ is a loop in the graph of } M \}.\]
As in [MT], cvx \( WPS = K(M) \); it also follows immediately from Lemma 3.2 and Corollary 3.4.

Define a matrix \( M_F \). Suppose that \( x^v \) appears in \( M_{ij} \) with coefficient \( \lambda \), and there exists a loop \( l \) with weight \( w \) and length \( t \) such that \( w/t \in F \) and the transition \( i \rightarrow j \) with weight \( v \) is part of \( l \). Then we put \( \lambda x^v \) into the \( ij \) position of \( M_F \). If no such loop exists, the contribution to the \( ij \) entry from \( x^v \) is zero. Summing over all \( v \) in \( \log M_{ij} \), we obtain \( (M)_F \), and this describes \( M_F \) completely.

If \( N \prec M \), we may define \( N_F \) in the obvious fashion (but associated to the graph of \( M \), not of \( N \)). Associated to the graph of \( M^k \), we define \( (M^k)_F \) (by Corollary 3.4(b), \( kF \) is the face of \( K(M^k) \) corresponding to \( F \)), and if \( N \prec M^k \), we can define \( N_{kF} \) (arising from the graph of \( M^k \)).

Lemma 3.5 (a) For all integers \( k \), \( (M_k)_F = (M^k)_F \).

(b) If \( N \prec M^r \) for some integer \( r \), then for all integers \( k \)

\[
N_{rF}(M^k)_F = (NM^k)_{r+1F}.
\]

Proof. (a). As \( K(M^k) = kK(M) \), the right side is at least well defined. Consider the \( ij \) entry of \( (M^k)_F \); this will be a sum of positive linear multiples of monomials with exponents \( w \) that can be decomposed as \( w = \sum_{s=1}^{k} v_s \), where (possibly after a permutation), \( x^{v_s} \) appears in the \( is_{s+1} \) entry of \( M \) and there exists a loop \( l_s \) of length \( t(s) \) and total weight \( w(l_s) \) containing the transition \( i_s \rightarrow i_{s+1} \) with weight \( v_s \) such that \( w(l_s)/t(s) \) belongs to \( F \). Now we construct a loop that contains the path \( i = i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_{k+1} = j \) out of the loops \( l_s \). Form the path \( i \rightarrow j; \) cyclically permute the loop \( l_k \) so that its initial state is \( i_{k+1} = j \), and snip off the last transition, \( i_k \rightarrow i_{k+1} \). Attach the rest of the loop \( l_k \) to the current path; the terminal point is now \( i_k \), and we see because the original path \( i \rightarrow j \) has its last transition \( i_k \rightarrow i_{k+1} \) with weight \( v_s \), a copy of the whole of \( l_k \) is embedded in the current path. Now start \( l_{k-1} \) at \( i_k \), snip off its last transition, and attach it at the current terminal point, \( i_k \); again loop \( l_{k-1} \) is embedded in the current path, and it terminates at \( i_{k-2} \). This process may be continued until we return to \( i_1 = i \), and the resulting loop is a union of the loops \( l_s \) (see the illustration below).

The resulting loop \( l \) has total length \( \sum t(s) \) (the sum of the lengths of the loops \( l_s \) and total weight \( \sum w(s) \). Now \( \sum w(s)/\sum t(s) \) is a convex combination of \( \{w(s)/t(s)\} \); as each of the latter belongs to \( F \), so does the former. Finally we note that a loop of length \( t/k \) corresponding to \( M \) has length \( t/k \) as a loop arising from the graph of \( M^k \), if \( k \) divides \( t \). Our loop \( l \) need not have length divisible by \( k \), but we can certainly iterate it \( k \) times. The resulting loop has the same weight per symbol, and can be viewed as a loop from the graph of \( M^k \), whose first transition has weight \( \sum v(s) = w \). The outcome of this is that if \( \lambda x^w \) appears in the \( ij \) position of \( (M^k)_F \), then it also appears in the \( ij \) position of \( (M^k)_F \).

Conversely, suppose that \( x^w \) appears in the \( ij \) position of \( M^k \) and \( w \) is the weight at a transition \( i \rightarrow j \) of a loop \( l \) (with respect to \( M^k \)) of length \( t \) such that \( w(l)/t \in kF \). Viewed with respect to the graph of \( M \), it decomposes into a number of loops of length \( kt \); obviously, each has its weight per symbol in \( F \). Write \( w = \sum_{s=1}^{k} v_s \) corresponding to a decomposition (at the transition \( i \rightarrow j \). Each \( v_s \) then appears in \( (M^k)_{i_s,i_{s+1}} \), and since this covers all possible ways of obtaining \( x^w \) in \( (M^k)_{ij} \), we deduce that whatever appears in the \( ij \) position of \( (M^k)_F \) also appears in the \( ij \) position of \( (M^k)_F \).

(b). By (a), it suffices to show \( N_{rF}(M_F) = (NM)_{r+1F} \). However, the preceding argument does not require the use of \( M^r \), but merely the presence of certain monomials in the same positions.

This permits us to define an ideal of \( E_b(G_M) \).

\[
I_F = \left\{ e = \hat{N}M^{-k} \in E_b(G_M) \mid (M^nN)_{(k+n)F} = 0 \right\}.
\]

We must check that this is well defined, that is, if \( (M^nN)_{(k+n)F} = 0 \), then \( (M^nN)_{(k+n+1)F} = 0 \). As \( N \) and \( M \) commute, this follows from Lemma 3.5(b) above. The rest of the properties for it to be an ideal are verified routinely. This allows us to define a positive ring homomorphism \( \pi_F \), given by

\[
\pi_F : E_b(G_M) \rightarrow E_b(G_{MF}) \quad \hat{N}M^{-k} \mapsto (\hat{N}_{kF})(\hat{M}_{kF})^{-k}.
\]
It is clear that if $N \prec M^k$, then $N_{kF} \prec (M^k)_{kF}$. The kernel is precisely $I_F$.

This construction begs the following questions: Is $\pi_F$ onto? If $a \in \text{Image}(\pi_F) \cap E_b(G_{M_F})^+$, does $a$ belong to $\pi_F(E_b(G_M)^+)$? Is $I_F$ generated by its positive elements? If $n = 1$, the answers to all three are affirmative (see [H5]). The answer to the first question is no in general, and the second and third questions are equivalent. As a very simple example for the first question, set

$$M = \begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix}.$$  

Then $K(M)$ is the interval $[0, 1/2]$; let $F$ be the face $\{0\}$, so that $M_F$ is the identity matrix of size 2. Thus $E_b(G_{M_F}) \cong \mathbb{R} \times \mathbb{R}$. However, it is easy to check that $E_b(G_M) \cong \mathbb{R}[1/(1 + \sqrt{x})]$ as rings, and there is only one map from this to the reals that sends the generator to zero; thus the image of $E_b(G_M)$ can only be one copy of $\mathbb{R}$. Hence the map $\pi_F$ is not onto.

Without being able to decide whether the ideals $I_F = \ker \pi_F$ (for each face $F$ of $K(M)$) are order ideals, we can show that there is an order ideal $J$ contained in $I_0 := \cap I_F$ (the intersection over all faces $F$) such that $I_0/J$ is nilpotent and moreover, every pure trace that is not faithful must kill at least one of the $I_F$. The methods are elementary (if somewhat tedious) in view of earlier results. Recall from [H5] the definition of $R_P$ (which is really just $E_b(G_M)$ for $M$ the $1 \times 1$ matrix $P$ with $P$ in $A^+$), as well as $p_F$, the kernel of $R_P \rightarrow R_{P_F}$ with $P$ a face of $cvx \log P$.

**Lemma 3.6** Let $P$ be an element of $A^+$, with $K = cvx \log P$, and assume the origin is contained in $\log P$ and the interior of $K$. Set $I = \cap F \in \{\text{facets of } K\} p_F$.

(a) There exists $m$ such that $I^m \subseteq (1/P)R_P$.

(b) Suppose that $Q$ belongs to $A^+$, $Q \prec P^k$ and $Q/P^k$ belongs to $I$. For all sufficiently large $s$, there exists $t$ such that $Q^t \prec P^{ik-s}$.

(c) For any $w$ in $\mathbb{Z}^d$, there exists $l$ such that $x^w \prec P^l$.

**Proof.** (a) The minimal prime ideals that contain $1/P$ are all of the form $p_F$ where $F$ is a facet [H5]; the hypothesis that 0 is in the interior of $K$ ensures that every such ideal contains $1/P$. Hence every element of $I/(1/P)$ is nil, and noetherianness of $R_P$ yields the $m$.

(b) It suffices to prove this with $s = 1$, since we may take powers. By (a), we may write $(Q/P)^k = (1/P)r$ where $r$ belongs to $R_P$. As $r$ is bounded by a multiple of 1, it follows that there exists a positive integer $j$ such that $Q^m P^j-kn < P_j-1$. By increasing $j$, we may assume it is divisible by $k$, and write $j = kj'$. As $Q \prec P^k$, $Q^j' \prec P^j$. Hence $Q^{m+j'} \prec P^{k(m+j')-1}$, as desired.

(c) Straightforward, and already in [H5].

**Proposition 3.7** Suppose $B$ and $M$ belong to $M_n A^+$, $B \prec M$ and $Q := \text{tr } B^{n!}$ and $P := \text{tr } M^{n!}$ satisfy the hypotheses of Lemma 3.6. Then for all positive integers $t$, there exists $m$ such that $B^m \prec M^{-t}$.

**Proof.** It obviously suffices to show the inequality for some positive integer $t$, since we may take powers and exploit $B \prec M$. By Theorem 5.3, there exists $j$ such that $(B^{n!})^j \prec Q(B^{n!})^{-j-1}$. We raise this to a very high power, $l$, which will be determined shortly; we obtain $(B^{n!})^{jl} \prec Q^l(B^{n!})^{(j-1)l}$.

Given $w$ in $\mathbb{Z}^d$, there exists a depending on $w$ such that $x^{-w} \prec P^a$. Hence if $Q \prec P^{r-s}$, we have $x^wQ \prec P^{r-s+a}$. Then $Q^r \prec x^wP^{r-s+a}$, and this is true for all sufficiently large $r$ (with fixed $a$ and $s$). For all sufficiently large $l$, it follows that $Q^l \prec x^wP^{l-s+a}$. Thus

$$(B^{n!})^{jl} \prec x^wP^{l-s+a}(B^{n!})^{(j-1)l} \prec x^wP^{l-s+a}M^{n!(j-1)l}.$$  

Abbreviate $l - s + a = t$. By the first paragraph of the proof of Theorem 2.8, there exists $w$ such that $x^wP^tM^{-nt-k'}$ belongs to $E_b(G_M)$ for some $k'$, in fact, $k' = n!$ will do but is extravagantly large. In other
words, there exists $b$ such that $\gamma^w P^1 M^b < M^{n!l+k'^b}$; The $a$ will be determined by this particular choice for $w$, and then require (on $s$) that $s - a > 1$. Combining the various inequalities,
\begin{align*}
B^{njl} &< M^{n!(j-1)!} M^{n!(l-s+a)+k'+b} \\
&= M^{n!(l-s+a+(j-1)!)+k'} \\
&= M^{n!(j-l+s+a)+k'},
\end{align*}
and since $k' \leq n!$, we are done.

Let $K$ be a partially ordered abelian group; a trace of $K$ is a nonzero positive group homomorphism, $\gamma : K \to \mathbb{R}$. If there is a distinguished element (such as an order unit) $u$, we say it is normalized (at $u$) if $\gamma(u) = 1$. A trace $\gamma$ is pure (or extremal) if it cannot be written as a positive linear combination of other traces, except in a trivial way; alternatively, if $\gamma$ is normalized at an order unit, if it cannot be expressed as a nontrivial convex linear combination of other traces. A trace (usually assumed to be pure) is faithful if its kernel contains no nonzero positive elements.

Proposition 3.8 Suppose $M$ is a primitive matrix in $M_n A^+$ such that the origin is interior of $K(M)$.

(a) If $NM^{-k}$ is an element of $E_b(G_M)$ such that $\pi_F(NM^{-k}) = 0$ for all facets $F$, then there exists a power of $NM^{-k}$ that belongs to $M^{-c}E_b(G_M)$ for some positive integer $c$ such that $M^{-c}$ is in $E_b(G_M)$.

(b) If $\tau$ is pure trace of $E_b(G_M)$ that is not faithful, then there exists a facet $F$ of $K(M)$ such that $\tau(I_F) = 0$.

Proof. (a) We may obviously replace $M$ by $M^k$ and so assume $k = 1$. We have that $(N_F)(M^{-1})_F = 0$ for all facets $F$ of $K(M)$; set $B$ to be the matrix obtained from $N$ by replacing all coefficients appearing in every polynomial entry of $N$ by 1 (of course, $B$ will not commute with $M$; fortunately, this does not matter). There exists a $c$ for which $M^{-c}$ belongs to $E_b(G_M)$ by Theorem 2.8. By the preceding result, $B^m < M^{m-c'}$ for all sufficiently large $c'$ for some $m$, depending on $c'$; we may choose $c'$ to be a multiple of $c$. Then $\pm N^m < M^{m-c'}$, whence with respect to $E_b(G_M)$, $(NM^{-1})^m$ is bounded above by a multiple of $M^{-c'}$, and it is easy to check that this forces $(NM^{-1})^m$ to belong to the ideal generated by $M^{-c}$ (which is also an order ideal).

(b) If $\tau(I_F)$ is not zero for all facets $F$, then $\tau$ does not kill their product, which is contained in their intersection. By (a), every element of the intersection raised to some power belongs to $M^{-c}E_b(G_M)$, hence $\tau(M^{-c}) \neq 0$. However, it is a triviality to see that this forces $\tau$ to be faithful.

4. Faithful traces

In this section, we shall determine all the faithful pure traces on $E_b(G_M)$, $E(G_M)$, and $G_M$, as well as on the order ideals of the latter. We shall also obtain partial results about the other pure traces.

Complete knowledge of the pure traces would enable us to determine the positive cones of the partially ordered groups involved. It is equivalent to describing the boundaries for the corresponding random walks, as discussed in the Introduction. Although the pure trace space for $E_b(G_M)$ is compact, and that for $E(G_M)$ is a natural copy of Euclidean space (when $M(1)$ is primitive), in general, the descriptions are not completely straightforward. There are complications arising from the fact that at some faces, $M_F$ need not be primitive.

In this section, we first describe some faithful, pure traces of $E_b(G_M)$, $E(G_M)$, and $G_M$, that are naturally obtainable from the limit process, and then show that all are of this form when $\Gamma = \mathbb{Z}^d$ and $M(1)$ is primitive. Subsequent sections will deal with the non-faithful traces, and complete descriptions are possible only in somewhat special cases.

Let $r = (r_i)$ be a point in $(\mathbb{R}^d)^{++}$, and let $N$ be a matrix in $M_n A$. Let $N(r)$ be the matrix of real numbers obtained by evaluating each entry at $r$. If $M$ has only entries from $A^+$ (and our convention is that
this is always the case), obviously $M(r)$ is a nonnegative real matrix. It will be primitive if and only if $M(1)$ is (recall that $M(1)$ is short for $M(1, \ldots , 1)$). We obtain a natural positive homomorphism, $G_M \rightarrow G_{M(r)}$, from the following diagram:

$$
\begin{array}{cccccc}
A^n & \xrightarrow{M} & A^n & \xrightarrow{M} & A^n & \xrightarrow{M} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R}^n & \xrightarrow{M(r)} & \mathbb{R}^n & \xrightarrow{M(r)} & \mathbb{R}^n & \xrightarrow{M(r)} & \ldots \\
\end{array}
$$

The map $A^n \rightarrow \mathbb{R}^n$ sends the column of polynomials $(a_1, \ldots , a_n)^T$ to the column of real numbers, $(a_1(r), \ldots , a_n(r))^T$. The map on the limit groups is clearly onto, the image of the positive cone of $G_M$ is all of the positive cone of $G_{M(r)}$, and there is a corresponding map, $E(G_M) \rightarrow E(G_{M(r)})$ sending $\hat{N}M^{-k}$ to $N(r)\hat{M}(r)^{-k}$. We can obtain at least one trace on $E(G_M)$ and $G_M$ as follows. Let $v(r)$ be a nonnegative, nonzero left eigenvector for $M(r)$, with eigenvalue $b$. Of course, if $M(1)$ is primitive, there is only one choice for $b$, namely $\beta(r)$, and up to scalar multiple, there is only one choice for $v(r)$ and this is strictly positive. We can define a trace, $V(r) : G_M \rightarrow \mathbb{R}$ by defining a trace on $G_{M(r)}$ and composing with the map in $(r)$. The trace on $G_{M(r)}$ is defined via the diagram:

$$
\begin{array}{cccccc}
\mathbb{R}^n & \xrightarrow{M(r)} & \mathbb{R}^n & \xrightarrow{M(r)} & \mathbb{R}^n & \xrightarrow{M(r)} & \ldots \\
\downarrow \gamma(r) & & \downarrow \frac{1}{b} \gamma(r) & & \downarrow \frac{1}{b^k} \gamma(r) & & \\
\mathbb{R} & & & & & & \\
\end{array}
$$

Here the row $\gamma(r)$ acts by left multiplication on the columns $\mathbb{R}^n$. If $b$ has multiplicity 1 as an eigenvalue of $M(r)$, or if $v(r)$ has been chosen so that it is a common eigenvector of the centralizer of $M$ (this can always be arranged), define $\gamma_r : E(G_M) \rightarrow \mathbb{R}$ via $\gamma_r(\hat{N}M^{-k}) = \alpha/b^k$, where $v(r)N(r) = \alpha v(r)$. Then $\gamma_r$ is clearly a trace, and is also multiplicative; despite its notation, it depends on the choice of eigenvector, $v(r)$, unless $M(1)$ primitive.

Proposition 4.1 If $V : G_M \rightarrow \mathbb{R}$ is a trace satisfying $V(e(g)) = \gamma(e)V(g)$ for $e$ in $E(G_M)$, $g$ in $G_M$, and $\gamma$ is a multiplicative trace of $E(G_M)$, then there exist $r$ and $v(r)$ such that $\gamma = \gamma_r$ and $V = V(r)$.

Proof. Restricting $\gamma$ to $A$ (obviously, $A$ is a unital subalgebra of $E(G_M)$), acting by multiples of the identity), we obtain our candidate for $r$, from $r_i = \gamma(x_i I)$; obviously, if $p$ belongs to $A$, then $\gamma(p I) = p(r)$. Now let $E_j$ denote the standard basis elements for $A^n$. Define $v_j = V([E_j, 1])$ and $b = \gamma(M)$. Then

$$
bv_j = V(\hat{M}[E_j, 1]) = V([ME_j, 1]) = V\left(\sum_i M_{ji}E_j, 1\right) = \sum_i M_{ij}(r)v_i = \sum_i v_i M_{ij}(r);
$$

in other words, $v = (v_j)$ is a left eigenvector for $M(r)$ with eigenvalue $b$; the entries are all nonnegative as $V$ is nonnegative. Now $G_M$ is spanned as an $A$-module by $[E_j, k]$, $k = 1, 2, \ldots$. As $\hat{M}[E_j, k] = [E_j, k + 1]$, $G_M$ is spanned as an $A[\hat{M}]$-module by $[E_j, 1]$. It follows that the trace $V$ is of the form described in $(r)$ (as the trace is nonzero, neither $b$ nor all of $v_j$ can be zero). Let $N$ be a matrix in the centralizer of $M$; then the multiplicativity of $V$ and $\gamma$ permit us to apply the same equations to $\hat{N}$ that we did for $\hat{M}$; we thus obtain that $v$ is a common eigenvector for $C_A(M)$, and $\gamma$ restricts to the corresponding eigenvalue of the matrix evaluated at $r$.  

•
Not all pure traces of $E(G_M)$ need be multiplicative—in fact, there need not be any multiplicative ones at all (take $d = 0, n > 1, M$ the identity matrix of size $n$; then $E(G_M) \cong M_n(R)$ with the entrywise ordering, so it has no real-valued multiplicative homomorphisms). However, if $M(1)$ is primitive, we show these ‘point evaluation’ traces exhaust the pure rays of traces. The following is a basic result that we use in various forms.

Theorem 4.2 ([H5, I.1]) Let $H$ be a partially ordered abelian group admitting an order unit, and let $R$ be a partially ordered unital ring having 1 as an order unit, and such that $H$ is an ordered $R$-module. Then for every pure normalized trace $V$ of $(H,u)$, there exists a unique multiplicative (and pure) trace $\gamma$ of $R$ such that for all $e$ in $R$ and $h$ in $H$, $V(eh) = \gamma(e)V(h)$.

Recall that an order-preserving group homomorphism $\phi : H' \to H$ between two partially ordered abelian groups is said to be an order-embedding if $\phi$ is one to one, and $\phi(h') \geq 0$ implies $h' \geq 0$.

Lemma 4.3 If the matrix $M$ is in $M_n A^+$, then there exists an order ideal $H$ in $G_M$ having an order unit $u$, such that there is an order-embedding, $\phi : E_b(G_M) \to \oplus^n H$.

Proof. Form the limit of ordered $A$-modules,

$$M_n A \overset{M \times}{\longrightarrow} M_n A \overset{M \times}{\longrightarrow} M_n A \overset{M \times}{\longrightarrow} \ldots;$$

(we are multiplying on the left). Observing that left multiplication by $M$ acts independently on the columns, we see that as ordered $A[M]$-modules, this decomposes as a direct sum of $n$ copies of $G_M$, even as $E(G_M)$-modules. If we replace every occurrence of $M_n A$ by $C_A(M)$, the resulting direct limit is simply $E(G_M)$, and the inclusion (obtained from $C_A(M) \subset M_n A$) is an order-embedding as $E(G_M)$-modules. Thus we obtain an order-embedding, $\Phi : E(G_M) \to \oplus^n G_M$; let $\Phi_i$ denote the projection to the $i$-th copy of $G_M$.

Set $H_i$ to be the order ideal of $G_M$ generated by $\Phi(1)$ (the image of the identity). Then $H = \sum H_i$ is an order ideal in $G_M$ (a sum of order ideals in a dimension group is an order ideal), and has $\sum \Phi_i(1)$ as its order unit. Set $\phi = \Phi|E_b(G_M)$, and observe that the range of $\phi$ is contained in $\oplus^n H$; the restriction of an order-embedding is obviously an order-embedding.

Denote by $H_M$ the order ideal $H$ obtained in Lemma 4.3.

Lemma 4.4 Every pure trace $\gamma$ of $E_b(G_M)$ is multiplicative and there exists a pure trace $V$ of one of the modules $H_i$ (Lemma 4.3) such that for all $e$ in $E_b(G_M)$ and all $h$ in $H_i$,

$$V(eh) = \gamma(e)V(h).$$

Proof. By Theorem 4.2, every pure trace (normalized at the identity) is multiplicative. By Lemma 4.3, there is an order-embedding $E_b(G_M) \to \oplus H_i$; by construction, the identity is sent to the natural order unit. By [GH, 1.4], there exists a trace of $\oplus H_i$ extending $\gamma$. Since the preimage of a face in the trace space of a partially ordered group is a face, it follows we may assume the extended trace is pure. A pure trace on an ordered direct sum must restrict to zero on all but one summand, and the restriction there must be pure. Hence we obtain a pure trace $V : H_i \to R$ for some $i$ such that $V(\pi_i(e)) = \gamma(e)$ for all $e$ in $E_b(G_M)$. By Theorem 4.2, there exists a pure trace of $E_b(G_M)$, $\gamma_0$ such that $V(eh) = \gamma_0(e)V(h)$. Applying this with $h = \pi_i(1)$, we deduce $\gamma_0(e) = \gamma(e)$, that is, $\gamma_0 = \gamma$, as desired.

Now suppose $M(1)$ is primitive, and let $e_0$ be the positive element obtained in Theorem 2.8. Let $\gamma$ be any multiplicative trace of $E_b(G_M)$. If $\gamma(e_0) \neq 0$, then $\gamma$ extends uniquely to a multiplicative trace on $E(G_M)$. Let $H = H_i$ as obtained in Lemma 4.4, and let $V$ be a pure trace of the latter satisfying the properties obtained there. It follows easily from Theorem 2.8 that $G_M = \cup_k e_0^k H_i$. We may extend $V$ to a trace $V'$ on $G_M$ (by defining $V'(e^{-k}h) = V(h)/\gamma(e)^k$) that satisfies all the properties needed for Proposition 4.1 to apply. Hence $\gamma$ and $V$ are determined by an eigenvector coming from a point evaluation.

Now let $\gamma$ be an arbitrary pure trace of $E(G_M)$. Its restriction to $E_b(G_M)$ is not zero and is pure by Theorem 2.8 and [HM, I.3]. Hence it is multiplicative.
Theorem 4.5 Suppose that $M(1)$ is primitive. Let $\gamma$ be either a pure trace of $E(G_M)$ or a pure faithful trace of $E_b(G_M)$. Then $\gamma$ is obtained from a point evaluation at $r$ and a left nonnegative eigenvector of $M(r)$ as in Proposition 4.1. If $V$ is any pure trace of $G_M$, then $V$ is also obtainable from a point evaluation, as in Proposition 4.1. All of these pure traces are faithful.

Proof. The first assertion is proved in the preceding paragraphs. To establish the second, restrict $V$ to any order ideal $H$ that has an order unit which is not killed by $V$ (there must exist a positive element that is not in the kernel of $V$; let $H$ be the order ideal generated by it). By Theorem 4.2, there exists a pure trace of $E_b(G_M)$, $\gamma$ with the usual properties. Now the restriction of $V$ to any order ideal containing $H$ is also not zero; it is easy to check that the corresponding pure trace of $E_b(G_M)$ must be the same. Hence we have that for every order ideal of the form $\sum \limits_{finite} e_0^{-k} H$, the restriction of $V$ is not zero and is affiliated with $\gamma$. Suppose $\gamma(e_0) = 0$; from $H = e^k(e_0^{-k} H)$, we deduce that for all $h$ in $H$, $V(h) = e^k(e) V(e^{-k} h) = 0$, a contradiction. Hence $\gamma$ extends to a pure trace on $E(G_M) = E_b(G_M)[e_0^{-1}]$; it follows easily that $V$ satisfies the hypotheses of Proposition 4.1 and we are done. Faithfulness is an automatic consequence of the primitivity of $M(r)$.

With some care, one can use similar techniques to prove analogous results even when $M(1)$ is not primitive; in this case, $E(G_M)$ should be replaced by $E_b(G_M)[e_0^{-1}]$ (where $e_0 = x^{-v_0} M^k$ for suitable choices of $v_0$ and $k$); of course, faithfulness will generally fail. We cannot avoid the non-primitive situation, because in much of what follows, the remaining pure traces of either an order ideal $H$ or of $E_b(G_M)$ itself emanate from the nonnegative left eigenvectors (and their relatives) of the facial matrices, $M_F$, which need not be primitive.

5. Attack of the perfidious traces

Here we discuss pure traces that are not faithful. A preliminary step concerns extensions of traces from subrings. If $S$ is a partially ordered ring (with 1 in the positive cone) and $s$ is a positive element, we may form the limit ordered ring $S[s^{-1}]$ as $\lim s : S \rightarrow S$ with the limit ordering,

$$S[s^{-1}]^+ = \{ a/s^k \mid as^l \in S^+ \text{ for some positive integer } l \}. $$

In general, $S[s^{-1}]^+ \cap S = S^+$, i.e., the inclusion $S \rightarrow S[s^{-1}]$ need not be an order-embedding.

Let $Q$ be a nonzero element of $A^+$. It is easy to see that there is a natural, order-preserving map $E_b(G_M) \subset E_b(G_{QM})$ and we identify the former with its image in the latter (however, the map is not generally an order-embedding, so some care may be necessary). It turns out to be much easier to compute the pure traces on order ideals of $E_b(G_{QM})$ and $G_{QM}$ for the appropriate choices of $Q$.

Lemma 5.1 Suppose that $Q$ is a nonzero element of $A^+$ and $\hat{Q} M^{-r}$ belongs to $E_b(G_M)$ for some integer $r$.

(a) $E_b(G_{QM}) \subseteq E_b(G_M)[(Q/M^r)^{-1}]$ and this inclusion is an order embedding (with the limit ordering on the latter).

(b) If $\hat{Q} M^{-r}$ is an order unit of $E_b(G_{QM})$, then

$$E_b(G_{QM}) = E_b(G_M)[(\hat{Q} M^{-r})^{-1}]$$

and the inclusion $E_b(G_M) \subset E_b(G_{QM})$ is an order embedding.

(c) Suppose that $\hat{Q} M^{-r}$ is an order unit of $E_b(G_M)$. For $f$ in $A^n$, if $(Q M)^m f \in (A^n)^+$ for some integer $m$, then there exists $m'$ so that $M^{m'} f \in (A^n)^+$.

Proof. (a). Select $e$ in $E_b(G_{QM})^+$, write $e = \hat{N} Q M^{-1}$ with $N$ in $C_A(M)^+$ (note that $C_A(M) = C_A(QM)$) and $N < (QM)^l$. From $\hat{Q} M^{-r}$ belonging to $E_b(G_M)$, it follows that there exists a positive integer $k$ such
that $QM^k \prec M^{k+r}$. Set $t = (k + r)l$, and define $N' = NM^{t-(r+1)l} = NM^{(k-1)l}$. Clearly $N$ lies in $C_A(M)^+$ and
\[ N' \prec (QM)^l M^{t-(r-1)l} = M^{(t-r)l} Q^l = (QM^k)^l \prec (M^{k+r})^l = M^l. \]

Hence $f = N'M^{-t}$ belongs to $E_0(G_M)^+$ and the formal equation,
\[ "N'M^{-t}(Q^{-1}M)^l = N(QM)^{-l}" \]

translates to $e = f(\hat{Q}M^{-r})^{-1}$. Hence the desired inclusion of rings, and this inclusion is clearly order-preserving. If $f$ belongs to $E_b(G_M)$ and $g = f(\hat{Q}M^{-r})^{-1} \in E_b(G_M)[Q/M^t]^+$, there exists an integer $m$ such that $(\hat{Q}M^{-r})^m f \in E_b(G_M)^+$, so if additionally $g$ belongs to $E_b(G_QM)^+$, then $g$ belongs to $E_b(G_QM)^+$ (as multiplication by $QM$ is an order automorphism of $G_QM$).

(c). It is sufficient to show $Qf$ belongs to $(A^n)^+$ implies $M^m f \in (A^n)^+$. From the proof of (a), we already have that $QM^k \prec M^{k+r}$. As $\hat{Q}M^{-r}$ is an order unit, there exists $t$ so that $M^{t+r} \prec QM^l$. Since we may increase either $k$ or $t$, we may assume that $QM^k \prec M^{k+r} \prec QM^k$. Thus for all $i$ and $j$, Log $(M^{k+r})_{ij} = \text{Log} Q + \text{Log} (M^k)_{ij}$. Write $P_{ij} = (M^{k+r})_{ij}$, and set $P'_{ij} = Q(M^k)_{ij}$. Then Log $P_{ij} = \text{Log} P'_{ij}$. Clearly $P'_{ij} f \in (A^n)^+$ for all $i$ and $j$. Hence $[H5, II.1]$ yields that there exists $n(i,j)$ such that $P^{n(i,j)} f \in (A^n)^+$. Setting $m' = \max n(i,j)$, we have $P^{m'} f \in (A^n)^+$. Hence $(M^{k+r})^{m'n} f \in (A^n)^+$.

(b). The second part is an immediate consequence of (c). Now $M^{t+r} \prec QM^l$ implies $M^r \hat{Q}^{-1}$ belongs to $E_b(G_{QM})$, so part (a) yields the desired equality.

Conditions that guarantee that the conclusion of (c) hold are obtained in the next result. If $S$ is a subset of $Z^d$ and $k$ is a positive integer, as usual, we define $kS = \left\{ \sum_{i=1}^{k} s_i \mid s_i \in S \right\}$.

**Proposition 5.2** Let $M$ be an element of $M_n A^+$, and let $S$ be a subset of $Z^d$ such that for every $i$ and $j$, there exists an integer $k(i,j)$ such that Log $M_{ij} = k(i,j) S$. Let $Q$ be an element of $A^+$ such that Log $Q = S$. Suppose that $f = (f_1, \ldots, f_n)^T$ is a column of size $n$ with entries from $A$ and there exists an integer $e$ such that the column $(QM)^e f$ consists of Laurent polynomials with no negative coefficients. Then there exists an integer $e'$ such that the same is true of the column $M^{d} f$.

**Proof.** It is sufficient to show that if $Qf$ has no entries with negative coefficients then $M^{e'} f$ has the same property for some $e'$. We recall a basic result $[H5, II.1]$, a consequence of which is the following: If $Q$ and $Q'$ are elements of $A^+$ with $k \text{Log} Q = \text{Log} Q'$, and $g$ is an element of $A$ for which there exists an integer $s$ with $Q^s g$ having no negative coefficients, then there exists $t$ such that $(Q')^t g$ has no negative coefficients. The hypothesis on $Qf$ asserts that $Qf_r$, its $r$th coordinate, has no negative coefficients for $1 \leq r \leq n$. The ‘$Q$’ will vary over Log $M_{ij}$. Applying this result a finite number of times, it follow that there exists an integer $N$ such that for all $i$, $j$, and $r$, for all $N' \geq N$, we have $M_{ij}^{N'} f_r$ has no negative coefficients. Now every entry of $M^{n^2N}$ will consist of a sum of products of $n^2N$ terms from among the $M_{ij}$'s. Thus each product will have at least one of the $M_{ij}$'s repeated at least $N$ times, and so each product will make any of the $f_r$'s have no negative coefficients. It follows that for all $i$, $j$, and $r$, $(M^{n^2N})_{ij} f_r$ will have no negative coefficients, and thus the same will be true of every entry of $M^{n^2N} f$.

Later (Theorem 5.6), we will show that if merely $\text{cvx} \text{Log} M_{ij}$ are all equal (that is, for all $i$ and $j$), then the conclusion applies—the result is that for such a matrix, some power satisfies the hypothesis of the preceding with all all $k(i,j)$ being one. However, if $\text{cvx} \text{Log} M_{ij}$ are multiples of a common $d$-dimensional lattice polytope, the conclusion can fail, even in one variable. (Straightforward examples exist.)

We now study $G_{QM}$ from a different point of view. If $Q$ belongs to $A^+$, we may view it as a matrix of size $1$, and thus form $E_b(G_Q)$; this is nothing else but $R_Q$ $[H5]$. In this case, all the pure traces have
been determined, together with the positive cone \([H3, \text{Theorem A or B}].\) For a lattice polytope \(K\) and \(f = f_{M^n}\) and describes the sequence of transitions, the path, \(p\), been determined, together with the positive cone \([H3, \text{Theorem A or B}].\) For a lattice polytope \(K\) and \(f = f_{M^n}\) and describes the sequence of transitions, the path, \(p\), such that \(n_k = n_i = i\) and \(n_{k'} = n_{j'} = j\), interchange the paths \((n_k, \ldots, n_{k'})\) and \((n_i, \ldots, n_{j'})\), call the resulting path \(q\); let the relation \(p \sim q\) generate the equivalence relation. This process will be called a “regrouping”, so if \(p \equiv q\), \(q\) is obtainable from \(p\) by a sequence of regroupings. Note that if \(p \equiv q\), then obviously \(x^{(p)} = x^{(q)}\), but the converse does not hold since cyclic permutations are not allowed in this equivalence relation.

Fix \(n\). We shall now there exists an integer (depending only on \(n\)) \(N\) such that for every path \(p = (n_1, n_2, \ldots, n_{N+1})\) of length \(N\), there exists a sequence of regroupings of \(p\) and \(j\) in \(\{1, 2, \ldots, N - k\}\) such that in the regrouped path, \(n_j = n_{j+n_1}\).
Lemma 5.4 For any positive integer \( n \), there exists an integer \( N \) (depending on \( n \)), such that for every path \( p = (k_1, k_2, \ldots, k_{N+1}) \) of length \( N \) on \( n \) symbols, there exists a regrouping of \( p \) and an integer \( j \) such that \( k_j = k_{j+n!} \).

Proof. Let \( \{1, 2, \ldots, n\} \) denote the symbols. Fix \( p \) of length \( N \), and suppose the conclusion fails for this path. A gap of length \( s \) in the occurrences of \( i \) describes the following segment of a path, \( k_t = i, \ldots, k_{t+s} = i \) with \( k_{t+j} \neq i \) for \( 1 \leq j < i \) (that is, a minimal loop with vertex \( i \) of length \( s \). We obtain an estimate for the number of occurrences of any one symbol. We note that there can be at most \( n! - 1 \) gaps of length \( 1 \) in \( p \) for any particular \( i \) (if there were more, we could regroup \( p \) to obtain a sequence of \( n! \) consecutive occurrences; similarly, there can not be more than \((n! - 1)/2\) gaps of length \( 2 \) for any particular \( i \), all the way up to at most \((n! - 1)/n) \) gaps of length \( k \). All the remaining gap lengths must be at least \( n + 1 \). Hence we obtain an upper bound of \((N + 1)/(n + 1) + (n! - 1)(1 + 1/2 + 1/3 + \cdots + 1/n)\) for the number of occurrences of \( i \). As there are \( n \) symbols and the path length is \( N \), we deduce

\[
N + 1 \leq \frac{n(N + 1)}{n + 1} + (n! - 1)n(\ln(n + 1)).
\]

Hence \( N + 1 \leq (n! - 1)n(n + 1)/(\ln(n + 1)) \). In particular, if we choose \( N + 1 > (n + 2)!/n(\ln(n + 1)) \), we are done.

The estimate on \( N \) obtained in the preceding is extremely crude. To begin with, \( n! \) may be replaced by \( \text{l.c.m.} \{2, 3, \ldots, n\} \). Even then there is a lot of looseness in the argument.

Proof of Theorem 5.3: First we prove this in the case of the generic matrix, with \( A = R[x_{ij}] \) and \( M = (x_{ij}) \). A monomial appearing in \( M_{ij} \) will be of the form \( x^{(p)} \) for a path on \( n \) symbols \( p = (m_1, m_2, \ldots, m_{N+1}) \), with \( m_1 = i \) and \( m_{N+1} = j \). By the preceding, up to a regrouping (which does not affect the monomial), there is a subpath of length \( n! \), \( p' = (m_t, \ldots, m_{t+n!}) \), with \( m_t = m_{t+n!} \). Form the path of length \( N - n! \), \( p'' \), by deleting \( (m_{t+1}, \ldots, m_{t+n!}) \). Clearly, \( x^{(p'')} \cdot x^{(p')} = x^{(p)} \). Now \( x^{(p')} \) appears in \( \text{tr} \ M^{n!} \) and \( x^{(p'')} \) appears in \( M^{N-n!} \), so \( x^{(p)} \) appears in \( P M^n \).

Now let \( M \) be an arbitrary matrix in \( M_n A^+ \). Say \( x^w \) appears in \( M^N \). Then we may write \( w = w_{i_1i_2} + w_{i_2i_3} + \cdots + w_{i_nj} \), with \( w_{st} \) in \( Z^d \) belonging to \( \text{Log} \ M_{st} \). Consider the path \( p = (i_1, i_2, \ldots, i_N, j) \). There exists a subloop \( i_1, \ldots, i_{t+n!} \) in (up to a regrouping) and we note that \( \sum_{s=0}^{n!-1} w_{i_{t+s+1}, i_{t+s+2}} \). Now we observe that the sum of the \( w \)’s remaining belongs to \( \text{Log} \ M^{N-n!} \), so the sum of the two belongs to \( (\text{Log} \ M^N)_{ij} \).

Despite its plausibility, for \( P = \text{tr} \ M^{n!} \), it is not true that the element \( M^{n!}/P \) of \( E_b(G_{PM}) \) is an order unit thereof.

If in Theorem 5.3, \( P = Q^{n!} \) for some \( Q \) in \( A^+ \), it is not generally true that \( M/Q \) itself belongs to \( E_b(G_{QM}) \), even though all powers beyond the \( n! \) do belong.

Example 5.5 A matrix \( M \) in \( M_9 A^+ \) \((d = 9)\) such that the element \( M^6/P \) \((P = \text{tr} \ M^6)\) of \( E_b(G_{PM}) \) is not an order unit thereof.

Let \( M = (x_{ij}) \) be the generic matrix in the 9 variables \( \{x_{ij}\} \). Then among the monomials in \( P \) are \( x_{22}^6 \).

It is clearly sufficient to show that for all positive integers \( s < k \) and \( j, \text{Log}(P^k M^j)_{11} \subseteq \text{Log}(P^{k-s} M^{j+6s}) \) (the desired conclusion follows on setting \( s = 1 \)). One of the monomials appearing in \( (P^k M^j)_{11} \) is \( x_{22}^6 x_{11}^j \). In order to obtain a monomial involving only \( x_{11} \) and \( x_{22} \) (and no other variables) in \( P^{k-s} M^{j+6s} \), we must have that all the \( x_{22} \) terms must come from the power of \( P \) (else an \( x_{12} \) or \( x_{13} \) "transition" term would appear, arising from \( M \)). Hence \( 6(k-s) \) is the largest exponent possible for \( x_{22} \) in such a monomial. As \( s > 0 \), the non-inclusion is verified. Obviously we could do this with fewer variables.

The FOG examples (2.3) also have this property. We shall see later that failure of \( M/P \) or any of its powers to be an order unit in the relevant bounded subring is generic.

Now that we have a special endomorphism, the next stage would be to find all the pure traces of \( E_b(G_{PM}) \) and the order ideals of \( G_{PM} \). This we shall do in the next section. For the remainder of this
section, we analyze in detail a class of matrices, those satisfying (**). We determine the pure traces, and precise criteria for eventual positivity. A matrix \( M \) in \( \mathbf{M}_n \mathbb{R}[x_i^{\pm 1}]^+ \) satisfies condition (***) if for some integer \( K \), for all \( i \) and \( j \), \( \text{cvx} \log(M^K)_{ij} \) are equal. We show that this implies a formally stronger condition.

**Theorem 5.6** If the matrix \( M \) in \( \mathbf{M}_n \mathbb{R}[x_i^{\pm 1}]^+ \) satisfies condition (**), then for each sufficiently large integer \( U \), the \( n^2 \) sets of lattice points, \( \text{Log}(M^U)_{ij} \) are all equal.

We first require a combinatorial “absorption” result, extending slightly [H8, Lemma 2.1]. Recall that if \( A \) and \( B \) are subsets of \( \mathbb{Z}^d \) (or any abelian group), then for positive integers \( p \) and \( q \), \( pA + qB \) denotes the set

\[
\left\{ \sum_{i=1}^p a_i + \sum_{j=1}^q b_j \mid a_i \in A, \ b_j \in B \right\}.
\]

**Lemma 5.7** (Absorption) Let \( m \) and \( n \) be positive integers, and let \( S \) and \( T \) be finite subsets of \( \mathbb{Z}^d \) such that \( mT \subseteq nS \) and \( m \text{cvx} T = n \text{cvx} S \). Then there exists an integer \( D \) such that for all \( D' \geq D \), \( D'S + mT = (D' + n)S \). Moreover, if \( k \) is any positive integer, then \( D'S + mkT = (D' + nk)S \). Finally, if we assume merely that \( m \text{cvx} T = n \text{cvx} S \) (and not \( mT \subseteq nS \)), then \( (D' + nk)S \subseteq D'S + mkT \).

**Proof.** Choose \( P \) in \( \mathbb{R}[x_i^{\pm 1}]^+ \) such that \( \log P = S \), and let \( Q \) in \( \mathbb{R}[x_i^{\pm 1}]^+ \) be such that \( \log Q = mT \). Then the inclusion yields that \( Q/P^n \) is an element of \( R_P \), and as the coefficients of \( Q \) are all nonnegative, it belongs to \( R_P^n \). By (e.g.) [HM, Mem; V], the equality of the convex sets guarantees that it is an order unit of \( R_P \). From the definition of order unit, there exists an integer \( D \) such that \( \log P^D Q = \log P^{D+n} \), and this obviously holds if \( D \) is replaced by \( D' \). Since all the coefficients are positive, the desired equality of sets results (with \( k = 1 \)). If \( k > 1 \), we simply note that \( D'S + 2mT = (D'S + mT) + (mT) = (D' + n)S + mT = (D' + 2n)S \), etc. The final statement comes from restricting to the subset of \( mT \) consisting of the extreme points of \( \text{cvx} mT \).

**Proof of Theorem 5.6.** In the cases that \( m = n \), we see that for \( m \) fixed, there is a unique minimal subset \( T \) satisfying the hypotheses, namely the set of extreme points of \( m \text{cvx} S \). This means that the integer \( D \) can be made to depend on \( m \) alone, not on the particular choice of \( T \). In the case of interest, \( S \) or \( S_{ij} \) will be \( \log M_{ij} \), and we may assume (by (**), and replacing \( M \) by a power of itself) that \( d_\text{cvx} S_{ij} := T \) is contained in the \( \text{Log} \) set of each entry of \( M \). Hence there exists an integer \( D \), which we rename \( N \) such that for all \( i \) and \( j \), \( \log P^N M_{ij} = \log P^{N+1} \).

Set \( m = 2(N - 1)(n^2 - 1) + N \), and choose \( U \geq n^2m \). We will show that \( \log (M^{U+1})_{ij} \) is independent of the choice of \( i \) and \( j \). For an integer \( K \), let \( p := \{(i, j)\} \) be a partition of \( K \) by \( n^2 \) nonnegative integers, indexed by \( \{(i, j)\} \), that is, \( K = \sum_{i,j=1}^{n} p(i, j) \). Let \( P(K) \) denote the set of such partitions of \( K \). If \( \{S_{ij}\} \) is a collection of finite subsets of \( \mathbb{Z}^d \), then corresponding to the partition \( p \), we may construct the set obtained as the sum of sets,

\[
S(p) := \sum_{i,j} p(i, j) S_{ij}.
\]

Now let \( q = (i(1), i(2), \ldots, i(K + 1)) \) be an ordered \( K + 1 \)-tuple of integers in \( \{1, 2, \ldots, n\} \). This will be used to denote a path of length \( K \) from the symbol \( i(1) \) to \( i(K + 1) \) by means of the sequence of transitions, \( (i(1), i(2)), (i(2), i(3)), \ldots, (i(K), i(K + 1)) \). From the path \( q \), we similarly obtain a set,

\[
S_{(q)} := \sum_{t=1}^{K} S_{i(t), i(t + 1)}.
\]

In what follows, \( S_{ij} \) will be \( \log M_{ij} \), and we shall show that for our choice of \( U = K \) (on the order of
that the set described in the Lemma yields that the set \( S \) contains \( U \geq n \). Hence the absorption property described in the Lemma yields that the set in (1) is automatically contained in the right side. To prove the reverse inclusion, we select a partition of \( U \) by \( n \) terms, \( p \equiv p(i, j) \). As \( U \geq n^2m \), there exists a pair \((i(0), j(0))\) such that \( p(i(0), j(0)) \geq m \).

Define the set \( D = \{(i, j) \mid p(i, j) \leq 2N\} \setminus \{(i(0), j(0))\} \), and consider the set
\[
T := p(i(0), j(0))S_{i(0),j(0)} + \sum_{D} p(i, j)S_{i,j}.
\]

We show that \( T \) is contained in a set arising from a path of length \( K := p(i(0), j(0)) + \sum_{D} p(i, j) \) and a partition of \( K \) of size \( n^2 \). Of course, there has been a standing assumption that \( S_{ij} \) are all nonempty. We contemplate the set
\[
\sum_{(i, j) \in D} p(i, j) \left( S_{i(0),j(0)} + S_{i,j} + S_{j(0),j(0)} \right) + \left( p(i(0), j(0)) - 2 \sum_{D} p(i, j) \right) S_{i(0),j(0)}.
\]

We remark that \( \sum_{D} p(i, j) \leq (n^2 - 1)2N \), so that the coefficient of \( S(i(0), j(0)) \) in (*) is at least \( N \). Hence the absorption property described in the Lemma yields that the set in (*) contains \( T \)—observe that the set \( S(i(0), j) + S(j, j(0)) \) is “absorbed” (in the sense of the final statement of Theorem 5.6) by \( NS(i(0), j(0)) \). Note that the expression in (*) is obtainable from a path of length \( K \).

Now let \( E = \{(i, j) \mid p(i, j) \geq 2N + 1\} \setminus \{(i(0), j(0))\} \). For \((i', j')\) belonging to \( E \), we consider,
\[
S_{i(0),j(0)} + S_{j(0),i'} + S_{i',j'} + S_{j',i'} + (S_{i',j'} + S_{j',i'}) + \ldots
\]
\[
+ (S_{i',j'} + S_{j',i'}) + \left\{ S_{i',j'} \right\} + S_{j',i'} + S_{i(0),j(0)}
\]

having \( [p(i', j') + 1]/2 \) occurrences of \( S_{i',j'} \). The former exceeds \( N \), and we conclude from the absorption result that this set contains \( 2S_{i(0),j(0)} + p(i', j')S_{i',j'} \). By concatenation (using \( T \) and the set \( E \)—note that \( D \) may be empty, but this poses no problem), we conclude that the set constructed from any partition is contained in one obtained from a path of the same length as the partition, at least if the path length is sufficiently large.

Now we note that the right side of (1) can be re-expressed as \( U(S_{ij}) \) (as follows easily from the absorption lemma, and the fact that it cannot be any larger). Next we show that
\[
\bigcup_{q \in Q(U+2)} S_{(q)} = \bigcup_{p \in P(U+2)} S(p).
\]

Let \( W = d_e \log M_{ij} = d_e S_{ij} \) (for all \( i \) and \( j \)). The left side contains \( W + (\bigcup S_{(q)}) + W \), where the union is over all paths of length \( U \). This is of course equal to \( 2W + U(\bigcup S_{ij}) \), and by the absorption lemma, this equals \( (U + 2)(\bigcup S_{ij}) \), and this equals \( \bigcup_{p \in P(U+2)} S(p) \). We thereby obtain
\[
\bigcup_{q \in Q(U+2)} S_{(q)} = \bigcup_{p \in P(U+2)} S(p) = (U + 2)(\bigcup S_{ij}).
\]
We conclude (on translating back to the matrix $M$) that if $U \geq n^2 m$, then

$$
\log \left( M^{U+2} \right)_{i(1),i(U+2)} = \bigcup_{q \in \{1,2,\ldots,n\}^{U+3}} S(q) = (L + 2) \left( \bigcup_{i,j} \log M_{ij} \right).
$$

Since $i(1)$ and $i(L + 2)$ are arbitrary, we deduce that $\log \left( M^{U+2} \right)_{i,j}$ is independent of the choice of $i$ and $j$.

**Corollary 5.8** Let $M$ and $X$ be commuting members of $\mathbb{M}_n A^+$.  
(a) If for some $k$, all the entries of $M^k$ have the same Log set as each other, then the same is true of $M^k X$.
(b) If for some $k$, all the entries of $M^k$ have the same Newton polytope, the same is true of $M^k X$.

**Proof.** (a) Set $S = \log (M^k)_{ij}$ (this is independent of the choice of $i$ and $j$, by hypothesis). Then $\log (M^k X)_{rc} = \bigcup_{t=1}^n (S + \log X_{tc})$; in other words, $\log (M^k X)_{rc} = S + \log \sum_{t} X_{tc}$ (the latter term is the Log set of the column sum), which is clearly independent of the row $r$. On the other hand, $\log (X M^k)_{rc}$ is independent of the column $c$, and since $X M^k = M^k X$, $\log (M^k X)_{rc}$ is independent of both $r$ and $c$.

(b) The obvious parallel argument works.

**Corollary 5.9** If $M$ satisfies (**), then for all integers $l$, for all $w$ in $\log \text{tr} M^l$, $(w,l)$ belongs to $T(M)$ (i.e., $x^w M^{-l}$ belongs to $E_b(G_M)$).

**Proof.** Without loss of generality, we may assume that $w = 0$. By Theorem 5.6, there exists $k$ such that $S$ defined as $\log (M^k)_{ij}$ is independent of the choice of $i$ and $j$. There exists a state $i$ such that $0$ appears in $\log (M^l)_{ii}$ (by hypothesis). Thus $S + 0 \subseteq \log (M^k+l)_{ii}$. By Corollary 5.8(a) (or the “sufficiently large” part of Theorem 5.6, with a possibly larger choice for $k$), $\log (M^{k+l})_{ij}$ is independent of $i$ and $j$, and thus $S \subseteq \log (M^{k+l})_{ij}$ for all $i$ and $j$. Hence $M^k \prec M^{k+l}$, so that $M^{-l}$ belongs to $E_b(G_M)$, as desired.

If $M$ satisfies (**), it is easy to see that for every face $F$, $M_F$ is primitive, in particular, there is exactly one primitive block per face. It is conceivable that this latter property would be sufficient for shift equivalence to a matrix satisfying (**), but this is not the case. Here is a simple example illustrating this phenomenon.

**Example 5.10** A primitive matrix $M$ in $\mathbb{M}_2 A^+$ with $d = 1$ such that for all faces $F$ of $K(M)$, $M_F$ has exactly one primitive block, but $M$ is not shift equivalent to a matrix satisfying (**).

Set

$$
M = \begin{bmatrix} 1 & x \\ 1 & x^2 \end{bmatrix}.
$$

At the two zero-dimensional faces, $M_F$ is a direct sum of the zero matrix with a nonzero size one matrix, so there is exactly one primitive block per face. We notice that the monomials that appear $(M^k)_{22}$ include \{x^{2k}, x^{2k-3}, x^{2k-5}, x^{2k-6}\} but exclude \{x^{2k-1}, x^{2k-2}, x^{2k-4}\} (this follows easily by considering the graph formulation of $M$). Hence $(M^{k+2})_{22}$ does not contain $x^{2k}$; this means that for all $k$, $M^k \neq M^{k+2}$. Hence $(0,2)$ does not belong to $T(M)$ (and all the more so $(0,1)$ since $T(M)$ is an additive semigroup). However, $\log \text{tr} M$ includes $0$, so as $\text{tr} M$ and $T(M)$ are shift invariants, no matrix shift equivalent to $M$ can satisfy (**). In $G_M$.

(According to Proposition 2.9, $(0,l)$ belongs to $T(M)$ for sufficiently large $l$; here $l \geq 4$.)

There is a more drastic example already in the literature. We require an elementary preliminary result (likely a special case of a result asserting that if $M$ and $M'$ satisfy (**)) and are algebraically shift equivalent, then they are shift equivalent; this is currently unknown).
We say two matrices, $M$ and $M'$, are Log $P$-shift equivalent, if there exists a polynomial $q$ in $A^+$ such that for some $m$, $Log q = Log P^m$, and the matrices $qM$ and $qM'$ are shift equivalent. The usual choice for $P$ here will be $tr M^n$ where $n$ is the size of $M$ (smaller powers might also be useful).

Lemma 5.11 Suppose $M$ is an $n \times n$ matrix with entries from $A^+$ and satisfies (**). If additionally, $M$ has zero as an eigenvalue of multiplicity $n - 1$, then $M$ is Log $P$-shift equivalent to a polynomial, where $P = tr M$.

Proof. The hypothesis ensures that the characteristic polynomial of $M$ is $(\lambda - P)\lambda^{n-1}$, so that for $n' \geq n$, the matrix $M^{n'}$ is rank one, and of course $P^{n'}$ is the only nonzero eigenvalue. By Theorem 5.6, there exists a power of $M$, say $M^N$ with $N \geq n$ such that all the Log sets of the entries of $M^N$ are equal, and since the trace is the only eigenvalue, this Log set must be $Log P^N$. Now select any row of $M^N$, say the first, call it $v = (q_1, \ldots, q_n)$; we have that $Log q_i = Log P^N$ for all $i$. Now define a column $w = (f_i) Tr$ with entries rational functions, as follows. First $f_1 = 1$, and in general, $f_i$ will be the $i 1$-entry of $M^N$ divided by $q_i$. We note that $v$ is a left eigenvector for $M^N$, and it is easy to see that because the latter is rank one, we have $M^N = vw$. To clear the denominators of the entries of $w$, multiply through by $q := \prod q_i$. Then $Log q = n Log q_1 = n N Log P$. Moreover, both $W := qw$ and $v$ all of their entries in $A^+$, and their product is $qM$. Hence $qM$ is shift equivalent to the polynomial $Wv$, which of course equals $qP^N$. 

Example 5.12 A primitive matrix $M$ in $M_2 A^+$ with $d = 1$ such that 0 is an eigenvalue and $T(M) = T([tr M])$ is maximal, but $M$ is not even Log tr $M$-shift equivalent to a matrix satisfying (**).

Proof. We take the matrix given in [H4, p.10] (there labelled $A$),

$$M = \begin{pmatrix} x^3 + 4x + 5 & x^3 + 2x^2 + 2x + 5 \\ 11x + 33 & 11x + 33 \end{pmatrix}$$

This has the property that $M = vw$, where $w = (x^2 - x + 5, 11) Tr$ and $v = (x + 1, x + 3)$, so that 0 is an eigenvalue. Here $\beta = tr M = (x + 2)(x^2 - 2x + 38)$. The argument in op. cit. shows (among other things) that $M$ is not shift equivalent to the polynomial $\beta$ (the only polynomial to which it could be shift equivalent). In fact, all that is necessary for the same argument to work for $qM$ (with some polynomial $q$ with positive coefficients) is that $q$ have a gap at the second highest coefficient (i.e., $q = x^r + ax^{r-2} + \ldots$; here $\beta$ has a similar gap). It then follows from Lemma 5.11 that $M$ cannot be even $\beta$-shift equivalent to a matrix satisfying (**).

We can also calculate the invariant $T(M)$ in this example. We observe that $M^2 = \beta M$, and it easily follows that the pair $(a, t)$ (with $a$ an integer since $d = 1$, and $t$ a positive integer) belongs to $T(M)$ if and only if $a$ belongs to $Log \beta$—but this simply says that $T(M)$ and $T(\beta)$ (considering $\beta$ as a size one matrix) are identical. In other words, in contrast to Example 5.10, $M$ cannot be distinguished from the shift equivalence classes of polynomials by $T(M)$.

Now we proceed to obtain complete descriptions of the positive cone and the extremal traces corresponding to any order ideal, in the case that (**) holds.

Let $P$ be an element of $A^+$ such that $Log M \subseteq Log P$ (in what follows, we may replace $M$ by a power of itself, if necessary). Then we may regard $M/P$ as an element of $M_n R_P^+$, and the direct limit

$$G_{M/P} = \hat{\text{lim}} \ R_P^+ \xrightarrow{\times M/P} R_P^+ \xrightarrow{\times M/P} \cdots$$

becomes an ordered $R_P$-module. There is a natural bijection between the ordered isomorphism classes of order ideals of $G_{M/P}$ that have an order unit and those of $G_{M/P}$ (except under further hypotheses on $M$, not all order ideals of the former admit order units, even though $R_P$ is noetherian). More generally, there is a
natural map, $G_{M/P} \to G_{PM}$, given by $[a/P^j,k]_{M/P} \mapsto [M^ja,j+k]$ for $a \in A^n$. It is easy to check that this is well-defined, one to one, and an order-embedding; moreover, it preserves the $P_R$ action and changes the action of multiplication by $M/P$ to multiplication by $M/P$. Noting that multiplication by $P$ is an order isomorphism of $G_{PM}$ and the union of the ranges of the image under powers of $P$ is all of $G_{PM}$, we see that solving $(\mathcal{G})$ for $MP$ is equivalent to solving the eventual positivity problem for $G_{PM}$.

We may form the endomorphism ring of $G_{M/P}$, and its bounded endomorphism ring. The latter is immediately seen to be naturally isomorphic (in all respects) to $E_b(G_{PM})$ (on the other hand, it may happen that the endomorphism ring of $G_{M/P}$ is the bounded subring).

An order ideal of $G_{M/P}$ is called uniform, if it can be written (up to order-isomorphism, as an $R_P[\widehat{M/P}]$-module) as the direct limit,

$$G_{M/P,I} := I^n \xrightarrow{\cdot M/P} I^n \xrightarrow{\cdot M/P} I^n \xrightarrow{\cdot M/P} \ldots,$$

where $I$ is an order ideal of $R_P$. Recall ([G]), that an element $a$ of a partially ordered abelian group $G$ is an order unit if for all $g$ in $G$ there exists a positive integer $n$ such that $g \leq na$. For $G = R_P$, this is the same as $1 \leq na$, or equivalently, $L(a) > 0$ for all pure traces of $R_P$.

Let $r_{k,i}$ denote the sum of the $i$th row of $(M/P)^k$, and as usual, $\langle s \rangle$ will denote the order ideal of $R_P$ generated by $s$ therein.

Proposition 5.13 Assume $M/P$ belongs to $M_n R^+_G$. If $G_{M/P}$ admits an order unit, then the order ideals generated (in $R_P$) by the row sums of powers of $M/P$ eventually stabilize; that is, for all sufficiently large $l$,

$$\langle r_{l,i} \rangle = \langle r_{l+1,i} \rangle \quad \text{for all } i$$

Conversely, if $(\dagger)$ holds, every uniform order ideal of $G_{M/P}$ admits an order unit.

**Proof.** Since every entry of $M/P$ belongs to $R_P$, it follows that for all $k$, $\langle r_{k+1,i} \rangle \subseteq \langle r_{k,i} \rangle$. Say the order unit of $G_{M/P}$ is of the form $W = [(u_1, \ldots, u_n)^T, k]$ for $u_i \in R_P$ and $k$ a positive integer. Since anything bigger than an order unit is an order unit, we may assume $u_i$ are themselves order units of $R_P$ and all equal to 1. As $W$ is an order unit, $[(1, 1, \ldots, 1)^T, k + 1] \leq KW$ for some positive integer $K$. Hence there exists $l$ such that $(M/P)^l(1, \ldots, 1)^T \leq K(M/P)^{l+1}(1, \ldots, 1)^T$. Note that this still holds if $l$ is increased (permitting $K$ to increase). Taking each entry, we obtain $r_{l,i} \leq K r_{l+1,i}$. Hence $\langle r_{l,i} \rangle \subseteq \langle r_{l+1,i} \rangle$, yielding equality for all larger $l$.

Suppose $(\dagger)$ holds; let $I$ be an order ideal of $R_P$ and form the corresponding uniform order ideal, $G_{M/P,I}$, of $G_{M/P}$. Since $R_P$ is noetherian, every order ideal admits an order unit (this is in [H5]), but follows from the fact an order ideal in a partially ordered ring having 1 as an order unit is an ideal in the ring); let $u$ be an order unit of $I$. If $k$ is any integer such that $(\dagger)$ holds for all $l \geq k$, we shall show that $U = [(u, u, \ldots, u), k]$ is an order unit of $G_{M/P,I}$. Now $(\dagger)$ entails that $(M/P)^k(1, \ldots, 1)^T \leq K(M/P)^{k+1}(1, \ldots, 1)^T$ (for some $K$); multiplying this by $u$ yields that $[(u, u, \ldots, u), k + 1] \leq KU$. This iterates and shows that $U$ is indeed an order unit for $G_{M/P,I}$.

If $PM^{-1}$ is in $E_b(G_M)$ (equivalently, $M/P$ is an order unit of $G_M$), then $(\dagger)$ holds; it also holds if every row sum of $M/P$ is itself an order unit. To see this, it is clearly enough to prove that the order ideal, $I$ in $P^n$ generated by $(M/P)^n$ is $I^n$ itself. Every order ideal of $I^n$ is of the form $\oplus J_t$. If $I = \oplus J_t$ with $J_t \neq I$ for some $t$, there exists a pure trace $\gamma$ of $I$ such that $\gamma(J_t) = 0$ (since $I/J_t$ is a nonzero dimension group with order unit). There is a pure trace $L$ of $R_P$ such that $\gamma(rs) = L(r)\gamma(s)$ for $r$ in $R_P$ and $s$ in $I$. However, $J_t$ is the order ideal generated by $(M/P)_{ij}I$ (as $j$ varies). Applying $\gamma$ to all of these and knowing they must all vanish, we deduce $L((M/P)_{ij}) = 0$ for all $j$. Hence the $t$th row sum vanishes at $L$ so cannot be an order unit, a contradiction.
Conversely, if not all row sums are order units, there exists a pure trace \( L \) of \( R_P \) that wipes out the \( t \)th row for some \( t \) (if a positive element is not an order unit, it is killed by some pure trace). By [H5, V.3, p. 45], there exists a pure trace, \( \gamma \), of the order ideal \( I \), such that \( \gamma(rs) = L(r)\gamma(s) \). Then \( \gamma \) applied to the \( t \)th coordinates of \( (M/P)^k I^m \) will be zero for all \( k \). This does not prevent \( G_{M/P,I} \) from having an order unit (because although \( u(1, \ldots, 1)^T \) will not be dominated by a multiple, \( (M/P)u(1, \ldots, 1)^T \), applying higher powers of \( M/P \) to both sides may change this). For example, set \( M = \begin{bmatrix} 1 & 2 \\ 3x & 4x \end{bmatrix} \) and \( P = 1 + x \). Neither row sum is an order unit (and of course this persists in all powers), but we note that the order ideal of \( R_P \) generated by the top row of \( (M/P)^k \) is the ideal \( (1/P)R_P \) and that generated by the bottom row is \( (x/P)R_P \) for any \( k \); hence the conditions of Proposition 5.13 are satisfied without any row sums being order units. It may be that this example behaves very nicely because if we change the coefficients (but not the monomials), we can transform the matrix into \( \begin{bmatrix} 1 & 1 \\ x & x \end{bmatrix} \) which is lag-1 related to a \( 1 \times 1 \) matrix, which means that all possible properties are satisfied (see section 11).

Call a column \( U \) in \( (\mathbb{R}[x^{x+1}])^n \) homogeneous if \( \text{Log} U \) are all equal. Suppose that for infinitely many \( m \), \( \text{Log} (M^m U)_i \) is independent of the choice of \( i \) in \( \{1, 2, \ldots, n\} \). This is analogous to a uniform order ideal admitting an order unit. Note that when this happens, we may take \( m = kn! \) and then it seems likely that \( \text{Log} (M^{kn!} U)_i + \text{Log tr} M^{n!} = \text{Log} (M^{(k+1)n!} U)_i \) for each \( i \) and sufficiently large \( k \).

Clearly, it is necessary that the sum of all the coefficients of \( M/P \) be an order unit, but this is far from sufficient. Note that one consequence of the stronger condition that all row sums be order units is that no row of \( L(M/P) \) (or of a power) can vanish identically for some pure trace \( L \) of \( R_P \). Conversely, if no row of \( L(M/P) \) vanishes merely for each pure trace \( L \) corresponding to extreme points of \( \text{cvx} \text{Log} P \) (there are only finitely many of them), then each row sum of all the powers are order units. Simply note that by [HM, V.3(d), p. 47], a positive element of \( R_P \) (in this case the relevant row sum) is an order unit if it is strictly positive at this finite set of pure traces.

In particular, the condition (***) is much stronger than (†). The condition (†) may fail, however, for essentially trivial reasons. It is unknown at present if sufficient for (†) to hold is that there exist a nonzero uniform order ideal of \( G_{M/P} \) that admits an order unit.

The pure traces on a uniform order ideal of \( G_{M/P} \) are relatively easy to describe. Begin with a pure trace of \( \gamma \) of the order ideal \( I \). Associated to \( \gamma \) is a (unique) pure trace \( L \) of \( R_P \) (hence \( L \) is a multiplicative positive function) such that for all \( s \) in \( R_P \) and \( t \) in \( I \), \( \gamma(st) = L(s)\gamma(t) \) (and either \( L \) restricts to a scalar multiple of \( \gamma \), or \( L(I) = 0 \) [HM, Lemma I.2(c), p. 4]. The matrix \( L(M/P) \) obtained by evaluating every entry of \( M/P \) at \( L \) is a nonnegative real square matrix (it need not be irreducible). By raising it to the \( n! \) power, we may assume all irreducible blocks are primitive. The Perron-Frobenius theorem applies and it has at least one nonnegative left eigenvector. Let \( v \) be an extremal left nonnegative eigenvector (that is, not a nonnegative combination of other nonnegative left eigenvectors, except in a trivial way) whose eigenvalue is not zero. This is supported on a union of the supports of primitive blocks, and the corresponding cut down square matrix is called indecomposable (e.g., \( \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \) has two extremal nonnegative left eigenvectors \((1, 1)\) and \((0, 1)\)—the former is supported on both blocks) and denote the eigenvalue \( \lambda \). We construct a trace \( \tau \equiv \tau _{\gamma,v} \) by means of the following limit diagram:

\[
\begin{array}{cccccc}
I^n & \xrightarrow{M/P} & I^n & \xrightarrow{M/P} & I^n & \xrightarrow{M/P} & \ldots, \\
\gamma^n & \downarrow & \gamma^n & \downarrow & \gamma^n & \downarrow & \\
\mathbb{R}^n & \xrightarrow{L(M/P)} & \mathbb{R}^n & \xrightarrow{L(M/P)} & \mathbb{R}^n & \xrightarrow{L(M/P)} & \ldots, \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
v & \xrightarrow{\lambda} & v & \xrightarrow{\lambda} & v & \xrightarrow{\lambda} & \ldots \\
\mathbb{R}
\end{array}
\]
In other words,
\[ \tau[(a_1, a_2, \ldots, a_n)^T, k] = \frac{v}{\lambda^{k-1}} \cdot (\gamma(a_1), \gamma(a_2), \ldots, \gamma(a_n))^T. \]

Notice that if \( L \) is a point evaluation trace on \( R_P \), then \( \gamma \) must be its restriction to \( I \) (up to multiplication by a scalar), and our hypotheses on \( M/P \) yield that \( L(M/P) = (M/P)(\tau) \) (evaluation a strictly positive \( d \)-tuple) is a primitive matrix, hence the “\( v \)” is unique. Thus \( \tau \) is merely point evaluation followed by hitting with a strictly positive left eigenvector; such \( \tau \) will also be called point evaluations.

At the other extreme, occurring when \( \gamma \) is not a point evaluation (and kills \( I^2 \)), there can be many extremal nonnegative left eigenvectors.

**Theorem 5.14** Let \( G_{M/P, I} \) be a uniform order ideal of \( G_{M/P} \) that admits an order unit. Then the pure traces of it are precisely of the form \( \tau = \tau_{\gamma,v}, \) every such is pure, and every \( \gamma \) will yield at least one such \( \tau \). A pure trace is a point evaluation if and only if \( \gamma = L | I \) (up to scalar multiple).

**Proof.** The diagram above commutes, and it follows that any such \( \tau \) is a trace (even if \( v \) is merely a nonnegative eigenvector, not just one obtained from an irreducible block—only the latter yield extreme traces). First, we show that each such trace is extreme. We use the purity criterion given in [GH2, Theorem 3.1]: If \( a \) and \( b \) are positive elements of a dimension group \( G \) (with order unit, although this is not essential), and \( \rho \) is a trace thereof, then \( \rho \) is pure (i.e., extreme) if and only if for all \( \epsilon > 0 \), there exists \( z \) in \( G^+ \) such that \( z \leq a, b \) and \( \tau(z) \geq \min \{\tau(a), \tau(b)\} - \epsilon \).

For \( G \equiv G_{M/P, I} \) and \( \rho \equiv \tau_{\gamma,v} \) it is easy to see that we may reduce to the case that \( a = [(A_1, \ldots, A_n)^T, k] \) and \( b = [(B_1, \ldots, B_n)^T, k] \) with \( A_j \) and \( B_j \) being positive elements of \( I \). Set \( A = (A_1, \ldots, A_n)^T \), and \( B = (B_1, \ldots, B_n)^T \). As usual, \( \lambda \) will denote the spectral radius of the relevant primitive block of \( L(M/P) \).

We may also assume that the sum of the coefficients of \( v \) is 1.

If \( c \) is a square \((n \times n)\) matrix or a column of size \( n \), denote its cut-down to the first \( m \) indices, \( c_{(m)} \). For convenience, we may reorder the index set \( \{1, 2, \ldots, n\} \) and so assume that the indecomposable block of \( L(M/P) \) emanates from which \( v \) emanates is supported on indices \( \{1, \ldots, m\} \); the primitive block with eigenvalue \( \lambda \) in the support of \( v \) (there is only one) is the support of the right eigenvector of \( L(M/P)_{(m)} \) with eigenvalue \( \lambda \), and we may assume the support is \( \{1, 2, \ldots, m'\} \) for some \( m' \leq m \); we denote this right eigenvector \( w \). In particular, \( v_i > 0 \) if and only if \( i \leq m \) and \( w_i > 0 \) if and only if \( i \leq m' \), and note that normalized powers of \( L(M/P)_{(m)} \) converge to the matrix \( wv_{(m)}/v_{(m)} \cdot w \). Also for convenience, \( \gamma^n(A) := (\gamma(A_1), \ldots, \gamma(a_n))^T \) in \( R^n \) will be denoted \( \gamma(A) \).

Suppose \( \tau(a) \leq \tau(b) \). If \( \tau(a) = 0 \), set \( z = 0 \) and we are done. Otherwise, since the ordered abelian groups with which we are dealing are real vector spaces, we may assume \( \tau(a) = \tau(b) = 1 \), so that \( v_{(m)} \cdot \gamma(A)_{(m)} > 0 \). Normalize \( w \) so that \( v_{(m)} \cdot w = 1 \). Thus \( (L(M/P)_{(m)})^I/\lambda^I \) converges to \( wv \). Hence \( (L(M/P)_{(m)})^I/\lambda^I C \) converges to \( (v \cdot C)w \) for \( C \) an \( m \)-tuple of real numbers. Applying this with \( C = \gamma(A_{(m)}) \) and \( C = \gamma(B_{(m)}) \), we deduce the existence of an integer \( l \) such that each entry of \( (L(M/P)_{(m)})^I/\lambda^I(\gamma(B)_{(m)})-\gamma(A)_{(m)}) \) has absolute value less than \( \epsilon \) (recall that \( \tau(a) = v \cdot \gamma(A)/\lambda^{k-1} \)).

Now consider the elements \( A' = (M/P)^{l+k}A/\lambda^l \) and \( B' = (M/P)^{l+k}B/\lambda^l \). By the result of the previous paragraph, \( \gamma(A'_j) - \gamma(B'_j) < \epsilon \) for each \( j \leq m \). Since \( \gamma \) is a pure trace of the dimension group \( I \), there exist \( z_j \) in \( I^+ \) such that \( z_j \leq A'_j, B'_j \) (as elements of \( I \)) and \( \gamma(z_j) > \min \{\tau(A'_j), B'_j\} - \epsilon \). Define \( Z \) in \( (I^n)^+ \) by setting its \( j \)th entry to be \( z_j \) if \( j \leq m' \) and zero if \( j > m' \). Then \( z = [\lambda^l Z, l + k] \) satisfies all the required properties to verify the extremal criterion.

Now we show that any limit of traces of the form \( \tau_{\gamma,v} \) is of the same form except that \( v \) need not be an extremal left eigenvector of \( L(M/P)_{(m)} \) (merely a strictly positive left eigenvector, not necessarily irreducible) and \( \gamma \) while being a trace of \( I \) need not be pure. That is, if \( \tau \) is a limit of these special traces, there exists \( L \) a pure trace of \( R_P \), a trace \( \gamma \) (not necessarily pure) of \( I \) such that \( \gamma(sa) = L(s)\gamma(a) \), and a nonnegative left eigenvector \( v \) of \( L(M/P) \) so that \( \tau \) is given as in (5.1).
If $\tau$ is such a limit, say of $\{\tau_k\}$ where $\tau_k$ depends on $\gamma_k$, $L_k$, and $v_k$, we first observe that for any element $u$ such that $\tau(u) > 0$ and $s$ in $R_P$, $L_k(s) = \tau_k(su)/\tau_k(u)$. Since each $L_k$ is a multiplicative trace of $R_P$, $L_k$ converge to a multiplicative (hence pure) trace of $R_P$, such that $\tau(su) = L(s)\tau(u)$. Similarly, $\gamma_k(a)$ is a scalar multiple of each of $\tau_k([ae_i, j])$ (except those $i$ for which the latter are zero), we find easily that $\gamma_k$ (normalized) must converge to a trace on $I$ (limits of such traces need not be pure, of course). By recalculating the coefficients of $\gamma_k$ that appear in $\tau_k([ae_i, j])$, we easily find the row $v$ that implements $\tau$, and it is easy to check it is an eigenvector.

Let $X$ denote the set of normalized pure traces of $G_{M/P,I}$ of the form described in (5.1), and let $\overline{X}$ denote its closure in the set of normalized traces; we denote the latter $S$ ($S$ is a Choquet simplex). We have just seen that $\overline{X}/X$ contains no pure traces. If there exists a pure trace $\psi$ not in $X$, then by the standard separation theorem for Choquet simplices, e.g., [AE, Theorem 1.1], of pure traces from compact sets of them, there exists an affine continuous function $\alpha : S \to \mathbb{R}$ such that $\alpha(\tau) \geq 1$ for all $\tau$ in $\overline{X}$ and $\alpha(\psi) \leq -1$. The image of $G_{M/P,I}$ in $\text{Aff}(S)$ is dense since the former is a real vector space. Hence there exists $a$ in $G_{M/P,I}$ such that $\tau(a) \geq 1$ for all $\tau$ in $\overline{X}$ and $\psi(a) \leq -1$. We will deduce a contradiction by showing that $a$ is positive (in fact it is an order unit, but this follows from the conclusion).

Write $a = [(a_1, \ldots, a_n)^T, k]$ with $a_i$ in $I$. Set $A = (a_1, \ldots, a_n)^T$ in $I^n$. By hypothesis, $v \cdot \gamma(A) > 0$ for all pure traces $\gamma$ (and the corresponding pure eigenvectors of $L(M/P)$, where $L$ corresponds to $\tau_{v, v}$. Fix a pure trace $\gamma$ of $I$, and its induced pure trace $L$ of $R_P$. The hypotheses assert that for all pure left eigenvectors $v$ of $L(M/P), v \cdot \gamma(A) > 0$. Hence (an easy argument even in this reducible case—see Appendix 2 for a more complicated version), there exists $m \equiv m(\gamma)$ such that the real column $L(M/P)^m \gamma(A)$ is strictly positive. For $L$ fixed, there are only finitely many pure traces $\gamma$ such that $\gamma(sj) = L(sj) \gamma(j)$ (just note that $I/\ker L$ is a finite dimensional real vector space). Hence we obtain $m$ depending only on $L$, not on $\gamma$, and moreover, the least entry of $L(M/P)^m \gamma(A)$ (as $\gamma$ varies over the finite set) exceeds zero, say is $\epsilon_L$. There thus exists an open subset of the (compact) space of pure traces on $R_P$ such that $L'$ in this set entails the least entry of $L(M/P)^m \gamma(A)$ exceeds $\epsilon_L/2$. These open sets cover the space, and by compactness, we obtain a finite subcover. Thus there exists a fixed $m$ such that every entry of $L(M/P)^m \gamma(A)$ is strictly positive. Hence $\gamma(((M/P)^mA)_i) > 0$ for all $\gamma$ and all $i$. Thus each $((M/P)^mA)_i$ is a positive element of $I$, so that $a$ is in the positive cone of $G_{M/P,I}$. Of course, this contradicts $\psi(a) < 0$, so no such $\psi$ exists.

The map $I \to G_{M/P,I}$ given by $b \mapsto [(b, b, \ldots, b)^T, 1]$ induces an affine (convex linear) map $S(G_{M/P}) \to S(I)$ (pick order units and normalizations to be compatible; the map on pure traces is then $\tau_{v, v} \mapsto \gamma$). A particular consequence of the previous result is that this map is onto and sends pure traces to pure traces. If $L(M/P)$ has only one nonnegative left eigenvector for each $L$, then the map is a homeomorphism on the pure traces, hence is a homeomorphism between the traces spaces. In particular, this applies if $M$ is a large enough power of a matrix satisfying (**) and $\text{Log} P = tr M$ (since $L(M/P)$ is then primitive). Something pleasant also occurs if (**) holds: every ideal of $G_{M/P}$ is uniform and admits an order unit.

Lemma 5.15 If every entry of some power of $M/P$ is an order unit of $R_P$, then every order ideal of $G_{M/P}$ is uniform.

Proof. We may replace $M/P$ by the appropriate power. Let $D$ be the order ideal of $G_{M/P}$. Denote the order ideal of $R_P$ generated by

$$\{ b \in R_P^+ \mid \text{there exists } B = (B_i)^T \in R_P^+ \text{ with } [B, k] \in D \text{ and } b = B_j \}$$

by $I_{k,j}$. Since each entry of $M/P$ is an order unit, we see immediately that $I_{k+1,i} = \sum_j I_{k,j}$ for all $i$ and $k$. Hence $I_{k,j} \equiv I$ is independent of $k$ and $j$, so $D = G_{M/P,I}$.

In the following, note the appearance of $G_M$, not simply $G_{M/P}$.

Theorem 5.16 If $M$ satisfies (**), then the collection of order ideals of $G$ are in a natural bijection with the order ideals of $R_P$ where $P = tr M_{n1}$, and the pure trace space of any order ideal of
Proof. Follows immediately from the previous results and standard results on extensions of traces.  

6. Return of the perfidious traces

The notion of uniformity in section 5 was useful for showing that at least some of the time, the pure (non-faithful) traces on order ideals of $G_{M,P}$ can be completely described in terms of the facial structure (Theorem 5.14). In this section, we obtain a partial description of pure traces but in a slightly more general setting, exploiting the idea of the argument involved in the proof of Theorem A1.3. The method involves the order ideal $I$ of $G$ all order ideals with order units are of this form. We may also consider the slightly larger order ideal, left eigenvectors of the cut-down matrix $M/P$ are then determined. In the case of $d > 0$ variables, we use this and try to reduce to this case.

We now describe one family of traces on order ideals (essentially the same as those described in Theorem 5.14) of order ideals of $G_{M,P}$; then we show that every pure trace is of this form when $\lambda(M/P) \neq 0$ ($\lambda$ being the trace). Let $\{I_i\}_{1 \leq i \leq n}$ be a set of order ideals of $R_P$, set $W = \bigoplus I_i E_i$. Assume that $W$ is invariant under the action of $M/P$. Let $H$ be the order ideal of $G_M$ generated by all choices for $[z_i E_i, 1]$ with $z_i$ varying over $I_i$. Since each $I_i$ admits an order unit, so does $H$. (It turns out that up to order-isomorphism, all order ideals with order units are of this form.) We may also consider the slightly larger order ideal, $\tilde{H} = \lim M/P : W \to W$, obtained by “inverting” $M/P$. It frequently happens that $\tilde{H} = H$. Now let $L$ be a pure trace of $R_P$ such that $L(M/P)$ is not a nilpotent matrix.

If $L$ is a faithful trace, it must be a point evaluation, given as $b \mapsto b(r)$ for some fixed $r \in (R^d)^{++}$. Since we are assuming $M(1)$ (and therefore $M(r)$ is primitive), there is then only one choice for the corresponding trace on $H$ — it must be faithful, so Theorem 4.5 applies. Hence we may assume $L$ is not faithful. Then, as in [HM, VII.5], there exists an order ideal $p$ such that $\ker L \cap R_P^+ = \ker p$ and $p$ corresponds to a face $F$ of $\text{cvx} \log P$, in the sense that $p$ is generated as an ideal (not simply as an order ideal) by $\{x^w / P \mid w \in \log P \setminus F\}$. If we define $P_F = \sum_{w \in \log P \setminus F} c_w x^w$ (where $P = \sum c_w x^w$), then the natural map $R_P \to R_{P_F}$ given by $g/P^k \mapsto g_{kF}/P_F^k$ (where $g_{kF}$ is the polynomial obtained from $g$ by removing all the terms $x^w$ for which $w \notin kF$) has kernel the prime ideal $p$, and the quotient ordering on $R_{P_F}$ agrees with its own natural ordering.

Let $I = \sum I_i$, and let $\phi : I \to I$ be a pure trace affiliated with $L$ (that is, $\phi(\alpha e) = L(\alpha)\phi(e)$ for $\alpha \in R_P$ and $e \in I$). Let $S = \{i \mid \phi(I_i) \neq 0\}$ and cut down the columns and $M/P$ itself, by deleting all the coordinates not in $S$. Let $(M/P)_S$ be the resulting $|S| \times |S|$ matrix. Form the corresponding $W_S$, $H_S$, $\tilde{H}_S$, etc.

Let $\rho$ be a positive eigenvalue of the real matrix $L((M/P)_S)$ for which there exists a nonnegative right eigenvector. Let $v = (v(i))_{i \in S}$ be an extremal nonnegative right eigenvector for $\rho$ (that is, $v$ is a nonnegative eigenvector and it cannot be expressed as a non-trivial sum of other other nonnegative left eigenvectors of $L((M/P)_S)$ for $\rho$). Define a trace $\gamma_S$ on $H_S$ using $v, \phi, \rho$, via $[f, k] \mapsto (v \cdot \phi(f))/\rho^{k-1}$. Then we define a trace on $\tilde{H}$ via the diagram:

Note that the trace is actually defined not just on $H$ but on the possibly larger order ideal, $\tilde{H}$. It is straightforward to show that any trace constructed in this manner is a pure trace, and the corresponding $\lambda(M/P) = \rho 
eq 0$. Our task is to show the converse.

We require a few elementary results about pure traces on dimension groups.

Lemma 6.1 Let $\{A_i\}_{i \in \Omega}$ be a set of order ideals in a dimension group $C$, and let $A = \sum A_i.$

(a) If $\gamma$ is a pure trace of $C$, then for each $i$, either $\gamma(A_i) = 0$ or $\gamma | A_i$ is a pure trace of $A_i$.

(b) Suppose that $|\Omega| < \infty$ and $\gamma$ is a pure trace of $A$ that kills $\cap A_i$. Then $\gamma$ kills at least one of the $A_i$.

(c) Suppose that all of the $A_i$’s, and $C$, admit an order unit. Let $\gamma$ and $\gamma'$ be pure traces of $C$. 

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such that $\gamma|A_1 = \gamma'|A_1$, and this is nonzero. Then $\gamma = \gamma'$.

(d) For each index $i$, let $\alpha_i$ be a trace on $A_i$, such that for all $i$ and $j$, $\alpha_i|A_i \cap A_j = \alpha_j|A_i \cap A_j$.

Then there exists a unique trace $\alpha$ of $A = \sum A_i$ such that $\alpha|A_i = \alpha_i$ for all $i$.

Proof: (a) This follows from the extremal criterion in [GH2, Theorem 3.1].
(b) First observe that $A_1/(A_1 \cap A_2) \oplus A_2/(A_1 \cap A_2) \cong (A_1 + A_2)/(A_1 \cap A_2)$ (since all things in sight are dimension groups, this type of result follows from the corresponding results for $\mathbb{Z}^k$ with the usual ordering). If $|\Omega| = 2$, any trace on $A_1 + A_2$ which kills $A_1 \cap A_2$ will thus decompose into the sum of its restrictions to the two factor groups; hence any such trace which is pure must kill one of them.

If $|\Omega| > 2$, proceed by induction. Suppose $\gamma(A_1) \neq 0$. From $\gamma(A_1 \cap (A_2 \cap \cdots \cap A_n)) = 0$, we deduce from the direct sum decomposition above, that $\gamma(A_2 \cap \cdots \cap A_n) = 0$, and now we apply induction to $\Omega' := \Omega \setminus \{1\}$.

(c) The difference $\gamma - \gamma'$ induces a bounded linear functional on the quotient $C/A_1$, which is of course a dimension group. Hence we may write the induced bounded linear functional as $\alpha v - \beta w$ where $\alpha$ and $\beta$ are positive real numbers and $v$ and $w$ are normalized traces on $C/A_1$. Viewing them as traces on $C$, we obtain $\gamma - \gamma' = \alpha v - \beta w$, so $\gamma + \beta w = \alpha v + \gamma'$. Now $\gamma + \beta w$ can be represented as a nonnegative measure over the extremal boundary (the measure is finite, not necessarily 1, since we have not bothered normalizing $\gamma + \beta v$), and since $\gamma$ is extremal, the measure must include a point mass at $\gamma$ (with some positive weight). For a Choquet simplex, the representing measure is unique, and if $\gamma' \neq \gamma$, it follows that the representing measure for $v$ must contain the point mass at $\gamma$. However, this entails $v|A_1 \neq 0$, a contradiction. Hence $\gamma = \gamma'$.

(d) As is well-known, $A$ itself is an order ideal in $C$ (as $C$ is a dimension group); moreover, it also follows that $A^+ = \sum A_i^+$. Given finite sets $\{a_j\}$ and $\{a'_j\}$ such that for all $j$, $a_j, a'_j \in A_i$ and $\sum a_j = \sum a'_j$, we wish to show that $\sum \alpha_j(a_j) = \sum \alpha_j(a'_j)$. From $\sum a_j = \sum a'_j$, it follows by the Riesz decomposition property that there exist $a_{jk}$ in $C^+$ such that for all $j$, $a_j = \sum k a_{jk}$, and for all $k$, $a_k^' = \sum j a_{jk}$. Note that $a_{jk} \leq a_j$ and $a_{jk} \leq a_k$, so that $a_{jk}$ belongs to $A_j \cap A_k$. We observe:

$$\alpha_j(a_j) = \sum_k \alpha_j(a_{jk}) = \sum_k \alpha_k(a_{jk}).$$

Thus $\sum_j \alpha_j(a_j) = \sum_j \alpha_k(a_{jk}) = \sum_j \sum_k \alpha_k(a_{jk}) = \sum_k \alpha_k(a'_k)$, as desired.

Choose a pure trace $\gamma$ on our order ideal $H$, which we may assume to be obtained from $W$ in the manner described earlier. Affiliated with $\gamma$ are pure traces $\lambda$ on $E_b(G_{PM})$ and $L$ on $R_P$ such that $\gamma(\varepsilon h) = \lambda(e)\gamma(h)$.
for all $e$ in $E_b(G_{PM})$ and $h$ in $H$, and $\lambda R_P = L$. We also assume for now that $\lambda(\tilde{M}/P) \neq 0$. A consequence of this assumption is that $\gamma$ extends to a trace on $\tilde{H}$, and so it makes sense to talk about $\gamma$ evaluated at elements of the form $\sum z_i E_i, k$ for $z_i$ in $I_i$; if we were restricted to dealing with $H$, then the $k$ could only be 1.

We shall prove the result first in the case that $W$ is of a special form. For $1 \leq i, t \leq n$, define $J_{i,t}$ to be the order ideal of $R_P$ generated by the $(i, t)$ entries of all powers of $M/P$. Since $R_P$ is noetherian and all of its order ideals are ideals, only finitely many powers are required to generate the $J$'s. Now for any nonzero order ideal $I$ of $R_P$, let $H$ be the order ideal (depending on a choice for $t$ in $\{1, 2, \ldots, n\}$) generated by $\{zE_i, 1 \mid z \in I\}$. Hence, we set $W = IE_1 \oplus J_{1,t}IE_1 \oplus J_{2,t}IE_2 \oplus \cdots \oplus J_{n,t}IE_n$ and we may take $H$ to be obtained from $[W, 1]$ (we are permitted to apply the shift, since we may invert $M/P$).

We may assume that $\gamma[[J_{i,t}IE_i, 1]] \neq 0$ (for some $i$ and $t$, since $\lambda(M/P)$). We perform an initial cut-down. Set

$$S = \{i \mid \gamma[[J_{i,t}IE_i, 1]] \neq 0\}.$$ 

For $s$ in $S$, we say that $s$ is an anomaly if $J_{s,t} \subseteq P$, i.e., $L(J_{s,t}) = 0$. (Note that we cannot conclude from $J_{s,t} \subseteq P$ that $\gamma[[J_{i,t}IE_i, 1]]$ is zero, since we do not know that $IE_i, 1$ belongs to the module of which $\gamma$ is a pure trace.) We will show that no anomalies exist later on, but for now we assume that no anomalies exist. (The proof that no anomalies exist uses the fact that when no anomalies exist, the traces are what they are supposed to be, hence this rather convoluted argument.) A consequence is that $L$ does not kill any of the order ideals $J_{i,t}$.

At this point, we realize that in order to avoid periodicity, we should have replaced $M/P$ by its $n$th power or any (fixed) higher power; this of course has no effect on the direct limit ordered modules, so we may do so. It follows that in all sufficiently high powers of $L((M/P)_S)$ all of the entries in column $t$ are not zero; hence we may assume the same of $L((M/P)_S)$, and moreover this is true for the $j$th column for any $j$ in $S$ such that $j$ and $t$ are states in the same irreducible block (with respect to the real matrix $L((M/P)_S)$, see the preambles to the proof of Theorem A1.3).

Now $P$ is prime as an ideal and since none of $J_{i,t}$ ($i \in S$) is contained in $P$, neither is their product $J_t = \prod_{i \in S} J_{i,t}$. In $R_P$, a product of order ideals is an order ideal, and thus $J_t$ is generated by positive elements as an ideal. There thus exists a positive element $r$ in $J_t \setminus P$. Obviously, $L(r) > 0$, and for $z$ in $I$ and $s \in S$, $[rzE_i, 1]$ is an element of $H$. If $s$ is another element of $J_{t} \setminus P$, then evaluating $\gamma([rzE_i, 1])$ in two different ways, we deduce $L(s)\gamma([rzE_i, 1]) = L(r)\gamma([szE_i, 1])$. Hence, the function $\phi_i : I^+ \to \mathbb{R}$ given by $\phi_i(z) = \gamma([rzE_i, 1]) / L(r)$ is independent of the choice of $r$, subject to $r \in J_t \setminus P$, and it is immediate that if $r$ is positive, then $\phi_i$ is a trace of $I$. Of course, we will end up proving that up to scalar multiples, all the $\phi_i$ are the same, and they are pure traces of $I$.

Let $\rho = \lambda(M/P)$ ($\rho$ is used to remind the reader of spectral radius). For every $i$ in $S$, we have $\gamma([rzE_i, 1]) = \frac{1}{\rho} \gamma(([M/P]_Sr z E_i, 1])$; expanding this, we deduce (writing $M/P = (a_{ij})$)

$$\phi_i(z)L(r) = \gamma([rzE_i, 1]) = \frac{1}{\rho} \sum_{j \in S} \gamma([nzE_j, 1]) L(a_{ji}) = L(r) \frac{1}{\rho} \sum_{j} \phi_j(z) L(a_{ji}).$$

In other words, the row $\Phi(z) := (\phi_j(z)L(r))$ is a nonnegative left eigenvector for $\rho$ of $(M/P)_S$. We will exploit purity of the original trace $\gamma$ to show that $\Phi$ is independent of the choice of $z$ (subject to $z \geq 0$ and $\Phi(z) \neq 0$), and is a pure nonnegative eigenvector of $L((M/P)_S)$. Let $Z$ denote the cone of nonnegative left eigenvectors of $L((M/P)_S)$ for the eigenvalue $\rho$. Then $\Phi : I^+ \to Z$ is additive, and obviously extends

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1 Products of order ideals in partially ordered commutative domains (that are even dimension groups admitting 1 as an order unit) need not be order ideals; however, in $R_P$, products of order ideals really are order ideals—this follows from the positive cone being generated by elements of the form $x^w / P^k$.\[40\]
to a positive homomorphism $I \to Z - Z$; of course $Z - Z$ is a subspace of the eigenspace for $\rho$. As is well-known, $Z$ is simplicial. This means that there is a basis for the vector space $Z - Z$, $\{v_q\}$ whose $R^+$ span is precisely $Z$. Pick some $z$ in $I^+$ such that $\Phi(z) \neq 0$, and relabel the basis so that $v_1$ has a component in $\Phi(z)$ relative to the basis. Let $\Psi : Z \to v_1 R^+$ be the projection relative to this basis. We use this to construct a trace $\gamma$ on $H_S$ (in fact, it is defined on a larger order ideal, in general) that is dominated by $\gamma$; purity of $\gamma$ entails equality.

Let $I_S$ denote $\lim(M/P)_S : I^{\#(S)} \to I^{\#(S)}$. Obviously, this contains $H_S$, even $\tilde{H}_S$. Our candidate for a trace on $I_S$ is given by

$$\Gamma([zE_i,k]) = \frac{\{|\Psi \Phi(z)|_i\}}{\rho^{k-1}} \quad \text{for } z \in I, i \in S, \text{ and } k \in \mathbb{N}.$$ 

The numerator is the $i$th entry of the vector $\Psi \Phi(z)$. To check that $\Gamma$ extends to a well-defined function, it is enough to check that $\Gamma([zE_i,k]) = \Gamma([(M/P)_S zE_i,k + 1])$. We compute:

$$\Gamma([(M/P)_S zE_i,k + 1]) = \sum_{j \in S} \Gamma([a_{ji} zE_j,k + 1]) = \frac{1}{\rho^{k}} \sum_j \{|\Psi \Phi(a_{ji} z)|_j\}$$

$$= \frac{1}{\rho^{k}} \sum_j \{|\Psi \Phi(L(a_{ji}) z)|_j\}$$

$$= \frac{1}{\rho^{k}} \sum_j L(a_{ji}) \{|\Psi \Phi(z)|_j\}$$

$$= \frac{1}{\rho^{k}} \rho \{|\Psi \Phi(z)|_i\}$$

$$= \frac{1}{\rho^{k-1}} \{|\Psi \Phi(z)|_i\} = \Gamma([zE_i,k]).$$

The fourth line comes from the fact that $\Psi \Phi(z)$ is a scalar multiple of a fixed eigenvector of $(M/P)_S$. Now that $\Gamma$ is known to be well-defined, it is immediate that it is also additive and positive, and so is a trace on $I_S$. Now we claim that $\Gamma|H_S \leq \gamma$. Suppose that $i$ belongs to $S$, $a$ belongs to $J_{i,t}$, $z$ belongs to $I$ and $az$ belongs to $R^T_P$. Then $az \in J_{i,t} I$, and we calculate:

$$\Gamma([azE_i,k]) = \frac{1}{\rho^{k-1}} \{|\Psi \Phi(az)|_i\} \leq \frac{1}{\rho^{k-1}} \{|\Phi(az)|_i\} = \frac{1}{\rho^{k-1}} \phi_i(az)$$

$$= \frac{1}{\rho^{k-1}} \cdot \frac{\gamma([razE_i,1])}{L(r)}$$

$$= \frac{1}{\rho^{k-1}} \frac{\gamma([azE_i,1])}{\gamma([azE_i,1])}$$

$$= \gamma([azE_i,k]).$$

So $\Gamma|H_S \leq \gamma$, and by purity of the latter, we must have equality (up to a scalar multiple); note that the assumption that $v_1$ actually appears in the decomposition of at least one $\Phi(z)$ ensures that $\Gamma$ is not zero. If instead of choosing $v_1$, we selected some other eigenvector, we would obtain a corresponding equality of the restriction with $\gamma$. It easily follows that there is only one $v_1$ that appears, and so $\Phi$ has one-dimensional range. Thus the $\phi$’s are all scalar multiples of each other. Let $\phi$ denote any one of them. To see that $\phi$ is a pure trace of $I$, suppose that $\phi' \leq \phi$ (we have not bothered to normalize any traces here, so we never have to deal with scalar multiples). Then $\phi'$ inherits the multiplicative property, $\phi'(az) = L(a)\phi'(z)$. Now define
\( \Gamma \) using \( \phi' \), and we similarly see that its restriction is dominated by \( \gamma \); hence equality holds, and so \( \phi' \) is a scalar multiple of \( \phi \).

We have deduced that (if no anomalies exist) \( \gamma \) (actually \( \gamma_S \), but this is obtained by factoring out an order ideal in the kernel of \( \gamma \)) is of the desired form, and it can be defined on a natural order ideal of \( \lim(M/P)_S : R^+_P \rightarrow R^+_P \).

Now we prove that no anomalies exist. We have defined \( I, S, L, p, \lambda \); suppose that \( s \) is an anomaly, that is, \( s \in S \) and \( J_{s,t} \subseteq p \). For an order ideal \( I_0 \) of \( R_P \) and \( s_0 \) in \( S \), define \( H(I_0, s_0, k) \) to be the order ideal of \( G_M \) generated by \( [I_0]E_{s_0,k} \), and \( \tilde{H}(I_0, s_0, k) \) to be the enlargement obtained by inverting \( M/P \). Notice that we have been working with \( H = H(I, t, 1) \). Now \( H(J_{s,t}E_{s_0,2}) \) is an order ideal of \( H \) and from the definition of \( S \), \( \gamma \) does not vanish on it. Since \( \gamma \) is a pure trace of \( H \) and the latter is a dimension group, \( \gamma_2 := |H(J_{s,t}IE_{s,2}) | \) is also a pure trace.

Now let \( i \) an element of \( S \) that is not an anomaly. Then there exists a positive element \( r \) in \( J_{i,t} \setminus p \) and necessarily \( [rzE_i, 2] \in H \) for all \( z \) in \( I \). From the computation,

\[
L(r)\gamma([J_{i,s}J_{s,t}IE_{i,2}], 2) = \gamma([rJ_{i,s}J_{s,t}IE_{i,2}], 2) = L(J_{s,t}J_{i,s})\gamma([rIE_{i,2}], 2) = 0 \quad \text{and} \quad L(r) \neq 0,
\]

we see that \( \gamma([J_{i,s}J_{s,t}IE_{i,3}], 2) = 0 \); extending this via \((M/P)^{-1}\), we see \( \gamma([J_{i,s}J_{s,t}IE_{i,k}], 2) = 0 \) for all \( k \). Hence \( \gamma_2 \) kills all the non-anomalous coordinates. The \( S_2 \) we obtain for \( \gamma_2 \) is a proper subset of \( S \), unless \( S \) consists entirely of anomalies. This last possibility cannot occur, since the trace of some power of \( M/P \) must be nonzero. If \( \gamma_2 \) admits anomalies, we may continue the process, and at some stage we arrive at a situation where there are no anomalies (as \( \gamma_2 \) is not zero, the same holds for its successors). Hence we assume that \( \gamma_2 \) admits no anomalies, \( \gamma \) does, and deduce a contradiction.

For \( \gamma_2 \) (which is defined on a smaller order ideal), there is a cut down to a smaller set \( S_2 \). By the preceding, \( S_2 \) consists of anomalies for \( S \), but by our induction assumption, admits no relative anomalies itself(!). Let \( T \) be the complement of \( S_2 \) in \( S \), and let \( U \) be a subset of \( S \). Suppose that \( r \) and \( z \) are positive elements of \( R_P \) such that \( rzI \subseteq J_{i,t}I \) for all \( i \) in \( U \), and for at least one such \( i, \gamma([rzE_i, 2]) > 0 \) (the first condition implies \([rzE_i, 2] \) belongs to \( H \), obviously a necessary condition for the second condition to make sense; indeed all the difficulties in the arguments here emanate from the apparent restrictions on domains of the various traces). From the equation \( \rho\gamma([rzE_i, 2]) = \gamma([rz(M/P)E_i, 2]) \) for all \( i \) in \( U \), we deduce (recalling that the entries of \( M/P \) are denoted \( a_{pq} \))

\[
(1) \quad \rho\gamma([rzE_i, 2]) = \sum_{j \in U} L(a_{pq})\gamma([rzE_j, 2]) + \sum_{j \in U \cap S_2} \gamma([a_{ji}rzE_j, 2]) \quad \text{for all } i \in U
\]

We make various choices for \( r \) and \( z \). First, set \( J_t = \prod_{i \in T} J_{i,t} \). Since \( p \) is prime (as an ideal in the ordinary ring sense), \( J_t \not\subseteq p \); since \( J_t \) and \( p \) are order ideals, there exists \( r_t \in J_t^+ \setminus p \); necessarily, \( L(r_t) > 0 \) (since \( \ker L \cap R^+_P = p \cap R^+_P \)). For fixed \( s \) in \( S_2 \), set \( U_s = T \cup \{s\} \). There exists \( z \in J_{s,t}I \cap R_P^+ \) such that \( \gamma([zE_s, 2]) > 0 \) (from the definition of \( S \)). Then we notice that \( rI \subseteq J_{i,t}I \) for all \( i \) in \( T \) and so \( rz \in J_{i,t}I \) for all \( i \) in \( U_s \). Now we see that \( \gamma([rzE_i, 2]) = L(z)\gamma([rE_i, 2]) = 0 \) for \( i \) in \( T \), and \( \gamma([rzE_s, 2]) = L(r)\gamma([zE_i, 2]) > 0 \). Plugging this into (1), we deduce \( L(a_{si}) = 0 \) for all \( i \in T \). Hence \( L(a_{si}) = 0 \) for all \( i \) in \( T \) and \( s \) in \( S_2 \). Thus if we write the real matrix \( L(M/P) \) according to the partition \( S = T \cup S_2 \), it appears in block upper triangular form,
Now we notice that for $\gamma_2$, the “$L$” and the “$\lambda$” are just the originals. Since $\gamma_2$ admits no anomalies, we can reconstruct it as follows. Cut down to $S_2$, form the order ideal $I_2 = \sum_{s \in S_2} J_{s,t}$. Then there exists a pure trace $\phi$ on $I_2$ and a left nonnegative eigenvector $v_2$ of $L((M/P)_{S_2})$ of $\rho = \lambda((M/P))$ that implements the trace. As a result of the block upper triangular form, we may enlarge $v_2$ to a nonnegative left eigenvector of $L((M/P)_S)$ simply by inserting to the left of $v_2$ as many zeroes as the cardinality of $T$. This permits us to define an extension of $\gamma_2$ to $H_S$, which wipes out the coordinates corresponding to $T$, and is clearly dominated by the original trace $\gamma$ (as $\gamma_2$ is the restriction of $\gamma$). Purity of $\gamma$ forces equality with this weird trace, but this is impossible as $\gamma$ does not wipe out any of the $T$-coordinates. So there are no anomalies.

To recap: If $\gamma$ is a pure trace with $\lambda(M/P) \neq 0$, and if the order ideal is generated by $[IE_i, k]$ for some integers $i$ and $k$, and $I$ an order ideal in $R_P$, then $\gamma$ is given by the prescription above; that is, we cut down, and locate a pure trace on $I$, a left nonnegative eigenvector for the cut-down real matrix $L((M/P)_S)$, and use it, exactly as in the case of $d = 0$ (c.f., the proof of Theorem A1.3). Notice by the way that the proof that no anomalous coordinates existed used an induction argument and the smaller order ideal was of the same type. The constraint on the order ideal may now be dropped by Lemma 6.1.

Now we can extend our characterization of the pure traces that do not wipe out $\overline{M/P}$ to arbitrary order ideals of $G_M$. We notice that an arbitrary order ideal of $G_M$ will be a sum of the order ideals of the type considered above; hence the restriction of $\gamma$ to at least one of these will be nonzero. This restriction is of the desired type—but we note that it can be extended to one of the same type (this was our original construction of candidate traces) which is also extremal. By Lemma 6.1(c), $\gamma$ must itself be of this type.

This completes the proof of the following.

**Theorem 6.2** Let $\gamma$ be a pure trace of an order ideal $H$ of $G_M$ such that $\lambda(\overline{M/P}) \neq 0$. Then $\gamma$ is obtained from the following prescription.

Let $S$ denote the complement of the set of coordinates $i$ such that $\gamma([zE_i, k]) = 0$ whenever $[zE_i, k]$ belongs to $H$. Form $H_S$ by throwing away all the coordinates not in $S$. There exists an order ideal $I$ of $R_P$ such that $H_S$ is contained in the order ideal of $G_M$ generated by $\sum_{i \in S}[IE_i, k]$ for some integer $k$. There exists a pure trace $\phi$ on $I$ and a left nonnegative eigenvector $v$ for the cut-down real matrix $L((M/P)_S)$ with eigenvalue $\rho = \lambda(\overline{M/P})$, such that the trace given by $[\sum_S z_i E_i, l] \mapsto v \cdot (\phi(z_i))/\rho^{l-1}$ restricts on $H_S$ to a multiple of $\gamma$; moreover, the extended trace is also pure.

Suppose that for all pure traces $\lambda$ of $E_b(G_{PM})$, $\lambda(\overline{M/P}) \neq 0$; then (as $\overline{M/P}$ is a positive element of $E_b(G_{PM}))$, $\overline{M/P}$ is an order unit. Thus there exist integers $k$ and $l$ so that

$$M^k P^l \prec M^{k+1} P^{l-1}$$

This is precisely the condition to be discussed in Proposition 7.5; it is a consequence of noetherianness of $E_b(G_M)$. It is usually easy to verify. In this case, by Lemma 5.1, the ordering on $G_M$ is completely determined by that of $G_M$. We have a situation in which all the pure traces on every order ideal are known, and in principal, this permits us to determine the positive cone of $G_M$.

Another consequence of (3) is that $(\overline{M/P})^{-1}$ is already in $E_b(G_{PM})$. This means that if $u = \sum [u_i E_i, k]$ is an order unit for $H$, so is its right shift, $\sum [u_i E_i, k + 1]$. In particular, $G_M$ admits an order unit (the column consisting of 1’s at level one). Conversely, if for all $i$, the order ideal generated by $[E_i, 1]$ contains $[E_i, k]$ for some $k$, then $\overline{M/P}$ is invertible in the bounded subring.

A sufficient condition for $G_M$ to admit an order unit, is that the sum of the columns of $M/P$ consist of order units; this is not necessary, nor does it imply that $(\overline{M/P})^{-1}$ is invertible—consider the example,

$$M = \begin{bmatrix} 1 + x & x \\ 1 + x & 0 \end{bmatrix}, \quad P = 1 + x.$$
One very special and particularly strong property is condition (**); see section 5. Then with \( P \) in \( A^+ \) such that \( \Log P = \Log M^k \), \( M^k/P \) is an order unit, and all the preceding applies. We give some examples.

Example 6.3 \( d = 1 \) and \( n = 4 \).

Here \( d = 1 \); set
\[
M = \begin{bmatrix}
x^2 & 1 & x & x^2 \\
1 & 1 + x^2 & x^2 & 1 \\
1 & 2 & x^2 + 2 & 1 \\
1 & 1 & 1 & x^2 + 2
\end{bmatrix}, \quad P = 1 + x + x^2.
\]

We observe that \((1 + x^2)I \leq M^2\), so that \( P^2I \leq M^2P^2 \); thus \( M^2 \) satisfies condition (3) above (with respect to \( P^2 \)) to guarantee that \( \overline{M/P} \) is bounded, so that \( \lambda(\overline{M/P}) > 0 \) for all pure traces of \( E_b(G_{PM}) \). (Obviously for the positivity problem, we may always replace \( M \) by any power of itself.) The pure traces of \( R_P \) are given by point evaluations, \( x \mapsto r \in [0, \infty] \). If \( r \) belongs to \((0, \infty)\), then \((M/P)(r)\) (the matrix obtained by evaluating all entries of \( M/P \) at \( x = r \)) is primitive (in fact, all of its entries are strictly positive); if \( r = 0 \), again primitivity results. In these cases, there is a unique nonnegative left eigenvector (up to scalar multiple), and it is strictly positive. On the other hand, if \( r = \infty \) (that is, \((M/P)(\infty) = \lim_{r \to \infty}(M/P)(r)\)) yields the pure trace \( L \), we have
\[
L(M/P) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

There are three nonnegative left eigenvectors, \((1 \ 0 \ 0 \ 0), (0 \ 1 \ 0), \) and \((0 \ 0 \ 0 \ 1)\). However, cutting down \( L(M/P) \) to its \((2, 2)\) entry, there is another candidate for an eigenvector that induces a trace, \((0 \ 1 \ 0)\). Whether this yields a trace depends on the choice of \((M/P^{-1})\)-invariant order ideal, \( W \), of \( R_P^4 \).

For example, if \( W = R_P^2 \), then no matter what our initial choice for eigenvector of \( L(M/P) \), the big ideal \( I \) will be all of \( R_P \) itself, and \( \phi \) must equal \( L \). Thus in this case, there are exactly three pure traces of \( H \) affiliated to \( L \), which is evaluation at \( \infty \).

On the other hand, let \( I \) be the ideal generated by \( \{x/P, 1/P\} \) in \( R_P \); this is an order ideal by \([H4, II.2A]\). Let \( W \) be the order ideal of \( R_P^4 \), given as \( I \oplus R_P \oplus \hat{I} \oplus \hat{I} \). It is routine to check that \((M/P)W \subseteq W \); in other words, \( W \) is invariant. Now let \( \phi \) be \( L \) itself; we note that \( L(I) = 0 \) (evaluate \( x/P \) and \( 1/P \) at \( \infty \)), so that we can cut down to the \((2, 2)\) entry. We obtain a pure trace that simply picks out the second entry of \( W \) and evaluates it at \( L \). Note that in this case, the corresponding eigenvalue of \( L(M) \) is \( L((1 + x^2)/P) \), which equals \( 1 \). This yields one additional pure trace affiliated to \( L \). For elements \( h = f/P^k \) in \( I \), we remark that \( \lim_{r \to \infty} h(r) = 0 \), and it follows from \( h \) being rational that \( \phi(h) = \lim_{r \to \infty} h(r)/r \) exists. This defines a linear functional \( \phi \) on \( I \) (obviously not extendible to all of \( R_P \)) which is a trace and it is routine to verify that \( L \) is affiliated to it. (Also verifiable: All other pure traces of \( I \) are restrictions of the point evaluations determined by \( x \mapsto r \in [0, \infty) \).)

Example 6.4 \( d = 2 \) and \( n = 3 \).

Here \( d = 2 \), \( M \) satisfies (**), and \( I \) is more interesting. Set
\[
M = \begin{bmatrix}
1 + 2x + 3y & 1 + 2x + 3y & 1 + x + y \\
1 + \frac{1}{2} + y & 1 + 2x + y & 1 + x + y \\
1 + x + y & 1 + 3x + y & 2x + y
\end{bmatrix},
\]

We can take \( P = 1 + x + y \). We note first that \( P^2I \leq M^2 \); then a computation reveals that for all \( 1 \leq i, j \leq 3 \), \( \cvx \Log(M^2)_{ij} = \Log P^2 \), and so \( M \) satisfies (**). Following \([H4, Example 1A, p. 7]\), set
$X = x/P, Y = y/P$, and $Z = 1 - X - Y$. Then by [op. cit.], $R_P = R[X, Y]$ (the pure polynomial algebra), and $R_P^+ = (X, Y, Z)$ (the $R^+$-semigroup generated additively and multiplicatively by $X$, $Y$, and $Z$). Set $I = (X, Y)^2$ (the square of the ideal generated by $X$ and $Y$). From [H4, Remark, pp. 46-47], $I/(X, Y)^3 \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 \cong \mathbb{Z}^2$ (where overlined objects are the images in the quotient group) with the direct sum ordering. Any pure trace of $I$ that is not a restriction of a pure trace of $R_P$ must annihilate $(X, Y)^3$ and so must be the projection onto one of the three summands. The pure traces of $R_P$ are given by point evaluations $(x, y) \mapsto (r_1, r_2) \in (\mathbb{R}^2)^+$ and their limit points; because of the change of variables, these correspond to $(X, Y) \mapsto (r, s)$, where $0 \leq r, s$ and $r + s \leq 1$. Observe that the pure trace $L$ of $R_P$ given by $(r, s) = (0, 0)$ annihilates $I$ (and this is the only pure trace of $R_P$ to do so).

The pure traces of $I$ are thus either point evaluations as indicated above, except that $L$ is excluded, or the projections onto the three summands of $I/(X, Y)^3$. The projections onto $X^2 \mathbb{Z}$ and $Y^2 \mathbb{Z}$ are limits of restrictions of point evaluations, whereas the projection onto $XY \mathbb{Z}$ is not. We observe that since $M$ satisfies $(\star \star)$, $L(M/P)$ must be primitive; moreover, by Lemma 5.15, when $M$ satisfies $(\star \star)$, every limit module of the form $H = \lim W \to W$ is uniform and so we can assume that $W = I^n$ for some order ideal $I$. So there really is not much loss of generality in taking $W = I \oplus I \oplus I$. Now

$$L(M/P) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

and its left nonnegative eigenvector is $v = (1, 1, \sqrt{3} - 1)$, with large eigenvalue $1 + \sqrt{3}$. There are precisely three choices for $\phi$ that are affiliated to $L$, as we have indicated above. If instead we had taken $W = R_P^3$, then $\phi = L$ is the only choice.

7. A FOGgy example and generic nonnoetherianness

Here we discuss the bounded endomorphism ring of the FOG matrix $M$ given earlier (Example 2.3); explicitly,

$$M = \begin{bmatrix} X & 0 & 1 \\ 1 & Y & 0 \\ 0 & 1 & 1 \end{bmatrix}. $$

We show that $E_0(G_M)$ is a commutative ring without zero divisors; it is not noetherian, and has a unique maximal order ideal. It has a face $F$ of dimension one for which the kernel of the corresponding map $E_0(G_M) \to E_0(G_{M_P})$ is contained in the kernel of a unique pure trace (one would expect it to be contained in the kernel of a continuum of pure traces), and the kernel is an order ideal without an order unit. Other somewhat weird properties are demonstrated.

To put this in context, consider what properties are satisfied by $E_0(G_M)$ when $n = 1$. In this case, $M = P$ a polynomial, and $E_0(G_M) = R_P$. Then $R_P$ is a commutative finitely generated algebra (in particular, it is noetherian, and thus every order ideal admits an order unit), and its positive cone is generated additively and multiplicatively by elements of the form $x^w/P$ where $x^w$ appears in $P$. For every face $F$ of the Newton polyhedron of $F$, the map $R_P \to R_{P_F}$ is onto and it follows that if $F$ is one dimensional, the kernel is contained in the kernel a “line segment” of pure traces.

We show that this example is typical if $d$ is at least 2. For instance, if $E_0(G_M)$ is noetherian, then $M$ must satisfy the strong condition discussed earlier in connection with calculations of traces (essentially that $M^{w^1}/P$ be an order unit in $E_0(G_{M_P})$ where $P = \text{tr } M^{w^1}$).

To avoid confusion between the generic $x^w$ (to denote a typical monomial in this case in two variables), and $X^k$ (the monomial in only the single variable $X$), we have used capital letters $X$ and $Y$ to denote the specific variables, and $x^w = X^{w(1)}Y^{w(2)}$ where $w = (w(1), w(2))$. To start the process, we verify the earlier statement that $T(M) = \{(w, k) \in (\mathbb{Z}^+)^2 \times \mathbb{N} \mid |w| \leq k - 3\}$; in other words, $x^w M^{-k}$ belongs to $E_0(G_M)$ (for $k > 0$) if and only if $w$ is a pair of nonnegative integers such that $|w| \leq k - 3$. We use the
notation \(|w|\) to denote the sum of the entries in \(w\), i.e., the total degree of the monomial \(x^w\). By using the graphical formulation of \(M\), we can check exactly which monomials appear in each entry of \(M^k\) (“deg” refers to total degree):

\[
\begin{bmatrix}
X^k & \text{all deg } \leq k - 3 \\
\text{all deg } k - 1 & \leq k - 4 \\
\text{all deg } k - 2 & \leq k - 4 \\
Y^k & \text{all deg } \leq k - 3 \\
\text{all deg } \leq k - 2 \\
Y^k - 1 & \text{all deg } \leq k - 4 \\
\text{all deg } \leq k - 3
\end{bmatrix}
\]

Now if \(k > 0\), \(x^w M^l < M^{k+l}\) for some positive integer \(l\) (i.e., \(x^w M^{-k}\) belongs to \(E_b(G_M)\)) forces \(|w| + l \leq k + l - 3\) (use the diagonal entries), i.e., \(|w| \leq k - 3\). Conversely, if \(|w| \leq k - 3\), it is easy to check that \(x^w 1 < M^k\). Thus \(x^w M^{-k}\) belongs to \(E_b(G_M)\) for \(k > 0\) if and only if \(w \geq (0,0)\) and \(|w| \leq k - 3\). In particular, if \(p\) is a real polynomial in the variables \(X\) and \(Y\), then \(pM^{-k}\) belongs to \(E_b(G_M)\) if \(\deg p \leq k - 3\) (when \(k > 0\)). Thus we have the following.

**Lemma 7.1** For \(k > 0\), \(x^w M^{-k}\) belongs to \(E_b(G_M)\) if and only if \(|w| \leq k - 3\) and \(w \geq (0,0)\). In particular, if \(p\) is a real polynomial in \(X\) and \(Y\) of total degree at most \(k - 3\), then \(pM^{-k}\) belongs to \(E_b(G_M)\).

Now we can get a fairly good description of the centralizer of \(M\) in \(M_n A\).

**Lemma 7.2** The characteristic polynomial of \(M\) is \((Z - X)(Z - Y)(Z - 1) = 0\), which is irreducible over \(A = \mathbb{R}[X,Y]\); thus the centralizer of \(M\) is commutative and has no zero divisors. The matrix \(M\) is \(GL(3,A)\)-conjugate to a companion matrix. The centralizer of \(M\) in \(M_n A\) is thus \(A[M]\), the polynomials in \(M\) with coefficients from \(A\).

The first part of the first statement is an easy calculation, and the rest of the first statement follows directly. The second statement is routine to verify, and the rest follows immediately.

Thus \(E(G_M)\) is naturally isomorphic to \(A[M,M^{-1}]\) (a subring of \(M_n \mathbb{R}(X,Y)\), the matrix ring over the rational function field), and under this identification, an element \(NM^j\) (where \(N\) is in \(A[M]\) and \(j\) is an arbitrary integer) is positive if and only if there exists a positive integer \(l\) so that all of the entries of \(NM^l\) have no nonnegative coefficients. Now as an \(A\)-module, \(A[M]\) is obviously free on the set \(\{1, M, M^2\}\). So every element of \(A[M]\) can be written uniquely in the form \(N = p_0 I + p_1 M + p_2 M^2\), where \(p_i\) belong to \(A\), i.e., are polynomials. Now we decide when an element \(NM^{-k}\) belongs to \(E_b(G_M)\).

We have previously noted that if \(p\) is a polynomial (with real coefficients) whose total degree is \(k - 3\) or less, then \(pM^{-k}\) belongs to \(E_b(G_M)\). Hence the vector space,

\[J := \{pM^{-k} \mid k \geq 2, \deg p \leq k - 3\}\]

is a (unitless) subalgebra of \(E_b(G_M)\). Now we show the following.

**Lemma 7.3** (a) \(E_b(G_M) = J + RI\)

(b) \(J\) is an order ideal of \(E_b(G_M)\), and it contains no order unit (relative to itself); in particular, \(J\) is not a finitely generated ideal of \(E_b(G_M)\).

(c) \(J^2 \subseteq M^{-3} J\).

To this end, suppose that \(NM^{-k}\) belongs to \(E_b(G_M)\) with \(N\) in the centralizer of \(M\). This means that for some \(l \geq 0\), \(NM^l < M^{l+k}\). We will obtain necessary and sufficient conditions for which \(N < M^k\); these may then be applied to \(NM^l < M^{l+k}\), and so we may assume \(l = 0\). We write \(N = p_0 I + p_1 M + p_2 M^2\) as before; then

\[N = \begin{bmatrix}
p_0 + Xp_1 + X^2p_2 & p_2 & p_1 + (X + 1)p_2 \\
p_1 + (X + Y)p_2 & p_0 + Yp_1 + Y^2p_2 & p_2 \\
p_2 & p_1 + (Y + 1)p_2 & p_0 + p_1 + p_2
\end{bmatrix}.
\]
From $N < M^k$ and using the form of the monomials appearing in $M^k$ displayed previously, we deduce the following:

1. $p_0 + Xp_1 + X^2p_2 = aX^k + q$ where $deg q \leq k - 3$, and $a$ is a real number;
2. $p_0 + Yp_1 + Y^2p_2 = bY^k + r$ where $deg r \leq k - 3$, and $b$ is a real number;
3. $deg(p_0 + p_1 + p_2) \leq k - 3$;

Subtract the $(2,2)$ entry from the $(1,1)$ entry; we deduce

(2.2) $p_1 + (X + 1)p_2 = cX^{k-1} + dX^{k-2} + eX^{k-3} + f$ where $c, d, e$ are real numbers and $deg f \leq k - 4$.

Subtracting the $(2,2)$ entry from the $(1,1)$ entry, we obtain $(X - Y)(p_1 + (X + Y)p_2) = aX^{k-1} - bY^{k-1} + (q - r)$. Since $X - Y$ is homogeneous and $q - r$ has total degree at most $k - 3$, $X - Y$ must divide $aX^{k-1} - bY^{k-1}$, which of course forces $a = b$. Obviously, $N' := N - aM^k < M^k$ in any case, and all of the diagonal entries of $N'$ are of degree at most $k - 3$. Now we show that $N'M^{k-1} - k$ belongs to $J$. Relabel the $p_i$ so that $N' = p_0 I + p_1 M + p_2 M^2$; all six conditions still apply, and additionally $a = b = 0$. Subtract the $(3,3)$ entry from the $(1,1)$ entry; we deduce $deg(X - 1)(p_1 + (X + 1)p_2) \leq k - 3$; this forces $deg p_1 + (X + 1)p_2 \leq k - 4$; in view of the condition from the $(1,3)$ entry, this forces $c = d' = e' = 0$. Similarly, subtracting the $(3,3)$ entry from the $(2,2)$ entry, we obtain $d = d'' = d'' = 0$. Subtracting the $(2,2)$ entry from the $(1,1)$ entry, we obtain $deg(X - Y)(p_1 + (X + Y)p_2) \leq k - 4$, and again this forces $t$ to be the zero polynomial.

From $deg p_1 + (X + 1)p_2 \leq k - 4$ and $deg p_1 + (X + 1)p_2 \leq k - 4$, we deduce that their difference, $(X - Y)p_2$ has degree at most $k - 4$, whence $deg p_2 \leq k - 5$. Thus $p_2 M^2 < M^k$. Since $deg p_2 \leq k - 5$, it follows from $deg p_1 + (X + 1)p_2 \leq k - 4$ that $deg p_1 \leq k - 4$. Hence $p_1 M < M^k$. From entry (3,3), $deg p_0 + p_1 + p_2 \leq k - 3$, which we see now forces $deg p_0 \leq k - 3$. Thus $p_0 I \equiv M^k$. So $N = aM^k + p_0 I + p_1 M + p_2 M^2$ and each of the four terms is dominated by $M^k$. Thus $NM^{k-1} - aI \in J$. Thus $E_b(G_M) = J + RI$ as desired.

Obviously, $J$ is generated as a vector space (or even as an abelian group) by its positive elements (positive multiples of $x^w M^{-k}$, where $|w| \leq k - 3$). The convex polytope $K(M)$ is simply the standard triangle, with vertices $(0,0), (1,0), (0,1)$; let $F$ be the face consisting of the edge joining $(1,0)$ to $(0,1)$.

It is easy to check that the map $E_b(G_M) \to E_b(G_{M^2})$ kills $J$; as $J$ is of codimension 1, the kernel of this map is exactly $J$, and so the map $E_b(G_M) \to E_b(G_M)/J$ is positive. As $E_b(G_M)/J \cong R$, it has a unique trace, which in turn induces a trace, necessarily pure, on $E_b(G_M)$. Any positively generated ideal that is the kernel of a pure trace is an order ideal. So $J$ is an order ideal.

Now we show that $J$ admits no order units (with respect to itself, obviously). Every element of $J$ is of the form $pM^{-k}$ where $k \geq 3$ and $deg p$. Suppose $U = pM^{-k}$ is an order unit of $J$. Since the operations of increasing an order unit (but still staying within $J$) and multiplying by scalars both preserve the order unit property, we may assume that $p = (1 + X + Y)^{-3}$. Let $v$ be an element of $(Z^+)^2$ and $a$ a positive integer such that $|v| \leq l - 3$, so that $x^v M^{-l}$ belongs to $J$. There thus exists a positive real number $K \equiv K(v, l)$ such that $x^v M^{-l} \leq pM^{-k}$. Set $l = k + 1$, and let $x^v$ vary over the monomials of degree $k - 2$ or less. We can add integer multiples of the corresponding inequalities and we obtain $(1 + X + Y)^{k-2} M^{-k} \leq K(1 + X + Y)^{k-3} M^{-k}$. Hence for some integer $m > 0$, $(1 + X + Y)^{k-2} M^{-l} < (1 + X + Y)^{k-3} M^{m-1}$. However, the $(1,1)$ entry of the left side contains the monomial $X^{k-2} M^{-m}$ whereas the $(1,1)$ entry of the right side does not (its monomials of degree $m + k - 2$ are $Y^i X^{m+k-2-i}$ with $i \leq k - 3$; a narrow miss). Hence $J$ has no order units. If $J$ were finitely generated as an ideal in a ring that has 1 as an order unit, it would have an order unit relative to itself (proof: decompose each member of the generating set as a difference of two positive elements; the sum of all of these elements would be an order unit, since every element of the ring is bounded above and below by a multiple of the identity).

To prove (c), it is sufficient to show the product of two of the generators of the form $pM^{-k}$ factors as
$M^{-3}$ times an element in $J$; this is trivial.

Next, we consider the positive kernels of the pure traces. If $R$ is an ordered ring in which 1 is an order unit, then any pure trace is multiplicative. If $\tau$ is such a pure trace, define $\text{Ker}^{+}_{\text{ord}} \tau$ to be the set of differences of the positive elements in the kernel of $\tau$; it is easy to see that $\text{Ker}^{+}_{\text{ord}} \tau$ is both an ideal and an order ideal, and is the largest order ideal of $R$ inside the kernel of $\tau$; moreover, $\tau$ induces a faithful pure trace on $R/\text{Ker}^{+}_{\text{ord}} \tau$. We first wish to determine the order ideals in $E_b(G_M)$ that are of the form $\text{Ker}^{+}_{\text{ord}} \tau$ for some pure trace $\tau$.

Define

$$J_X := \{ pM^{-k} \mid p \in A, \ deg p \leq k - 3, \ \partial p/\partial X \neq 0 \}$$

$$J_Y := \{ pM^{-k} \mid p \in A, \ deg p \leq k - 3, \ \partial p/\partial Y \neq 0 \}.$$

It is clear that $J_X$ and $J_Y$ are ideals of $E_b(G_M)$ contained in $J$. In fact, each is of the form $\text{Ker}^{+}_{\text{ord}} \tau$ for suitable $\tau$. Let $F$ be the face of $K(M)$ (the standard triangle) consisting of the line segment $Y = 0$, $0 \leq X \leq 1$. Then $M_F$ is obtained by simply replacing $Y$ by 0, and the map $\pi_F : E_b(G_M) \to E_b(G_{M_F})$ clearly kills $J_Y$ and is one to one on $J \setminus J_Y$. If $\tau$ is a faithful pure trace of $E_b(G_{M_F})$ (necessarily arising from a positive left eigenvector of $M_F$), then it easily follows that for $\tau := \tau \circ \pi_F$, $\text{Ker}^{+}_{\text{ord}} \tau = J_Y$. Similarly, we obtain $J_X$ as the positive kernel of a trace factoring through the map which sends $X$ to 0. The singleton face consisting of the origin, corresponds to sending both $X$ and $Y$ to 0, and we see easily that the positive kernel of this is $J_X + J_Y$. Of course, we have seen already that $J$ is similarly obtained from the face described by $X + Y = 1$. It turns out that this describes all the positive kernels of pure traces, and this can be used to characterize all the pure traces of $E_b(G_M)$.

Let $\tau$ be a pure trace of $E_b(G_M)$. If $\tau$ is faithful, it arises from a left eigenvector as in Theorem 4.5. If not, it kills a positive element. If we go through the procedure of 3.7–3.8, (taking a suitable power of $M$ and dividing by a monomial in $x$ so that the Log sets of the diagonal entries generate $\mathbb{Z}^2$ as a semigroup), we see that $\tau$ if it kills anything, must kill $(XM^{-3})(YM^{-3})(M^{-3})$, and so it must kill one of the three parenthetic expressions. However, if it kills the third one, it kills $J^2$, hence kills $J$, and as $J$ is of codimension one, $\tau$ would have to be the map $E_b(G_M) \to E_b(G_M)/J$ corresponding to the diagonal edge. If it fails to kill $M^{-3}$, then we can invert the latter, and $\tau$ will extend to a positive multiplicative linear functional, also called $\tau$, on $E_b(G_M)[M^3]$; the latter contains $\mathbb{R}[X,Y,M^{\pm1}]$, and it is straightforward to see that $E_b(G_M)[M^3]$ actually equals the former.

In any event, $\tau(X)$ and $\tau(Y)$ are now defined. If they are both not zero, then $\tau$ is faithful; if say $\tau(X) = 0$, we see that $\tau$ is now defined on $E_b(G_M)[M^3] = \mathbb{R}[X,Y,M^{\pm1}]$ factored out by sending $X \to 0$, which yields $\mathbb{R}[Y,M^{\pm1}_F]$ where $F$ is the face described by $X = 0$. Since $M_F$ is itself in companion matrix form, $\mathbb{R}[Y,M^{\pm1}_F] = E_b(G_{M_F})[M^0_F]$. Thus $\tau$ induces a multiplicative trace on $E_b(G_{M_F})$. If $\tau(Y) = 0$, we can factor this out and repeat, and end up with $\tau$ factoring through the quotient induced by the singleton face $(0,0)$, whose kernel is exactly $J_X + J_Y$; if $\tau(Y) \neq 0$, then the trace induced by $\tau$ on $E_b(G_{M_F})$ is faithful, and thus the positive kernel of the original $\tau$ is $J_X$.

Lemma 7.4 (a) Every nonzero order ideal of $E_b(G_M)$ that is of the form $\text{Ker}^{+}_{\text{ord}} \tau$ for some pure trace $\tau$ is one of $J$ (corresponding to $F$ being the diagonal edge or either vertex of the diagonal edge of $K(M)$), $J_X$ (corresponding to the vertical edge), $J_Y$ (corresponding to the horizontal edge), and $J_X + J_Y$ (corresponding to the vertex $(0,0)$).

(b) Every proper order ideal of $E_b(G_M)$ is contained in $J$.

(c) If $e$ is a positive element of $E_b(G_M)$ not in $J$, then $e$ is an order unit of $E_b(G_M)$.

Part (a) has been proved above. Towards (b), let $I$ be a proper order ideal, then $E_b(G_M)/I$ is partially ordered ring with the identity element as order unit; a pure, hence multiplicative trace thus exists. Hence $I$ is in the positive kernel of a pure trace of $E_b(G_M)$, so must be one of $J_X$, $J_Y$, $J_X + J_Y$, or $J$. However, all of them are contained in $J$, and we are done.
Now (c) is trivial. If \( e \) were not an order unit, being positive, it would generate a proper order ideal, hence belong to \( J \) (by (b)), a contradiction.

Lemma 7.4(c) is of course the analogue of [HM, V.3(d)] which holds for \( R_P \); it asserts that there is a finite set of maximal order ideals (one for each vertex of the Newton polyhedron of \( P \)) such that if \( e \) is in \( R_P^+ \) does not belong to any of them, then \( e \) is an order unit.

In view of the structure of \( J \), it is reasonable to ask if the positive cone of either \( E_b(G_{M}) \) or \( J \) is generated by terms of the form \( x^w M^{-k} \) and the identity (i.e., can a positive element be expressed as a positive linear combination of products of such elements). Multiplying by suitable powers of \( M \), this is equivalent to asking that if \( N \) is a matrix in \( \mathbb{M}_n \) centralizing \( M \) all of whose entries have no negative coefficients, then there exists \( l \) such that \( NM^l \) is a positive linear combination of terms of the form \( x^v M^j \) \((j \text{ being nonnegative}, v \geq 0, \text{ with no restriction on the degrees of the monomials})\). This would be the precise analogue of the result for \( R_P \) [HM, Section I], which asserts that the positive cone of \( R_P \) is generated multiplicatively and additively by terms of the form \( x^w / P \).

Unfortunately, this property fails here. As a simple example, begin with \( N := M^3 - 1 \). We see by direct calculation that this is positive, and so \( e := NM^{-3} \) is a positive element of \( E_b(G_M) \), and its image modulo \( J \) is 1. If \( e \) were in the positive cone generated by \( x^w M^{-k} \) and the identity, we see that since all the generators but the identity are in \( J \), it would have to be written as a positive element of \( J \) plus 1. However, it is clearly in the form of a negative element of \( J \) plus 1, which is a contradiction. It follows of course that \( eM^{-3} \) is an element of \( J \) which is also not so expressible.

In any event, we can write down all the pure traces of \( E_b(G_M) \). The faithful ones correspond to evaluating \( X \) and \( Y \) at nonzero real numbers and are implemented by eigenvectors as in Theorem 5.14. The non-faithful ones factor through the appropriate facial map and are obtained as faithful traces on the appropriate quotients, which are obtained by either evaluating at least one of \( X \) or \( Y \) at 0, or by killing \( J \). Note that the natural order structure on the faces is not precisely reflected in the corresponding ideals—e.g., the diagonal edge is disjoint from the origin, but the order ideal corresponding to the former \((J)\) contains the order ideal corresponding to the latter \((J_X + J_Y)\).

In this example, what seems to cause the most effects (the weirdest of which is nonnoetherianness) is the fact that \( XM^{-1} \) is not in \( E_b(G_M) \), even though \( X \) appears in \( \text{tr} M \). Many of these phenomena hold much more generally. For example, although for \( M' = (1 + X + Y)M \), the facial ideals are ordered corresponding to the faces, \( E_b(G_{M'}) \) is still not noetherian, even though it now contains lots of elements, e.g., it contains \( \mathcal{I} R_P \) where \( P = 1 \).

There is a quite strict necessary condition in order that \( E_b(G_M) \) be noetherian; in the one variable case, this condition is always satisfied (by Proposition 2.9—and presumably \( E_b(G_M) \) is noetherian in this case), but with more variables, it fails generically.

**Proposition 7.5** Let \( M \) belong to \( \mathbb{M}_n A^+ \), and let \( P \) be an element of \( A^+ \) such that \( \log P = \log \text{tr} M \). If every order ideal of \( E_b(G_M) \) admits an order unit, then there exists \( m \) such that \( P^{m+1} M^{n+1} \prec P^m M^{n+1} \). In particular, this applies if \( E_b(G_M) \) is noetherian.

**Proof.** First, replace \( M \) by \( M^n \), so that \( \log P = \log \text{tr} M \), and \( \log P^k = \log \text{tr} M^k \) for all \( k \). By dividing by any monomial \( x^w \) such that \( x^w M^{-1} \) belongs to \( E_b(G_M) \), we may assume that 0 is in \( \log P \) as well. By Theorem 5.3, we may further replace the \( M \) by some higher power, in order to guarantee that \( M^2 \prec QM \), where \( \log Q = \log \text{tr} M \) and \( \log Q = i \log P \) for some integer \( i \).

Now the first claim in the proof of Theorem 2.8 asserts that there is a point \( w \) in \( \mathbb{Z}^d \) and an integer \( k \) such that for all \( 0 \leq s \leq t \), \((w + \log \text{tr} M^s, t + k) \) belongs to \( T(M) \). That is, \( x^{w+s} M^{-(k+t)} \) belongs to \( E_b(G_M) \) for all \( v \) in \( \log \text{tr} M^s = \log Q^s \) for all \( s \) between 0 and \( t \). Since 0 belongs to \( \log P \) and hence to \( \log Q \), and thus \( \log Q^s \subseteq \log Q^t \), we thus have that everything of the following form also belongs to \( E_b(G_M) \),

\[
x^w M^{-k} pM^{-t} \quad \text{for any } p \text{ in } A \text{ with } p < Q^t.
\]
Define $J_t$ to be the order ideal of $E_b(G_M)$ generated by $x^uv M^{-(k+t)}$ where $v$ ranges over Log $Q^t$. Obviously, $x^uv M^{-(k+t)}$ is an order unit for $J_t$. We claim that for $t > 1$, $J_t \subseteq J_{t+1}$. This follows from $M^{t+1} < QM^t$—multiply both sides by $x^u Q^t M^{-(2t-1-k)}$ to obtain that $x^u M^{-(k+t)} Q^t M^t$ is in the order ideal generated by $x^u M^{-(k+t)} Q^{t+1} M^{t-1}$.

Hence $\{J_t\}$ is an increasing family of order ideals of $E_b(G_M)$; set $J$ to be their union. Obviously $J$ is an order ideal of $E_b(G_M)$. If it had an order unit, that order unit would have to belong to one of the $J_t$, and we would deduce that for some $t$, $J_t = J_{t+1}$. However, this means that there would exist a positive integer $K$ such that $x^u M^{-(k+1)} Q^{t+1} M^{t-1} \leq K x^u M^{-(k+t)} Q^t M^t$ in $E_b(G_M)$. This translates to $x^u Q^{t+1} M^u \leq x^u Q^t M^{u+1}$ in $M_n A^+$ for some positive integer $u$, and hence $Q^{t+1} M^u \leq Q^t M^{u+1}$. We can certainly increase the powers $t$ and $u$ appearing there, and after we translate back to our original choice for $M$ and $P$, we obtain the desired result.

In a partially ordered noetherian ring for which $1$ is an order unit, every order ideal admits its own order unit (every order ideal is an ideal, therefore requires finitely many generators as a right ideal; express each of the generators as a difference of positive elements in the order ideal, and add the positive elements; the result is an order unit for the ideal).

For example, if $M = \begin{bmatrix} x & 1 \\ y & 1 \end{bmatrix}$, then it is an easy exercise to check that for $k \geq 2$, $\log(M^k)_{1,1}$ is \{ \{(k, 0)\} \cup \{ (a, b) \in (\mathbb{Z}^+)^2 \mid a + b \leq k - 2 \} \}. We note that $(x + y + 1)^k M^k$ always contains a $y^{k+1} x^k$ term in the $(1, 1)$ entry whereas $(x + y + 1)^k M^{k+1}$ does not, this for every $k$. Since $\log((x + y + 1)^2) = \text{cvx} \, \text{tr} \, M^2 \cap \mathbb{Z}^2$, it follows that the criterion of Proposition 7.5 fails in this example, and hence $E_b(G_M)$ is not noetherian. Similar computations apply to $M$ of the previous example, showing again that $E_b(G_M)$ is not noetherian.

It is an easy exercise to verify that every order ideal admitting an order unit is equivalent to the ascending chain condition on order ideals. I still don’t know if the condition obtained here, or the stronger condition (that cvx $T(M)$ be closed) is sufficient to imply every order ideal has an order unit, although this seems plausible. If $M$ is any primitive matrix in $M_n A^+$ and $P = \text{tr} \, \log(M^m)$, then $M^{n+1} + P$ satisfies the necessary conditions of Lemma 7.4.

If $l$ is a loop of length $k$ determined by the directed weighted graph (with weights in $\mathbb{Z}^d$) corresponding to $M$, define $\text{wt}(l)$ to be the average value of the weights of the arcs that constitute it; that is, take the sum of the weights and divide by $k$. This is the “weight per symbol” of the loop (MT). Let $K(M)$ be the normalized convex polyhedron associated to $M$, as usual. For a face $E$ of $K(M)$, define

$$S(E) := \{ i \in \{ 1, 2, \ldots, n \} \mid \text{there exists a loop } l \text{ containing } i \text{ such that } \text{wt}(l) \in E \}$$

Say a matrix $M$ in $M_n A^+$ is drawn if for $P = \text{tr} \, M^m$, $P^{m+1} M^n_{m+1} \leq P^m M^n_{m+1}$ for some integer $m$. So if every order ideal in $E_b(G_M)$ has an order unit, e.g., if $E_b(G_M)$ is noetherian, then $M$ is drawn. Call $M$ tight if $QM^k M^l$ belongs to $E_b(G_M)$ for some $k$ wherein $\log(Q) = \partial_k K(M)$; this is equivalent to the convex hull of the range of $T(M)$ (c.f., section 2) being closed. From the absorption lemma 5.7, it follows that tight implies drawn.

First we give a necessary condition in order that $M$ be drawn. Let $K$ be a (compact) convex polyhedron in $\mathbb{R}^d$ (the definitions, with minor modifications, can be made to work in infinite dimensions), and let $F$ be a face, and suppose that $F$ is the convex hull of two disjoint closed faces, say $F = \text{cvx} \, \{ E, F' \}$. We say the triple $(F; E, F')$ has parallel faces if no matter how an arbitrary element $f$ of $F$ is written as a convex combination of elements of $E$ and $F'$, say $f = \alpha e + (1 - \alpha) f'$ (with $e \in E$, $f' \in F'$, and $\alpha$ in $[0, 1]$), the coefficient $\alpha$ is unique. In case $F$ is a simplex, or if one or the other of $E$ or $F'$ is a singleton (i.e., a face consisting of an extreme point), then of course this property holds, because the $e$ and $f'$ themselves are unique. An example of a complemented face $F = E \vee F'$ wherein even the uniqueness of the coefficient fails occurs with $E$ and $F'$ being two line segments in the plane (placed so that each is still an edge of the convex hull) that are not parallel. On the other hand, if there exists a codimension one affine subspace, $V$,
of the affine span of $F$ such that $E$ and $F$ are contained in translates of $V$, then uniqueness of the coefficient holds, even though uniqueness of the decomposition may fail (as it does for any nontrivial trapezoid, with $E$ and $F'$ being the parallel edges).

The principal use of the following occurs when $E$ and $F'$ consist of the endpoints of an edge $F$.

Lemma 7.6 Let $M$ be a primitive element of $\mathbf{M}_nA^+$ that is drawn. Let $i$ be any state, and let $F$ be a face of $K(M)$ and suppose that $(F; E, F')$ has parallel faces. For any state $i$, if there exists a loop $l$ containing $i$ such that $w(t)$ lies in $E$, then there exists a loop $l'$ containing $i$ such that $wt(l') \in F \setminus E$.

Proof. If not, for some $i$, there exists a loop passing through $i$ with weight in $E$, and every loop passing through $i$ having weight in $F$ has weight in $E$. Say $w$ in $E$ is realized by one such loop $l$; by the facial property, it is easy to see that we may assume that $l$ is minimal, so its length is $n$ or less. Hence there exists an integer $l \leq n$ such that $tw$ appears in $\log (M^t)_{ii}$; as a consequence, $w' := n!w$ appears in $\log (M^{n!})_{ii}$, and for any positive integer $k$, $kw'$ appears in $\log (M^{n!k})_{ii}$.

Pick an extreme point, $f'$, of $F$ not in $E$ (necessarily, $f'$ belongs to $F'$). Then $(k + 1)n!f'$ appears in $\log P^{k+1}$. Thus

$$(k + 1)n!f' + kn!w \in \log \left( P^{k+1} M^{kn!} \right)_{ii}.$$

Write this as

$$z := (2k + 1)n! \left( \frac{k + 1}{2k + 1} f' + \frac{k}{2k + 1} w \right) \in (2k + 1)n!E.$$

By hypothesis, there exists $k$ so that $z$ (which obviously depends on $k$) belongs to $\log (P M^{(k+1)n!})_{ii}$. By the facial property, $z$ belongs to the $F$-component of this, i.e., $z$ belongs to

$$\log \left( P^{k} \right) \log \left( M^{(k+1)n!} \right)_{ii}.$$

Thus we may write $z = (kn!)f_1 + (k + 1)n!e_1$ where $f_1$ and $e_1$ belong to $F$ and there is a loop passing through $i$ of length $(k + 1)n!$ of weight $e_1$. By hypothesis, $e_1$ belongs to $E$.

When we decompose $z/(2k + 1)n!$ according to the first decomposition, we see that the coefficient of the $E$ term is $k/(2k + 1)$. However, when we decompose $f_1$ into its $F'$ and $E$ components, we see that the coefficient of the $E$ component in the second decomposition of $z/(2k + 1)n!$ is at least $(k + 1)/(2k + 1)$, arriving at two decompositions with distinct coefficients, contradicting the parallel faces hypothesis.

Here is a sufficient condition for $M$ to be tight, which as we saw above, is stronger than being drawn. This is usually very efficient, as it only involves minimal loops (a distinct advantage over the previous necessary condition).

Lemma 7.7 Let $M$ be a primitive element of $\mathbf{M}_nA^+$. Sufficient for $M$ to be tight is the following:

For all states $i$, for all minimal loops $l$ passing through $i$, for all $v$ in $\partial_c K(M)$, there exist minimal loops $l'$ and $l''$ such that $l'$ passes through $i$, $wt(l) = wt(l')$, $wt(l'') = wt(l)$, and $l''$ has a state in common with $l''$.

Proof. This is practically tautological. It is sufficient to show that $n!v + \log (M^{n!})_{ij} \subseteq \log (M^{2n!})_{ij}$ for all extreme points $v$. Decompose an arbitrary path of length $n!$ from $i$ to $j$ as the union of a bunch of minimal loops plus a short path with no repetitions of states from $i$ to $j$ passing through into minimal loops. Then perform the obvious surgery on the loops to “multiply” by $x^n$ ($n!$ is excessively large; lem$\{2, 3, 4, \ldots, n\}$ would do just as well) and see the new path of length $2n!$.

Here are a few easy illustrative examples. They become almost transparent when written as graphs.
Example 7.8 Let

\[
A = \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix}, \quad B = \begin{bmatrix} 0 & x + y & 0 \\ y & 0 & x \\ x & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & x + y & 0 \\ y & 0 & x \\ x & 0 & 0 \end{bmatrix}
\]

Then \(A\) and \(B\) fail to be drawn, but \(C\) is tight, hence drawn. All are primitive.

Proof. Primitivity is obvious for \(A\), and it follows for \(B\) and \(C\) from the fact that there are minimal loops of lengths 2 and 3 in matrices of size 3.

For \(A\), there are three minimal loops with average weights the three vertices, respectively, of the standard triangle in \(\mathbb{Z}^2\). The only loop (minimal or not) containing state 1 (the top entry) with weight in the face, \(F\), spanned by \(\{(0, 1), (1, 0)\}\) is the self-loop with weight \((1, 0)\), hence the conditions of Lemma 7.6 are violated (here \(E = \{(1, 0)\}\) and \(F' = \{(0, 1)\}\)).

Regarding \(B\), there are three minimal loops of lengths 3, 2, and 1 with average weights respectively \((1, 0)\), \((0, 1)\), and \((0, 0)\) respectively. With face \(F = \text{cvx} \{(0, 0), (0, 1)\}\), the only loop containing state 3 and having weight in \(F\) is the self-loop, and again the conditions of Lemma 7.6 are violated.

The matrix \(C\) is obtained from \(B\) by moving the self-loop from state 3 to state 1 (moving it to state 2 would also yield a tight matrix). Now we note that the three minimal loops have a state in common, and it follows that the criterion of Lemma 7.7 is satisfied.

Along these lines, there is also a necessary condition for a matrix to be tight. (Recall that an edge of a convex polyhedron is a one-dimensional face.)

Lemma 7.9 If \(M\) is a primitive tight matrix in \(M_n A^+\), then for each minimal loop \(l\) of weight \(v\) in \(\partial_i K(M)\), for all states \(i\) which \(l\) passes through, and for all \(w\) in \(\partial_i K(M) \setminus \{v\}\) such that \(\text{cvx} \{v, w\}\) is an edge of \(K(M)\), there exist minimal loops \(l'\) and \(l''\) such that \(\text{wt}(l') = v\), \(l'\) passes through \(i\), \(\text{wt}(l'') = w\), and there is a common state through which \(l'\) and \(l''\) pass.

The proof of this is more or less straightforward.

A more drastic condition than tightness is that \(PM^{-n!}\) belong to \(E_b(G_M)\) (where \(\text{Log} P = \text{Log} \text{tr} M^{n!}\) as usual). By Lemma 5.1, this forces \(PM^{-n!}\) to be an order unit of \(E_b(G_M)\). Much more is true.

If \(R\) is an ordered ring, and \(r\) is a positive central element (which may be a zero divisor, then we may form what we shall abusively call \(R[r^{-1}]\), obtained as the limit of repeated multiplication by \(r\), with the resulting direct limit ordering. In case \(r\) is a zero divisor, the ring is obtained by factoring out from \(R\) all elements that multiply some power of \(r\) to zero (this is a two-sided ideal), and inverting the image of \(r\) (possible because \(r\) and its image are central). If \(r\) has the property that \(r r' \geq 0\) implies \(r' \geq 0\) in \(R\), then of course \(R \rightarrow R[r^{-1}]\) is a relative order embedding (in the sense that if \(s\) in \(R\) has positive nonzero image in \(R[r^{-1}]\), then \(s\) already is in \(R^+\)). We usually assume that any element \(s\) such that \(r m s = 0\) is not positive, which is what happens below.

In the present situation, \(R = E_b(G_M)\); let \(Q\) be a polynomial such that \(QM^{-k}\) belongs to \(E_b(G_M)\). In general, there is no guarantee that \(r r' \geq 0\) implies \(r' \geq 0\) for \(r'\) in \(E_b(G_M)\). Related to this is the natural map \(E_b(G_M) \rightarrow E_b(G_{MP})\); (a) of interest is when is it a relative order embedding, or at least (b) every pure trace on the former lifts to the latter. We conjecture that (b) is always true; if this were the case, we would obtain more information on traces on \(E_b(G_M)\).

Proposition 7.10 Let \(M\) be a primitive matrix in \(M_n A^+\), and suppose that for some (hence any) \(P\) in \(A^+\) such that \(\text{Log} P = \text{Log} \text{tr} M^{n!}\), \(PM^{-n!}\) belongs to \(E_b(G_M)\). The following hold:

(a) If \(u\) is a column with entries from \(A^n\) such that \(P^f u\) belongs to \((A^n)^+\) for some integer \(f\), then there exists \(k\) such that \(M^k u\) belongs to \((A^n)^+\).

(b) As ordered rings, \(E_b(G_{MP}) = E_b(G_M)[(P/M^{-n!})^{-1}]\), and the map \(E_b(G_M) \rightarrow E_b(G_{MP})\) is an order embedding inducing a bijection on order ideals.
(c) All traces of $E_b(G_M)$ extend uniquely to traces of $E_b(G_{MP})$.

Proof. (a) By Theorem 5.3, for all sufficiently large $k$, $M^{kn!} \prec PM^{(k-1)n!}$. From $PM^{-n!}$ in $E_b(G_M)$, we obtain that for all sufficiently large $l$, $PM^{(l-1)n!} \prec M^{n!}$. Thus with $m \geq \max \{k, l\}$, we have $PM^{(m-1)n!} \prec M^{n!m} \prec PM^{(m-1)n!}$. Hence for all $i$ and $j$, we have that

$$\log (M^{n!m})_{ij} = \log P + (M^{n!(m-1)})_{ij}.$$ 

Set $N = M^{n!m}$, and write $u = (u_s)^T$; from $P^j u_s$ in $A^+$, it obviously follows that for all $i$ and $j$, $(P(M^{n!m-1}))_{ij}^Tu_s$ also belongs to $A^+$. By [H2, Theorem 2.2], there exists a positive integer $h_0 \equiv h_0(i, j, s)$ such that $(N^h)_{ij}^Tu_s$ belongs to $A^+$. Set $h = \max_{i, j, s} \{h_0(i, j, s)\}$.

Now consider $N^{h^2}$. Every entry is a sum of products of $h^2$ entries of $N$, hence every such product must contain a product of at least $h$ identical terms, i.e., $(N^2)^h$ for some $i, j$. Each such product thus renders any $u_s$ positive (all coefficients nonnegative), so that every entry of $N^{h^2}$ will do the same. Thus $N^{h^2}$ will belong to $(A^m)^+$, so $n = h^2 m n!$ will do.

(b) Suppose $e = N(MP)^{1-k}$ is in $E_b(G_{MP})$; then there exists $l$ such that $N(MP)^l \prec (MP)^{l+k}$; we may increase $l$ as much as we like. As in the beginning of the proof of (a), $(MP)^{l+k} \prec M^{l+k+n(l+k)}$ (for sufficiently large choice of $l$). Hence $NP^l M^{-(k+n(l+k+1))} := e'$ belongs to $E_b(G_M)$. Then $e' \cdot (M^{n!}/P)^{l+k} = e$ expresses $e$ as an element of $E_b(G_M)((P/M^{-n!})^{-1})$. Now part (a) shows that the two possible positive cones on the latter are equal (since we just apply it to all columns of the relevant matrices. It is an easy exercise to check that the map is an embedding, and it then follows from (a) that it is an order embedding. It is also easy to see that two distinct order ideals in $E_b(G_M)$ will remain so in the larger ring.

(c) If $A \rightarrow B$ is an order preserving map of unperforated partially ordered abelian groups with $A$ admitting an order unit which is also an order unit for $B$ such that if any element of $A$ that becomes an order unit for $B$ is then an order unit for $A$, then traces extend [H8, Lemma 1.4]. Uniqueness follows from uniqueness for the pure traces, which follows from the fact that the extension of the traces is uniquely determined by the its value on the inverted element.

The really interesting property is (c); that is because (c) is possibly true for all primitive matrices $M$. If this were the case, then we could analyze the pure traces on order ideals of $G_{MP}$ and then get information about the traces of corresponding order ideals of $G_M$ (pure traces may extend in more than one way in this more general setting, as in the standard FOG example; however, we would obtain the trace space as a quotient of the known trace space modulo the obvious equivalence relation). Moreover, traces on order ideals of $G_{MP}$ and $E_b(G_{MP})$ are easier to analyze, because we can use $R_P$. We notice that $M^{n!}/P$ is a bounded endomorphism of $(R_P)^n$, and the limit is order isomorphic to $G_{MP}$. The matrix $M$ is drawn if and only if the automorphism of $G_{MP}$ induced by $M^{n!}/P$ is an order unit in $E_b(G_{MP})$ (which entails that $E_b(G_{MP})$ is not generally obtainable from $E_b(G_M)$ by inverting $P/M^{n!}$). The traces of $R_P$ are completely known, and in almost all cases, the traces of order ideals factor through the latter.

A plausible argument (that comes very close to succeeding) that would prove that traces do extend (from $E_b(G_M)$ to $E_b(G_{MP})$) is the following. Pick a pure trace of $E_b(G_M)$; if it is faithful, it is given by a point evaluation and an eigenvector, and it is clear how to extend it. Otherwise, it must kill one of the facial ideals $I_F$ (Proposition 3.8), hence if we pick the smallest $F$ (largest $I_F$) for which this holds, we obtain a trace on $E_b(G_M)/I_F$. The natural map $E_b(G_M)/I_F \rightarrow E_b(G_{MF})$ is clearly one to one and an order-preserving. If we knew that traces extend here, we would be done. It is quite likely that this map is even an order embedding, in which case of course, trace extension is immediate. If the range is one dimensional (as occurs in the FOG example, for any face $F$), then trace extension is automatic.

The problem is that we do not know what the quotient ordering really looks like; explicitly, if $N$ is in the centralizer of $M$, $N \prec M^{-k}$, and $N_{kF}$ has all of its entries in $A^+$, does it follow that there is another matrix $N' \prec M'^{-k}$ commuting with $M$ such that all entries of $N'$ have no negative coefficients and $N'M'^{-l}$.
has the same image as $NM^{-k}$; in other words, do positive elements of $E_b(G_{MF})$ that come from $E_b(G_M)$ lift to positive elements? (To complete the argument, if the trace does extend to $E_b(G_{MF})$, then modulo some primitive blocks of $M_F$ which it kills, it is a faithful trace and would then extend to an even larger ring than $E_b(G_{MF}/P_{HF})$ which is more than enough to extend it to $E_b(G_{MF})$.)

It also would be desirable to answer the following questions, for $M$ a primitive matrix in $M_nA^+$:

1. If $M$ is drawn, then is it tight?
2. If $M$ is tight, i.e., for some $k$, $QM^{-nk}$ (where $\log Q = \partial_e k K(M)$ and $Q$ has no negative coefficients) is in $E_b(G_M)$, then is it an order unit?

Regardless of the answer to (2), the element $Q$ satisfies Proposition 7.10(a) (virtually the same argument, since the crucial tool [H2, Theorem 2.2] was proved in just the right generality to obtain this). This in turn leads to another question, which unfortunately has a negative answer.

One of the most interesting results in the study of $R_F$ is the fact that in $R_P$, if $u$ is an order unit and $r$ is an arbitrary element, then $ur \geq 0$ implies $r \geq 0$ (an alternative formulation of [op. cit.]). For an ordered ring $R$, let us call this the order unit cancellation property. This also holds for the ordered rings arising from certain characters of compact Lie groups, although it is not explicitly stated in this fashion in [H8]. The comment of the previous paragraph says that in $E_b(G_M)$ (and somewhat more generally), order unit cancellation holds for elements of the form $QM^{-nk}$ (at least when they are order units). It would be useful to know that order unit cancellation holds for all order units in $E_b(G_M)$ (this would give a quick proof of trace extension in the case that $M$ is tight). Unfortunately, it fails, even for size two matrices in one variable!

As a general comment on order unit cancellation, in any unperforated ordered ring with 1 as order unit, if $ur \geq 0$ ($u$ an order unit) and there exists a positive integer $k$ such that one of $kur \pm r$ is positive, then $r$ is positive. (See [H2, Chapter 2].) So order unit cancellation is very close to holding in general, and for a generic order unit, it should hold.

In this example, $E_b(G_M)$ was commutative and had zero divisors. It is possible that order unit cancellation holds if $E_b(G_M)$ has no zero divisors (which is the generic case in more than one variable — $E_b(G_M)$ will have no zero divisors if the characteristic polynomial of $M$ is irreducible away from zero).

The result that $E_b(G_M)$ noetherian implies $M$ is drawn was the main reason for introducing the property of being drawn. Does a converse hold? Perhaps the converse should be put in the form, if $M$ is drawn, then every ascending chain of order ideals in $E_b(G_M)$ terminates, equivalently, every order ideal admits an order unit. There are some (restrictive) cases in which a converse of some sort is true.

Suppose that the conclusion of Proposition 7.10(b) were true. If we knew that every matrix with entries in $R_P$ (that commutes with $M^{n!}/P$) is bounded by a multiple of the identity (essentially this means that if $N$ is a matrix in $M_nR_P$, possibly commuting with $M^{n!}/P$, then $\pm N(M^{n!}/P)^k \leq K(M^{n!}/P)^{k+1}$, entrywise in $R_P$ for some positive integers $k$ and $K$), then $E_b(G_{MP})$ is just the centralizer of $M^{n!}/P$ in $M_nR_P$ with the inverse adjoined. Since $R_P$ is noetherian, the centralizer of a matrix with entries from it is a finitely generated $R_P$-module (follows easily from the fact that if $V$ is a finite type module over a commutative noetherian ring, then the endomorphism ring of $V$ is of finite type as a module; to see that just
apply \( \text{Hom}[-,V] \) to a presentation of \( V \), and observe that the functor is one-sided exact). Hence in this case, \( E_b(G_{MP}) \) is noetherian; moreover, the bijection between the order ideals shows that every order ideal in \( E_b(G_M) \) has an order unit.

This argument requires that every matrix (centralizing \( M^{n!}/P \) inside \( \text{M}_n(R_F) \)) induce a bounded endomorphism. Certainly if \( (**\prime) \) holds, this is true, so that \( E_b(G_{MP}) \) is noetherian. (This is hardly surprising—it is easy to check directly if that \( (**\prime) \) holds, the order ideals of \( E_b(G_M) \) are in bijection with those of \( R_F \), hence satisfy the ascending chain condition, hence every order ideal admits an order unit).

**Traces, faces, and extensions.** A natural and useful result would be a determination of all the pure, i.e., multiplicative, traces on \( E_b(G_M) \) (say for \( M \) primitive). We already know that the faithful pure traces are given by Perron eigenvectors obtained from point evaluation—yielding a parameterization of the faithful pure traces by \( (\mathbb{R}^d)^{++} \). In the case that \( n = 1 \), it was shown in [H5, V.1] that these are dense, and we can obtain the limit points by following certain families of paths (exponentials of rays) obtained from the dual of the Newton polytope. For \( n > 1 \), density fails. However, by Proposition 3.8, a non-faithful pure trace must kill at least one of the ideals, \( \ker \pi_F \) for some face, thereby inducing a positive multiplicative map on the factor algebra, \( E_b(G_M)/I_F \).

**Q1** Is \( I_F \) an order ideal of \( E_b(G_M) \)?

We do not know whether \( I_F \) is an order ideal, so that although we can impose a pre-ordering on \( E_b(G_M)/I_F \) (with positive cone the image of the positive cone of \( E_b(G_M) \)), we cannot deduce many properties of this factor algebra. For example, in some cases, it is possible to prove directly that \( E_b(G_M) \) is a dimension group (for instance, if \( M \) is in companion matrix form with no repeated factors in its characteristic polynomial—it should be possible to prove \( E_b(G_M) \) is a dimension group even without the companion matrix hypothesis), and if \( I_F \) were an order ideal, then \( E_b(G_M)/I_F \) would be a dimension group. The result (Proposition 3.8(a)) that \( \cap I_F \) is nil modulo an ideal of the form \( M^{-c}E_b(G_M) \) (and this ideal is an order ideal) is a partial and useful result in this direction.

**Q2** Does every pure trace on \( E_b(G_M) \) that kills \( I_F \) extend to a pure trace on \( E_b(G_{MF}) \) (via \( \pi_F : E_b(G_M) \to E_b(G_{MF}) \))? 

**Q2’** Is the map \( E_b(G_M)/I_F \to E_b(G_{MF}) \) induced by \( \pi_F \) an order-embedding?

An affirmative answer to question Q2’ would imply an affirmative answer to Q2. A serious problem is whether we can use \( \pi_F : E_b(G_M) \to E_b(G_{MF}) \) more fully. We would like to conclude that every pure trace on \( E_b(G_M)/I_F \) (meaning, a pure trace on \( E_b(G_M) \) that kills \( I_F \)) extends to \( E_b(G_{MF}) \). The reason is the following.

The matrix \( M_F = \bigoplus M_{F,\alpha} \) is a direct sum of irreducible matrices. The bounded endomorphism ring is just the direct product (with the product ordering) of the bounded endomorphism rings \( E_b(G_{MF,\alpha}) \), and modulo irreducible matrices which are not primitive (which can be dealt with separately), we are back in the primitive case, except this time in effectively fewer variables (the number of variables measured by the dimension of the affine span of \( F \), which is always smaller than the original number). Every pure trace on each \( E_b(G_{MF,\alpha}) \) of course induces a pure trace on \( E_b(G_M) \), although of course different choices of primitive component, \( M_{F,\alpha} \), need not yield different pure traces (e.g., if for two \( \alpha \)'s, the corresponding \( M_{F,\alpha} \) have the same \( \beta \) function, then the faithful pure traces on \( E_b(G_{MF,\alpha}) \) for one of the \( \alpha \)'s will restrict to the same pure trace on \( E_b(G_M) \) as their counterparts from the other choice for \( \alpha \)). However, this a rather minor point. Since the effective number of variables drops in going down to a face, there is an obvious opportunity for an induction argument (in some cases, there is also a drop in the size of the primitive/irreducible matrix, which can be exploited).

If we consider non-faithful pure traces \( \dagger \) of one of the \( E_b(G_{MF,\alpha}) \) (and assume for simplicity that \( M_{F,\alpha} \) is primitive and not just irreducible), then of course we can apply the same process. However, all we know about the polytope \( K(M_{F,\alpha}) \) is that it is a non-empty subset of \( F \) — it is easy to construct examples wherein it is an arbitrary convex polyhedral subset of \( F \) — the only constraint being on the union over \( \alpha \), namely that \( \text{cvx} \cup_{\alpha} K(M_{F,\alpha}) = F \) (every possibility can be realized with this constraint).
If for fixed $\alpha$, $K(M_{F,\alpha})$ has a proper face $G$, then of course we can iterate the process and consider the facial matrix of $M_{F,\alpha}$ corresponding to $G$ (I am not going to complicate the notation further), and so on. Note that $G$ might be in the interior of $F$, so that it bears no relation to any faces of $F$ or $K(M)$ (except that it has strictly smaller affine dimension than that of $F$).

As an amusing observation, if $M$ satisfies the separation condition of section 10, ($\#$), (which asserts that the ratio of the second largest absolute value of the eigenvalues of $M(r)$ to the Perron eigenvalue is bounded above away from $1$, as $r$ runs over $(\mathbb{R}^d)^{++}$, then for at least one $\alpha$, $K(M_{F,\alpha}) = F$. If $M$ satisfies the fairly strong condition that for each extreme point $w$ of $K(M)$, some power of $x^w M^{-n_1}$ belongs to $E_b(M_{F,\alpha})$ (this condition implies the formally weaker condition $P^{k-1}M^{kn_1} \prec P^kM^{(k-1)n_1}$, which is implied by noetherianness of $E_b(M_{F,\alpha})$), then for every $\alpha$, $K(M_{F,\alpha}) = F$, which forces the “$G$” above to be a face of $F$; moreover, this fairly strong condition is inherited by subsequent iterates of this process, so that every iterate only involves faces of $K$, not just faces of lower dimensional convex subsets. The extremely strong condition (***) forces there to be only one block per face, and the block is full size.

If we knew that question Q2 had an affirmative answer, then every pure trace on $E_b(M_{F,\alpha})$ would either be faithful, or factor through one of the $E_b(M_{F,\alpha})$ for some face $F$; moreover, a finer description would be obtained by applying the same process (reduction to a relative face), inductively to the relevant irreducible component, until we reach a situation where the induced trace is faithful, whereupon it corresponds to a point evaluation and a corresponding Perron eigenvector. The next step would be to decide given two pure traces constructed by this iterated (relative) face business restrict to equal traces on $E_b(M_{F,\alpha})$; this looks relatively easy.

Q3 For $P = \log \tr M^{n_1}$, do traces extend from $E_b(M_{F,\alpha}) \to E_b(M_{F,\alpha})$? Is this true for arbitrary $P$ in $A^+$?

This holds for $n = 1$ (not explicitly stated, but deducible from results in [H5]), and is also true whenever Q2 has an affirmative answer, as a study of the faces and the relative faces arising from the iterated process described above reveals. If we could show that the first part were true, then we could probably obtain that the second part were true, and we would get some results with respect to Q2. The idea is that the $E_b(M_{F,\alpha})$ is an algebra over $R_P$, and every pure trace on $E_b(M_{F,\alpha})$ induces a pure trace of $R_P$ and the latter are completely understood. These would play the role of point evaluations, except now we are working over a compact space (in fact a compactification of $(\mathbb{R}^d)^{++}$, and not just $(\mathbb{R}^d)^{++}$, and things seem easier.

With the first choice for $P$, the condition that some power of $P M^{-n_1}$ belong to $E_b(M_{F,\alpha})$ implies the necessary condition for noetherianness of $E_b(M_{F,\alpha})$, namely $P^{k+1}M^{kn_1} \prec P^kM^{(k+1)n_1}$. In this case, the traces on $E_b(M_{F,\alpha})$ are completely determined in section 5 (since in this case, $M/P$ is an order unit of $E_b(M_{F,\alpha})$).

The usual (and effectively the only) way to show that pure traces extend, say from $\pi : (A, u) \to (B, \pi(u))$ where $u$ and $\pi(u)$ are order units and $\pi$ is order preserving, is to show that $\pi^{-1}(B^{++})$ is contained in the positive cone of $A$ (which is in turn equivalent to $\pi^{-1}(B^{++}) = A^{++}$). Thus to answer Q3, it would be sufficient to show that if $NM^{-k}$ belongs to $E_b(M_{F,\alpha})$ and $PM^k \prec PN \prec PM^k$, then there exists $l$ such that $NM^l$ is nonnegative. For Q2, the problem is to show the same type of result with $E_b(M_{F,\alpha}) \to E_b(M_{F,\alpha})$. Attempts to attack this and Q3 via local methods (i.e., by means of the faces) boil down to the following: Is it true that if $\pi_F(e) \geq 0$ (or is an order unit) with respect to the ordering on $E_b(M_{F,\alpha})$ (or one of its factor rings coming from an irreducible block of $M_{F,\alpha}$), then there exists $e'$ in $E_b(M_{F,\alpha})^{++}$ with $\pi_F(e') = \pi_F(e)$. In other words, do positive elements (or even just order units) lift to positive elements? This is closely connected with Q1.

A trivial case in which this holds occurs when the range of $\pi_{F,\alpha}$ is one dimensional at each $\alpha$—of course, the range simply consists of the scalars, and positive scalars obviously lift to positive elements. (This occurs in the FOG example.) Anyway, when this lifting property holds for all faces, then both Q2 and Q3 have an affirmative answer.

8. Linking $M$ & $M$:

As usual, let $M$ be a primitive $n \times n$ matrix with entries in $A^+$, Laurent polynomials in $d$ variables. Let
Let \( v = (v(1), \ldots, v(d)) \) be a nonzero element of \( \mathbb{R}^d \), and let \( B = (B_1, B_2, \ldots, B_d) \) be an element of \( (\mathbb{R}^d)^{++} \). Define the path in \( (\mathbb{R}^d)^{++}, \) \( X := X_{B,v} : \mathbb{R}^+ \to (\mathbb{R}^d)^{++} \) via \( X(t) = (B_1^{v(1)}, B_2^{v(2)}, \ldots, B_d^{v(d)}) \); thus \( log X \) is a ray in \( \mathbb{R}^d \) passing through \( \log B \) with directional derivative \( v \).

Now consider the entries of \( M(X_{B,v}(t)) \), particularly as \( t \to \infty \). Let \( K = K(M) \); then the vector \( v \) exposes a face \( F \); more precisely, let \( s = \sup \{ v \cdot k | k \in K \} \) (this will be abbreviated \( \max v \cdot K \); then \( F \) defined as \( \{ k \in K | v \cdot k = s \} \) is a nonempty face of \( K \). Each \( M_{i,j}(X_{B,v}(t)) \) is a formal polynomial in the variable \( t \) ("formal" means a sum \( \sum r_i t^{\alpha(i)} \) where \( r_i \) are real and \( g(i) \) real) with no negative coefficients. For any face \( F \), we may find a vector \( v \) exposing \( F \) relative to \( K \) that has integer coefficients (since all the vertices of \( K \) are lattice points, and we can find rational solutions to systems of linear equations with rational coefficients if we can find a real solution; clear the denominators). So let us assume \( v \) actually has integer entries. This means that each \( M_{i,j}(X_{B,v}(t)) \) is a Laurent polynomial with positive coefficients; we can of course work with rational entries as well, which would correspond to polynomials fractional exponents (and in this case, we would have to replace the \(-1\) appearing in the big Oh term by \(-1/n!\)).

Hence we may write \( M_{i,j}(X_{B,v}(t)) = g_{i,j}(B)t^{d(i,j)} + O(t^{d(i,j)-1}) \) for large \( t \), where \( g_{i,j} \) is a Laurent polynomial in \( t \) variables with positive coefficients (so \( g_{i,j}(B) \) is an abbreviation for \( g_{i,j}(B_1, B_2, \ldots, B_d) \); however, we will hold \( B \) constant for a while), \( d(i,j) \) is the degree (in \( t \)) of \( M_{i,j}(X_{B,v}(t)) \), and of course \( d(i,j) = \max \{ v \cdot w | w \in \Log M_{i,j} \} \).

Now we can recover \( F \) as an asymptotic limit, as follows. Recall that \( F \) is a direct sum of irreducible matrices and a zero matrix. The blocks are the irreducible components or the singleton sets whose state does not belong to any irreducible block. We say two states \( i \) and \( j \) belong to the same block if either \( i = j \) or the \( (i,j) \) entry of \( F \) lies in some irreducible block.

Lemma 8.1 For all choices of \( B \) in \( (\mathbb{R}^d)^{++} \) and rational \( v \) in \( \mathbb{R}^d \setminus \{0\} \), if \( (M_F)_{i,j} \neq 0 \), then

\[
M_{i,j}(X_{B,v}(t)) = (M_F)_{i,j}(B)t^{d(i,j)} + O(t^{d(i,j)-1/n!})
\]

Remark. The expression \( M_F(B) \equiv M_F(B_1, \ldots, B_d) \) of course makes sense, since \( M_F \) was defined as a matrix with entries in \( A^+ \). The \( n! \) term could be replaced by the least common multiple of \( \{2, 3, \ldots, n\} \), but there is no obvious advantage. We could even allow the \( v \) to have real entries, and then there is some \( \delta > 0 \) that replaces the \( 1/n! \) term, independently of \( i \) and \( j \).

Proof. Fix \( i, j \) and expand \( M_{i,j} = \sum_{\Log M_{i,j}} \lambda_z x^z \) where \( \lambda_z > 0 \). For each \( z \) in the Log set, either \( \lambda_z x^z \) appears in \( (M_F)_{i,j} \) or zero does, and the former occurs if and only if there is a cycle \( i \to j \to \cdots \to i \) whose first weight is \( z \) and whose average weight, \( a \), is in \( F \), i.e., satisfies \( v \cdot a = s := \max v \cdot K = v \cdot F \).

Suppose \( i \) and \( j \) belong to the irreducible block \( \alpha \) (just a name; \( M_{F,\alpha} \) is the corresponding irreducible matrix). If \( (M_F)_{i,j} \) is not zero, then there exists a cycle \( i \to j \to \cdots \to i \) whose first weight is \( y \) (i.e., \( \lambda_y x^y \) appears in \( (M_F)_{i,j} \)) and average weight \( a \) lying in \( F \). We may replace the first arc \( i \to j \) by one with weight \( z \) and we obtain an average weight for the new cycle, \( a + (z - y)/l \) where \( l \) is the length of the cycle. Since \( v \cdot a = \max v \cdot K \), it follows immediately that \( v \cdot z \leq v \cdot y \) for and \( z \) in \( \Log M_{i,j} \). Moreover, if equality holds, then \( \lambda_z x^z \) does appear in \( (M_F)_{i,j} \).

On the other hand, suppose \( \lambda_z x^z \) and \( \lambda_y x^y \) appear in \( (M_F)_{i,j} \). Interchanging the roles of \( z \) and \( y \), we see that \( v \cdot y \leq v \cdot z \), so equality holds.

Thus, if \( (M_F)_{i,j} \neq 0 \), \( \lambda_z x^z \) appears in \( (M_F)_{i,j} \) if and only if \( v \cdot z = \max v \cdot \Log M_{i,j} \). The latter is precisely \( d(i,j) \). Everything now follows.

A minor problem that arises is that \( \delta t \) can certainly happen that \( (M_F)_{i,j} = 0 \) for some entry of an irreducible block, in which case the previous result tells us nothing. This will be remedied below.

Let \( u : (\mathbb{R}^d)^{++} \to \mathbb{R} \) be an algebraic function over \( \mathbb{R}[x_i^{\pm 1}] \) (that is, \( u \) satisfies a polynomial, \( u^n + \sum_{i=0}^{n-1} p_i u^i = 0 \) where all the \( p_i \) are rational functions with real coefficients). We say that an algebraic function \( u \) exhibits *fractional polynomial behaviour* if (i) \( u \) is strictly positive as a function on \( (\mathbb{R}^d)^{++} \),
and (ii) there exists an element $J$ in $A_\mathbb{Q}^+$ such that $u/J$ is bounded above and below (away from zero) on $(\mathbb{R}^d)^{++}$. If additionally $J$ can be chosen in $\mathbb{R}[x_i^{\pm 1}]$, then $u$ is said to exhibit polynomial behaviour.

For example, by Lemma 3.2, if $\beta$ is the large eigenvalue function of the $n \times n$ primitive matrix $M$ in $\mathbb{M}_n[\mathbb{R}[x_i^{\pm 1}]]$, then $\beta^{n!}$ exhibits polynomial behaviour (take $J = \text{tr} M^{n!}$) and of course $\beta$ itself exhibits fractional polynomial behaviour (take $J = \sum x^w$ where $w$ varies over exponents of the form $v/n!$ where $v$ runs over $\partial_i \log \text{tr} M^{n!}$). For many problems dealing with finite equivalence, we need to show that $M$ has a left or right eigenvector (for the eigenvalue function $\beta$) all of whose entries exhibit fractional algebraic behaviour. This does not always occur (the FOG examples), but it will under some hypotheses, given later.

A more general property for the entries of the eigenvector is fractional rational asymptotic behaviour; this means there exist $J$ and $J'$ in $A_\mathbb{Q}^+$ such that $u.J'/J$ is bounded above and below away from zero on $(\mathbb{R}^d)^{++}$.

Now let $C := (c_1, \ldots, c_n)$ be a nonnegative left eigenvector for $M$ at the eigenfunction $\beta$; we may choose the $c_i$ to be both algebraic and real analytic on $(\mathbb{R}^d)^{++}$, e.g., by using the method of adjoints. In general, as we shall see, we cannot choose $c_i$ to have fractional polynomial behaviour (we obtain conditions on $M$, which it turns out are generic if $d > 1$, that prscribe fractional polynomial behaviour). Nonetheless, $C(X_{B,v}(t))$ does exhibit this growth (when the entries are viewed as analytic functions in $t$) (as we have chosen $v$ to be rational—drop the “algebraic” if $v$ is not necessarily rational), because it is the left eigenvector of the matrix $M(X_{B,v})$ (viewed as a matrix whose entries are polynomials in fractional powers of $t$), and an analytic algebraic function in one variable is of the form $L^{a/b} + O(t^{(a-1)/b})$ for some integers $a$ and $b$ and nonzero constant $L$—[A, pp. 290–295]. The point sets of the paths $X_{B,v}$ and $X_{B',v}$ are equal for sufficiently large $t$ if and only if $\log B - \log B' \in v\mathbb{R}$, i.e., the paths may be taken to be parameterized by elements of $(\mathbb{R}^d)^{+/v\mathbb{R}}$.

In particular, we may write $c_j(X_{B,v}(t)) = h_j(B)t^{m(j)} + O(t^{m(j)-1/n!})$ where $h_j$ is a function (it is not true that $B \rightarrow h_j(B)$ is real analytic in this context), $h_j(B) > 0$, and $m(j)$ is a rational whose denominator divides $n!$ and probably divides the least common multiple of $\{2, 3, 4, \ldots, n\}$. Obviously, $C(X_{B,v})$ is a left nonnegative eigenvector for $M(X_{B,v})$ with eigenvalue $\beta(X_{B,v})$ (all are functions of $t$).

Now we may write $\beta(X_{B,v}(t)) = f(B)t^s + O(t^{s-\delta})$ from [A, pp. 290–295] and Lemma 3.1, and the fact that $\beta(X_{B,v})$ is the large eigenvalue of the primitive matrix (viewed with entries from $t$) $M(X_{B,v})$. (Again, while $f(B) > 0$, we are not claiming that $B \rightarrow f(B)$ is real analytic.) Finally, let $d(i,j)$ denote $\deg M(X_{B,v})_{ij}$.

Call the $(i,j)$ entry of $M(X_{B,v})$ normal if $d(i,j) = s + m(j) - m(i)$. We sometimes refer to $(i,j)$ being normal by abuse of notation.

Lemma 8.2

(a) If $i$ and $j$ are in the same block and $(M_F)_{ij} \neq 0$, then the $(i,j)$ entry is normal.

(b) If the $(i,j)$ entry is normal, then $(M_F)_{ij} \neq 0$.

Proof. (a) For any $i$ and $j$, $d(i,j) \leq s + m(j) - m(i)$. Suppose that $(M_F)_{ij} \neq 0$. There exists a loop $i = j_0 \rightarrow j = j_1 \rightarrow \cdots \rightarrow j_l = i$ with average weight $a$ which belongs to $F$ (i.e., $v \cdot a = s$). Thus $d(i,j) + \sum_{k=0}^{l-1} d(j_k,j_{k+1}) = ls$. The left side is dominated by a telescoping sum, $ls + \sum (m(j_{k+1}) - m(j_k)) = ls$, with equality if and only if each $d(j_k,j_{k+1}) = s + m(j_{k+1}) - m(j_k)$; hence each of the consecutive pairs appearing in the path are normal, in particular, $(i,j)$.

(b) From irreducibility and the definition of the blocks of $M_F$, there exists a path $j = j_0 \rightarrow j_1 \cdots \rightarrow j_l = i$ with each entry, $(M_F)_{j_k,j_{k+1}}$ not zero. By (a), $d(j_k,j_{k+1}) = s + m(j_{k+1}) - m(j_k)$. It follows easily that if $a$ is the average weight of the loop obtained by concatenating the path with $i \rightarrow j$, then $v \cdot a = s$, so that $a$ belongs to $F$. Hence $(M_F)_{ij} \neq 0$.

A useful tool is conjugation with a suitable diagonal matrix whose entries are monomials. This forces the path of matrices, $(M(X_{B,v}(t)))_{t \rightarrow \infty}$, converge, but to a matrix generally different than $M_F(B)$; it is the local version of the trick to be used in section 10, and is a form of the H-transform used in Markov chains.
Let $\Delta = \text{diag}(t^{m(1)}, \ldots, t^{m(j)})$; then $M' := \Delta M(X_{B,v})\Delta^{-1}$ has $C' := C(X_{B,v})\Delta^{-1}$ as its left eigenvector, and of course $M'$ still has as its entries Laurent polynomials in fractional powers of $t$. However, $C'$ consists of analytic and algebraic functions that are bounded above and below (in $t$). This forces all the column sums of $M'$ to be bounded above and below away from zero by a multiple of $\beta(X_{B,v})$, in particular, all the entries are bounded above by a multiple of this, so that their growth is always at most $O(t^s)$. The path of matrices $M'(t)/t^s$ need not converge to $M_F(B)$ as $t \to \infty$ (since it can happen that some of the entries not in a nonnull block can grow as $t^s$—examples in several variables of this phenomenon are unfortunately ubiquitous, as we shall see, and the phenomenon represents an obstruction of an unpleasant sort; convergence does occur in the entries for which $M_F$ is not zero, by Lemma 8.1.

In particular, the limit point of the path of matrices is a nonnegative real matrix which is entrywise at least as large as $M_F(B)$ and therefore, the spectral radius of the latter is dominated by the limit matrix, call it $M_\infty$. However, it is easy to see that $M_\infty$ has $f(B)$ (the asymptotic value of $\beta$) as an eigenvalue and the corresponding left eigenvector is strictly positive, by construction. It follows immediately that $f(B)$ is the spectral radius of $M_\infty$ (proof: if not, by the Perron-Frobenius theorem applied to right eigenvectors, there exists $r > f(B)$ and a nonzero nonnegative right eigenvector $w$ for eigenvalue $r$; since $r \neq f(B)$, the inner product of the strictly positive left eigenvector for $f(B)$ with the nonnegative $w$ must be zero, which is obviously impossible).

When we view $M(X_{B,v}(t))$ as a matrix with entries which are Laurent polynomials (with rational exponents) in the single variable $t$ (or $t^{1/k}$ for some integer $k$) and no negative coefficients, conjugation with the diagonal matrix $\Delta$ is a (very) strong shift equivalence, and it is usually very easy to check that specific properties are preserved by this transformation.

Let $\alpha$ and $\alpha'$ correspond to two distinct blocks (one of which might be null). We say $\alpha$ is linked to $\alpha'$, denoted $\alpha \subset \alpha'$ relative to $B$, $F$, etc., if there exists $i$ in $\alpha$ and $j$ in $\alpha'$ such that

$$\lim_{t \to \infty} \frac{c_j(X_{B,v}(t)) M_{j,i} X_{B,v}(t)}{c_i(X_{B,v}(t)) \beta(X_{B,v}(t))} \neq 0;$$

in other words, $m(j) + d(j, i) = m(i) + s$ (the left side is always less than or equal to the right, as follows from the eigenvector equation). If $i$ and $j$ belong to different blocks, $\alpha$ and $\alpha'$ respectively, then normality of the pair $(j, i)$ implies $\alpha' \subset \alpha$. (It perhaps is unfortunate that I called this property linking, which suggests a symmetric relation—the relation is practically anti-symmetric.)

A linkage means that there is a contribution to the eigenvector product that vanishes when we go to $M_F$, but does not vanish in the (asymptotic) limit, as $t \to \infty$. (So the asymptotic version of the eigenvector will not be an eigenvector for $M_F$, nor will the asymptotic limit of the eigenvalue necessarily be an eigenvalue for $M_F$ when linkages exist.)

Consider the example

$$M = \begin{bmatrix} 1 + x & 1 \\ y & x + y \end{bmatrix}.$$ 

Here $K(M)$ is the standard triangle, the left eigenvector is $(1, 1)$ and $\beta = 1 + x + y$. The face $F$ spanned by $(1, 0)$ and $(0, 1)$ is exposed by the vector $(1, 1)$ (a coincidence that it is the left eigenvector too), and $M_F$ is simply $\text{diag}(x, x + y)$. If we take the path $X(t) = (B_1 t, B_2 t)$ (any $B_i > 0$), we see that the first block is linked to the second, but not conversely. The limiting value of $\beta$ is of course $B_1 + B_2$, but the asymptotic spectral radius of the top block is $B_1$, strictly less than this—this is what permits the linking to take place. On the other hand, the second block has "full" value, and is not linked to anything.

Certain types of shift equivalence preserve linking with an obvious proof; specifically, the one outlined above, conjugating with a diagonal matrix whose entries are monomials—this type of conjugation obviously preserves $d(j, i) + m(j) - m(i)$ and $s$, so links are preserved and no new ones introduced.
Another simple example is FOG (section 2),

$$M = \begin{bmatrix} x & 0 & 1 \\ 1 & y & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$ 

Here $K(M)$ is the standard triangle, and we set $F$ to be the diagonal edge of the triangle. Then the exposing vector $v$ is $(1, 1)$, and so the relevant paths are $X(t) := X_{B,v}(t) = (B_1t, B_2t)$. As $\beta \geq \max \{1, x, y\}$ on $(\mathbb{R}^2)^{++}$, it follows that $\beta \geq k(1+x+y)$ for some $k > 0$, along a path of this type, $\beta(X(t)) \geq k(B_1 + B_2)t$, but $M_F = \text{diag}(x, y, 0)$, so its spectral radius at $B$ is $\beta(M) = \max \{B_1, B_2\}$; for some choices of $B$ is the limiting value obtained as the spectral radius. Note that there are three blocks in $M_F$ (labelled 1, 2, 3), and the third is the null block.

To see which pairs of blocks are linked, we have to do some actual matrix computations. The first (easy) observation is that the row $[(\beta - 1), (\beta - 1)(\beta - x), 1]$ is a left eigenvector of $M$ for the eigenvalue $\beta$. The space of paths of interest are parameterized by $X_b(t) = (t, bt)$ (where $b = B_2/B_1$). We have to determine the rate of growth of the entries of the eigenvector along these paths. The third entry is constant, so there is no problem there. Next, we observe that $\beta - 1 \geq k(x + y)$ for some $k > 0$ (on $(\mathbb{R}^2)^{++}$), and on $(\mathbb{R}^2)^{++}$, $\beta \geq \beta(0, 0) = r$, where $r$ is the real root of $r^3 - r^2 - 1 = 0$. As $r > 1$, it follows that $\beta - 1 \geq r - 1$, and so $\beta - 1$ is bounded below by a multiple of $1 + x + y$; since $\beta$ is bounded above by a multiple of the same thing, we have that $\beta - 1/(1 + x + y)$ is bounded above and below away from zero on $(\mathbb{R}^d)^{++}$. Hence along $X_b, (\beta - 1) \circ (X_b(t)) = c(b)t + O t^{1-\delta}$, for some value $c(b) > 0$.

Now something odd happens when we try to do the same process with $\beta - x$. The characteristic polynomial of $M$ translates to $(\beta - x)(\beta - (\beta - y))(\beta - 1) = 1$. Let $\lambda \equiv \lambda(t)$ be $(\beta - x)(X_b)$. Then we have $\lambda(\lambda - (b - 1)t)(\lambda - (1 - t)) = 1$. If we write $\lambda(t) = kt^r + O t^{<r}$ (and possibly $r$ depending on $b$, with $k > 0$) which of course we may, since $\lambda$ is a nonnegative real analytic algebraic function of $t$, then we see there are three cases to consider.

If $b > 1$, the middle term $(\lambda - (b - 1)t)$ is negative and the other two positive if $r < 1$, while if $r > 1$, the growth of the first and last terms is at least $t^{2r} > t$, while the middle term is at least $t$, so the product cannot be bounded (1). Hence $r = 1$, and this last argument also shows that $\lambda - (b - 1)t$ is of order $1/t^2$.

If $b \leq 1$, the middle term and the first term are positive, and so the third term must be positive; hence $r \leq t$. The first term is of order $t^r$, and the third is of order $t$. If $b = 1$, the second term is of order $t^r$, which forces $2r + 1 = 0$, i.e., $r = -1/2$. If $b < 1$, the second term is of order $t$, which forces $r + 2 = 0$, i.e., $r = -2$.

The fact that $r$ changes as $b$ varies indicates that $\beta - x$ does not have fractional polynomial behaviour, as is easy to check. In any event, the eigenvector has growth rates $(t, t^2, t^0)$ if $b > 1$, $(t, t^{1/2}, t^0)$ if $b = 1$, and $(t, t^{-1}, t^0)$ if $b < 1$.

Now we can check which blocks are linked. Notice that $\lambda$ grows as $t$. Regardless of the value of $b$, block 3 is linked to block 1. If $b > 1$, then block 1 is linked to block 2. If $b < 1$, then (weirdly) block 2 is linked to block 3, the null block. There are no other links.

Lemma 8.3 Every null block is linked to some other block.

Proof. If $\{i\}$ is a null block of $M_F$, the equation, $\sum_j c_jM_{ji} = \beta c_i$ entails $\max_j \{m(j) + d(j, i)\} = s + m(i)$. As $d(i, i) < s$ (else $\{i\}$ would not be a null block), there exists $j$ unequal to $i$ such that $m(j) + d(j, i) = s + m(i)$; thus $\{i\}$ is linked to the block containing $j$.

Lemma 8.4 If $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_k$ is a chain of linkages, then the blocks $\alpha_1, \alpha_2, \ldots, \alpha_k$ are distinct.

Proof. Otherwise, we may suppose that $\alpha_1 = \alpha_k$, $k > 1$, and no smaller value of $k$ is possible. We have pairs $(i_1, j_2), \ldots, (i_{k-1}, j_1)$ where $i_l$ and $j_l$ belong to $\alpha_l$, and each $(j_{l+1}, i_l)$ is normal. If $\alpha_l$ is a singleton
(this means that the block size is one, e.g., if \( \alpha_1 \) is a null block), then of course \( j_1 = i_1 \). Otherwise, there exists a path \( j_1 = j_{l,1} \rightarrow j_{l,2} \rightarrow \cdots \rightarrow j_{l,l(l)} = i_l \) where each ordered pair \((j_{l,m}, j_{l,m+1})\) is normal and all states appearing in the path belong to \( \alpha_l \). Thus for each \( l \), either \( j_l = i_l \) or there is a path of this type from \( j_l \) to \( i_l \). Hence we obtain a loop \( i_1 \rightarrow j_2 \rightarrow \cdots \rightarrow j_1 \rightarrow \cdots \rightarrow i_1 \) (the last step may simply be \( \rightarrow j_1 = i_1 \) if \( \alpha_1 \) is a singleton), where each transition is a normal pair. As we have seen in the previous argument, this implies that the average degree of the loop is \( s \), so that all the states appearing in the loop belong to the same block, a contradiction.

Proposition 8.5 The transitive relation on blocks generated by linkages is antisymmetric. The maximal elements are not null, and \( \rho(M_{F,\alpha}) = f(B) \) if and only if \( \alpha \) is a maximal element with respect to this relation.

Proof. The previous result yields antisymmetry, and the penultimate result yields that null blocks cannot be maximal.

From the relation \( \sum c_j M_{ji} = \beta c_1 \), we have that for large \( t \),

\[
    f(B)t^{s+m(i)} \left( 1 + o \left( t^{s+m(i)} \right) \right) = \sum' c_j (X_{B,v}) M_{ji} (X_{B,v})
\]

where the prime denotes the sum only over those \( j \) such that \((j, i)\) is normal. If \( i \) belongs to a maximal block, then \((j, i)\) being normal entails that \( j \) also belongs to the same block, \( \alpha \), and thus \( (M_F)_{ji} \neq 0 \). It follows from dividing by \( t^{s+m(i)} \) that \( \sum_{j \in \alpha} h_j(B) (M_F)_{ji} = f(B) h_i(B) \). Hence \( H_{\alpha} := (h_j(B)) \) is a strictly positive left eigenvector for \( f(B) \) of \( (M_{F,\alpha}) \). A nonnegative matrix with a strictly positive left eigenvector must have that eigenvalue as its spectral radius (by applying the Perron Frobenius theorem to the right eigenvectors). Hence \( f(B) \) is the spectral radius of \( (M_{F,\alpha}) \).

Conversely, if \( i \) belongs to an irreducible block, we apply the following, Lemma 8.6, to the matrix obtained by replacing all nonnormal entries by zero (and the others by their asymptotic limits); it is useful to perform the conjugation with the \( \Delta \) matrix as described earlier in this section.

Lemma 8.6 Let \( A \) be an square matrix with nonnegative real entries which is irreducible. Let \( Z \) be a nonnegative real matrix which partitions as

\[
    Z = \begin{bmatrix} A' & R \\ S & T \end{bmatrix}.
\]

If \( A \leq A' \) and \( Z \) admits a strictly positive left eigenvector with eigenvalue equaling that of the spectral radius of \( A \), then \( S \) consists entirely of zeroes.

Proof. Let \( r \) be the spectral radius. By conjugating \( Z \) with the diagonal matrix arising from the eigenvector (and simultaneously conjugating \( A \) with the corresponding block of the diagonal matrix), we may assume the left eigenvector for \( Z \) is just \((1 \ 1 \ \ldots \ 1 \ 1)\). Hence all the column sums of \( A' \) are bounded above by \( r \), and at least one of them is strictly less than \( r \) if \( S \) is not the zero matrix. Obviously, the same applies to the columns sums of \( A \). An irreducible matrix with some columns less than \( r \) and none exceeding it must have spectral radius strictly less than \( r \) (for primitive matrices, this follows from the strict monotonicity of the spectral radius function; for irreducible matrices, apply the primitive result to \((1 + A)/2\), which is primitive). Thus \( S \) zero, so that \( A = A' \).

In the FOG example, we saw that linkages between the various blocks (arising from a fixed face) depended on the choice of the strictly positive parameter, \( B \). This phenomenon is a consequence of the fact that the rates of growth—i.e., the exponent of \( t \)—of the entries of a strictly positive eigenvector along the path \( X_{B,v} \) depend (for \( v \) fixed) on the choice of \( B \). One situation in which this phenomenon cannot occur is when the matrix \( M \) satisfies the growth condition on the entries of a left eigenvector of section 10. Say \( M \)
satisfies (%) if it admits a left eigenvector whose entries satisfy fractional rational algebraic behaviour. In fact, a formally weaker condition is sufficient (it turns out of course that the “formally weaker” condition is equivalent to (%)).

Proposition 8.7 Let $M$ be a primitive matrix in $M_n A^+$, and suppose that $M$ admits a left positive eigenvector, $U = (U_i)$ for the analytic eigenfunction $\beta : (R^d)^{++} \to R^+$, with the property that for each $i$, there exist elements of $A^+_Q$, $p_i$ and $q_i$ such that $U_i q_i / p_i : (R^d)^{++} \to R^+$ is bounded above and below away from zero. Then for every face $F$ (and corresponding choice of exposing vector $v$), the linkages between the blocks of $M_F$ are independent of the choice of parameter $B$.

**Proof.** We observe that for an element of $A^+_Q$, $p$, $\lim_{t \to \infty} \log p(X_B,v(t))/\log t$ is exactly $\max v \cdot K(p)$, where $K(p)$ denotes the convex hull of $\log p$, i.e., the Newton polyhedron of $p$. It follows that for each $i$,

$$\lim_{t \to \infty} \log U_i(X_B,v(t))/\log t = (\max v \cdot K(p_i)) - (\max v \cdot K(q_i)).$$

This is obviously independent of the choice of $B$. Since the partition of the states into blocks depends only on $F$ (and not the path, $X_B,v$), and the presence or absence of linkages between blocks depends only on the rates of growth of the entries in $M$ along the paths (i.e., $d(i,j)$), which of course is $\lim_{t \to \infty} \log M_{ij}(X_B,v(t))/\log t$ and the entries of the eigenvector, and all of these are independent of the choice of $B$, it follows that linkages themselves are so independent.

In particular, this yields a fairly simple criterion for (%) to fail; namely, if there is a block which is maximal for one value of the parameter $B$ and not maximal for another (it being understood that the exposing vector $v$ is fixed as usual). This provides yet another proof that the FOG example fails to satisfy (%), using the diagonal face of the standard triangle. In more than one variable, failure of (%) is generic; on the other hand, in one variable, (%) holds.

The condition on the entries given in the preceding, that there is a choice of eigenvector so that $U_i$ has fractional polynomial growth (without assuming analyticity or algebraicity) is formally weaker than (%). However, if it holds, then consider the new eigenvector $(U_i / U_1)$; the first entry is the constant 1 as a function on $(R^d)^{++}$; if $U' := (U_i')$ is an algebraic and analytic left strictly positive eigenvector for $\beta$ (one exists!), then by uniqueness of the Perron eigenvector for each point of $(R^d)^{++}$, we must have $U / U_1 = U' / U_1$; hence the former is analytic and algebraic as well. Obviously, if the entries of $U$ satisfy fractional algebraic growth, then so do the entries of $U / U_1$. Thus the apparently weaker formulation is equivalent to (%).

The condition (%) is close to implying the other condition needed in section 10, namely (#) (which asserts that the large eigenvalue is uniformly bigger than the second largest eigenvalue over all points of $(R^d)^{++}$). Condition (%) asserts that for every face, there is a block whose spectral radius is the spectral radius of the facial matrix, independently of the choice of $B$, although some other blocks may hit the spectral radius, and if they do so at one point, they do so uniformly as well; on the other hand (#) is equivalent to there being a unique (irreducible) block of maximal spectral radius, and it must be aperiodic (that is, primitive).

Obviously, (%) is an invariant of finite equivalence. It would be interesting to find a matrix $M$ which satisfies (%), but $M \cdot \text{Tr}$ does not. (Such an example would have to be at least $3 \times 3$, because of a simple argument connecting the eigenvector of a size 2 matrix with that of its transpose.)

Now to deal with intertwiners. Suppose that $MY = YM'$, where all the relevant matrices have entries from $A^+$. We will show that $Y$ induces nontrivial intertwining between at least one pair of blocks in $M_F$ and $M'_F$ respectively (with maximal spectral radius for some value of the parameter), and as much as possible, the intertwining preserves linkages. Of course, $\beta = \beta'$ entails that $K(M) = K(M')$, so that we may use the same vector $v$ to expose the same face $F$ of these polytopes. Fix the parameter $B$, and the corresponding path $X_{B,v}$.

We now deal with the paths of matrices, $Y(X_B,v), M(X_B,v), \text{and } M'(X_B,v)$; we abbreviate the first one $Z$ (or $Z(t)$ when convenient), and we abbreviate $X_{B,v}$ to $X$ if no confusion results. A left eigenvector
(which we take to be strictly positive, algebraic, and analytic) $C$ of $M$ yields an eigenvector $C' := CY$ of $M'$. If we set $\Delta = \text{diag} (\ldots, h_i(B)t^m(i), \ldots)$, then on setting $N = \Delta M(X)\Delta^{-1}$, we have that $C(X)\Delta^{-1} = (1,1,\ldots,1)(1 + o(1))$ (for large values of $t$) is a left eigenvector for $N$, and similarly we may conjugate $M'(X)$ with a diagonal matrix with monomial entries to obtain $N'$, still intertwinnable with $N$ whose eigenvector has the same asymptotic behaviour in each coordinate. The intertwiner induced by $Z$ will be called $Z'$, so that $NZ = Z'N'$.

Now $C(X)\Delta^{-1}Z'$ is of the form $t^{a/b}(1 + o(1))(1,\ldots,1)$ for some integers $a$ and $b$, as $Z'$ has entries in $A^+_Q$. Dividing $Z'$ by the monomial $t^{a/b}$, we may assume in what follows that $a = 1 = b$.

With our current eigenvectors, the $(i,j)$ entry of $N$ (or of $N'$) is normal if and only if $d(i,j) = s$, since $m(i) = m(j) = 0$. The column sums of $Z'$ are all $1 + o(1)$; hence if $d(i,j) < s$, the contribution of $N_{ij}$ in the product $NZ'$ is $o(t^s)$. Thus $(NZ')_{kl} = \sum'_{m} N_{km}Z'_m kl + o(t^s)$, where $\sum'$ is the sum, as $m$ varies, of the normal entries $N_{km}$. Similarly, $(Z'N')_{kl} = \sum'_{m} Z'_m N'_{ml} + o(t^s)$.

Divide each of $N$ and $N'$ by $t^s$, so that $\beta(X_{B,v}) = f(B) + o(1)$. Now the equation $NZ' = Z'N'$ consists of matrices with bounded entries as $t \to \infty$, and the limits exist. The limiting matrices satisfy $N_{\infty}Z'_{\infty} = Z'_{\infty}N_{\infty}$, the column sums of $Z'_{\infty}$ are all $1$, and $\rho(N_{\infty}) = \rho(N'_{\infty}) = f(B) \neq 0$. We also observe that $N_{\infty}$ agrees with $M_F$ in the entries appearing in the blocks of the latter, and moreover, there are no new cycles appearing in $F$; hence the characteristic polynomials of $N_{\infty}$ and $M_F$ are equal.

Lemma 8.8 Let $T$ and $T'$ be square nonnegative real matrices, and let $S$ be a nonnegative real matrix such that $TS = ST'$, and suppose that $T$ and $T'$ admit strictly positive left eigenvectors $V$ and $V'$ respectively such that $VS = V'$. Then there exist irreducible blocks, $T_\alpha$ and $T'_\alpha$ of $T$ and $T'$ respectively such that $\rho(T_\alpha) = \rho(T')$ and the restriction of $S$, $S_\alpha$, to the corresponding subspace is nonzero and intertwines $T_\alpha$ and $T'_\alpha$.

Proof. If $U$ is an $T$-invariant cone (with $T$ acting from the right), then $U' := US$ is an invariant cone for $T'$. All the blocks above and below a block with full spectral radius must be zero (this applies to both $T$ and $T'$, and follows from the fact that the left eigenvector is strictly positive). Hence we obtain an obvious invariant cone for $T$, that spanned by $\{e_i \mid i \in \alpha, \rho(T_\alpha) = \rho\}$; call this cone $U_\rho$. We similarly define $U'_\rho$, its counterpart for $T'$.

It is a triviality to check that $U_\rho S \subseteq U_\rho$. Let $T_\rho$ and $S_\rho$ denote the restrictions of $T$ and $S$ to the vector space spanned by $U_\rho$, and $T'_\rho$ will denote the restriction of $T'$ to $U'_\rho$; then each of the three matrices is nonnegative and they satisfy $T_\rho S_\rho = S_\rho T'_\rho$. Of course $T_\rho$ and $T'_\rho$ are just matrix direct sums of the blocks of $T$ and $T'$ respectively that have full spectral radius. Now $T_\rho S_\rho$ is not zero. To see this, let $V_\rho$ be the corresponding truncation of the eigenvector to the states appearing in the blocks of spectral radius $\rho$. Since all blocks in $T$ above and below a block of maximal spectral radius must be zero, it follows immediately that $V_\rho$ is an eigenvector for $T_\rho$, with eigenvalue $\rho$, and it is obviously strictly positive. It follows from the original intertwining relation that $V_\rho S_\rho = V'_\rho$, whence $S_\rho$ is not zero.

We have thus reduced the original problem to the case wherein both matrices are direct sums of irreducible matrices all with the same spectral radius, and it is easy to verify that in this case the conclusion holds.

We note that conjugation with the diagonal matrices in the earlier constructions amounts to strong shift equivalence (over the Laurent polynomial ring in fractional powers of $t$), since the entries are monomial.

A few additional tidbits follow from the properties of intertwiners. If for some value of the parameter $B$ (with $v$ held fixed), an intertwiner induces a nontrivial intertwining involving the blocks $\alpha_1,\ldots,\alpha_k$ and $\alpha'_1,\ldots,\alpha'_k$ respectively, then it induces a non-trivial intertwiner involving the same pairs of blocks for all values of $B \gg 0$. This follows from the continuity in $B$ of the asymptotic values of polynomials—that is, if $p$ is a Laurent polynomial with positive coefficients, and $p(X_{B,v}(t)) = P(B)t^s + o(t^s)$, then $B \mapsto P(B)$ is continuous, in fact, it is real analytic.

In general, if $q : (R^d)^{++} \to R$ is algebraic, real analytic, and bounded above and below, and we define
a function $Q$ via $B \mapsto Q(B)$ where $q(X_{B,v}(t)) = Q(B) + o(1)$ $(Q(B)$ exists and is strictly greater than zero, as $q(X(t))$ is bounded real analytic and algebraic), then it is not true that $Q$ is real analytic. To see this, simply take the function $\beta$ of the FOG example, and set $q = \beta/(1 + x + y)$.

We have shown that for $no N$ with entries in $A^+$ centralizing $M$ is it true that $NC$ belongs to $(A^3)^+$. The row $V$ of course is the Perron eigenvector for the eigenvalue $\beta$.

We first note that $V \cdot C = (\beta - 1)(\beta - y)$, which equals $1/(\beta - x)$ (this uses the fact that $(\beta - x)(\beta - y)(\beta - 1) = 1$). As before in this section, let $X_b(t) = (t, bt)$. For a positive real analytic and algebraic function $f$ and parameter $b$, define

$$
\psi_f(b) = \lim_{t \to \infty} \frac{\log f(X_b(t))}{\log t}
$$

(if the limit exists). If $f = \beta - x$, we have already seen that $\psi_f(b)$ is 2 if $b > 1$ and $-1$ if $b < 1$. Hence for $g = 1/(\beta - x)$, $\psi_g(2) - \psi_g(1/2) = -3$. Now if $N$ centralizes $M$, then $V \cdot NC = \beta_N V \cdot C = \beta_N/(\beta - x)$; as we are also assuming $N$ has entries in $A^+$, $\beta_N$ has fractional polynomial growth, so that $\psi_{\beta_N}(b)$ is constant in $b$. Thus

$$
\psi_{\beta_N g}(2) - \psi_{\beta_N g}(1/2) = \psi_g(2) - \psi_g(1/2)
$$

$$
= -3.
$$

In addition, we have that $NC$ has entries in $A^+$. Let $U = (p_1, p_2, p_3)^T$ be an arbitrary column with entries in $A^+$. Then $V \cdot U = (\beta - 1)p_1 + (\beta - 1)(\beta - x)p_2 + p_3 := h$. Since $p_i$ are positive as functions on $(\mathbb{R}^2)^+$, we have that $\psi_h(b) = \max \{ \psi_{\beta - 1}(b) + \psi_{p_1}(b), \psi_{\beta - 1}(b) + \psi_{\beta - x}(b) + \psi_{p_2}(b), \psi_{p_3}(b) \}$. We had already seen that $\psi_{\beta - 1}(b)$ is the constant 1, and $\psi_{p_3}(b) = \max(1, 1) \cdot \log p$ for $p$ in $A^+$. It easily follows that $\psi_h(2) - \psi_h(1/2) \geq 0$ (the point is, that the only term within the set of three that is not constant in $b$ is $\psi_{\beta - x}(b)$; however, this is non-decreasing in $b$). With $U = NC$, we have shown that this function applied to $h$ is $-3$, a contradiction. So no such $N$ exists. (Of course, $1/(\beta - x)$ is strictly positive on $(\mathbb{R}^2)^+$, so there was no obvious reason why $N$ could not exist.)

**Companion matrices.** These are relatively simple to deal with, and except degenerately satisfy all the usual properties. Suppose $M$ is a companion matrix with entries from $A^+$ (that is, $M$ is in the form of companion matrix, not merely that $M$ is algebraically shift equivalent to one). So

$$
M = \begin{bmatrix}
0 & 0 & 0 & \ldots & p_n \\
1 & 0 & 0 & \ldots & p_{n-1} \\
0 & 1 & 0 & \ldots & p_{n-2} \\
\vdots & \ddots & \ldots & \vdots \\
0 & 0 & \ldots & 1 & p_1
\end{bmatrix}
$$

where $p_i$ belong to $A^+$. Let $\beta$ denote the largest eigenvalue function, as usual. If $p_n = 0$, we may remove the first row and column, and obtain a smaller size companion matrix strongly shift equivalent to the original; so there is no harm in assuming $p_n \neq 0$. Necessary and sufficient for $M$ to be primitive is
that \( \gcd \{ i \mid p_i \neq 0 \} = 1 \). The left eigenvector function for \( \beta \) is \((1, \beta, \beta^2, \ldots, \beta^{n-1})\), and in particular, \( M \) satisfies (\%). Note that \( K(M) \) is the convex hull of \( \bigcup \frac{1}{i} \log p_i \) (with the convention that \( \log 0 \) is the empty set), as is easy to check by calculating the graph of \( M \). The right eigenvector is almost as easy to calculate. We first notice that the characteristic polynomial of \( M \) is of course \( \chi(\lambda) := \lambda^n - \sum_{i=1}^{n} p_{n-i} \lambda^{i-1} \); define \( \chi_j(\lambda) = (\lambda^n - \sum_{i=1}^{n-j} p_{n-i} \lambda^{i-1}) / \lambda^{n-j} \) (that is, truncate the characteristic polynomial, and divide by the appropriate power of \( \lambda \) so that the degree is \( j \)). In particular, \( \chi_1(\lambda) = \beta - p_1 \) and \( \chi_2(\lambda) = \beta(\beta - p_1) - p_2 \) (the \( \chi_j \) can be defined recursively as well, via \( \chi_{j+1} = \lambda \chi_j - p_{j+1} \), as is well known). Then the right eigenvector function for \( \beta \) is \((1, \chi_1(\beta), \chi_2(\beta), \ldots, \chi_{n-1}(\beta)) \) Tr.

To verify that \( M \) Tr satisfies (\%), we observe that \( \chi_j(\beta) = \beta^{j-n} (\sum_{i=1}^{n-j} p_{n-i} \beta^{i-1}) \). Since \( \beta \) is bounded above by \( n \) below by a polynomial (possibly with fractional exponents), so is each \( \chi_j(\beta) \). (It is conceivable that in general, \( M \) satisfying (\%) implies \( M \) Tr does as well.)

The graph of a companion matrix consists of loops with a common vertex (we assume that not all the \( p_i \) are zero). It follows that there is exactly one nonnull block at every face; the block will be primitive (i.e., aperiodic, since nonnull blocks are automatically irreducible) provided the gcd of the surviving periods of cycles is one. Sufficient for this is that the trace, \( p_1 \), behave asymptotically like \( \beta \).

Now suppose that \( w \) is in \( \log p_j \). Then we claim that \( \lambda^w M^{-j} \) belongs to the bounded subring (in other words, \( (w, j) \) belongs to \( T(M) \)). It suffices to show that \( \lambda^w M^{-j} \prec M^n \) — but this is clear from the graph of \( M \) — any path of length \( n - j \) beginning at a state not in a cycle corresponding to \( p_j \) must simply run along the arrows of weight 1 until it hits a state on the cycle, where upon the latter can be folded in to create a path of length \( n \).

An even stronger property is satisfied by \( M \). Let \( P = \text{tr} M^{nl} \); then \( PM^{nl} \prec M^{2nl} \) (viz., section 7).

The minimal loops all pass through the same distinguished state (in this case, state \( n \)) and are obtained from the various \( p_i \), and \( P \) is obtained from all loops of length \( nl \). (This can also be obtained by modifying the criterion of Lemma 7.7 for this stronger property.) It is at least plausible that a primitive companion matrix with the block at each face being aperiodic is shift equivalent to a matrix satisfying (**).

9. Local and global characterization of (\#):

In this section, we express one of the many properties we have discussed, (\#), in terms of the behaviour of the facial matrices \( M_F \). Earlier, the property (\#) was defined in terms of the existence uniform gap in the spectrum; explicitly, if \( \beta_2(r) \) denotes the second largest absolute eigenvalue of the real matrix \( M(r) \) for a strictly positive \( d \)-tuple \( r \), and then (\#) is the property that

\[
\inf_{r \in (\mathbb{R}^d)^+} \frac{1}{\beta_2(r)} > 0
\]

(or simply that \( \beta_2/\beta \) is uniformly bounded away from 1 over \( (\mathbb{R}^d)^+ \)). In this formulation, it is expressed globally. However, it admits a relatively simple local formulation: (\#) holds if and only if for every face \( F \) of \( K(M) \), \( M_F \) admits just one irreducible block whose spectral radius is that of \( M_F \) and moreover, this block is primitive.

This suggests a weakening of (\#), defined in local terms only, namely \textit{weak} (\#) holds if for every face \( F \) of \( K(M) \), \( M_F \) contains only one block of maximal spectral radius. (The block need not be primitive.)

Weak (\#) also admits a global characterization, wherein instead of taking \( \beta_2(r) \) as the second largest absolute value of the eigenvalues of \( M(r) \), take it to be the eigenvalue closest to \( \beta(r) \) (that is, to minimize \( |\alpha - \beta(r)| \), where \( \alpha \) varies over the eigenvalues of \( M(r) \)).

We begin by examining the connections between \( M_F \) and \( M \). Set \( N = M^{nl} \), and \( Q = \text{tr} M^{nl} \). We have that \( N/Q \) lies in \( E_b(G(MQ)) \) by Theorem 5.3, although it need not be true that all of the entries of \( N/Q \) lie in \( R_Q \). However, it is easy to check that all of the symmetric functions of the matrix \( N/Q \) do lie in \( R_Q \). We have that \( K(N) = n! K(M) \), and let \( F \) be a face of \( K(M) \), so that \( F' := n!F \) is a face of \( K(N) \). For a square matrix \( A \) (with entries in some commutative ring) denote by \( f_A \) the characteristic polynomial of
A, det(zI − A), using the variable z. As the coefficients of $f_{N/Q}$ are the symmetric functions of $N/Q$, obviously $f_{N/Q}$ belongs to the polynomial ring $R_{Q}[z]$.

Proposition 9.1 Let $M$ be a primitive matrix with entries in $A_{Q}^{+}$. Then $M$ satisfies (#) if and only if for all faces $F$ of $K(M)$, $M_{F}$ has (uniformly) one block of maximal spectral radius, and it is primitive.

Proof. Let $\pi_{F} : R_{Q} \to R_{Q_{n_{1}F}}$ be the usual quotient map, and use the same symbol to denote the induced map $R_{Q}[z] \to R_{Q_{n_{1}F}}[z]$ on the polynomial rings. We claim that $\pi_{F}(f_{N/Q}) = f_{N_{n_{1}F}/Q_{n_{1}F}}$. There is no assumption that all the entries of $N_{n_{1}F}/Q_{n_{1}F}$ lie in $R_{Q_{n_{1}F}}$, merely that the symmetric functions of the matrix do; this and the claim are immediate consequences of the definitions. If $'$ denotes differentiation with respect to the variable $z$, then it follows obviously that $(\pi_{F}(f_{N/Q}))' = f'_{N_{n_{1}F}/Q_{n_{1}F}}$.

Now suppose that (#) fails. Then there is a sequence of points in $(R^{d})^{++}, \{r_{i}\}$ such that $(\beta(r_{i}) - \beta_{2}(r_{i}))/\beta(r_{i}) \to 0$; thus $(\beta^{n_{1}}(r_{i}) - \beta_{2}^{n_{1}}(r_{i}))/\beta^{n_{1}}(r_{i}) \to 0$. By taking a subsequence of $\ln r_{i}$ (as in [H5, p. 125–126]), we may find a path of the form $X(t) = \exp(tv + w)$ (where $v$ and $w$ are in $R_{d}^{d}$) such that $\lim_{t \to -\infty}(\beta^{n_{1}}(X(t)) - \beta_{2}^{n_{1}}(X(t))) = 0$. Let $F$ be the face of $K(M)$ exposed by $v$ (so that the maximum, $\nu \cdot K(M)$, is attained precisely at $F$); then $n_{1}F$ is exposed by $v$ in $K(N)$.

Now we wish to show that $H(t) := (f'_{N/Q}/Q(X(t)) = 0$ as $t \to \infty$. If $g$ is a monic polynomial of degree $n$ with roots $\{b, b_{2}, \ldots, b_{n}\}$ with $|b| \geq |b_{1}|$, then $g'(b) = \prod_{i \geq 2}(b - b_{i})$, and of course $|g'(b)| \leq |b|^{n-2}\inf_{i \geq 2}|b - b_{i}|$. Since $\beta^{n_{1}}/Q$ is bounded on $(R^{d})^{++}$, we infer that $|H(t)| < K_{1}|\beta^{n_{1}}(X(t)) - \beta_{2}^{n_{1}}(X(t))|$. Since $(\beta^{n_{1}}/Q)^{n_{1}}$ are both bounded on $(R^{d})^{++}$, we deduce $|H(t)| \leq K|\beta^{n_{1}}(X(t)) - \beta_{2}^{n_{1}}(X(t))|$, and this goes to zero.

Now consider each of the coefficients (in $z$) of $f'_{N/Q}$ along $X(t)$; they are elements of $R_{Q}$, there is a pure trace $\alpha$ such that for all $c$ in $R_{Q}$, $\lim c(X(t)) = \alpha(c)$ and $\alpha$ induces a faithful pure trace on $R_{Q_{n_{1}F}}$. The pure faithful traces on the latter are given by point evaluations, so we can find a point $s$ in the corresponding positive orthant such that $\alpha$ is obtained by evaluating the elements of $R_{Q_{n_{1}F}}$ at this point. Automatically, $N_{n_{1}F}(s)$ is a multiple eigenvalue (since we know already that $\beta_{n_{1}}$ is an eigenvalue of $N_{n_{1}F}$). In particular, $N_{n_{1}F}$ cannot have just a single irreducible block with maximal spectral radius, hence the same is true of $M_{F}$.

Conversely, suppose that $M_{F}$ has two blocks (at some real point of the underlying positive orthant) with maximal spectral radius, and that (#) holds for $M$. Then the same is true for $N_{n_{1}F}$ and $N = M^{n_{1}}$ respectively. Recalling our $g$ from several paragraphs above, we see that $|g'(b)| = \prod_{i \geq 2}|b - b_{i}| \geq (\inf \{|b - b_{i}|\})^{n_{1}-1}$. From (#), we have that for $b = \beta^{n_{1}}(r)/Q(r)$, $|f'_{N/Q}(b)|$ is uniformly bounded away from zero on $(R^{d})^{++}$. The same thing then applies to the derivative of the characteristic polynomials of the matrices corresponding to the quotients, i.e., $N_{n_{1}F}/Q_{n_{1}F}$, so the latter cannot have a multiple root for the large eigenvalue at any positive real evaluation.

The first and second matrices in Example 7.8 do not satisfy (#), but the third one does.

10. Finite equivalence For matrices satisfying (%) & (#)

Here we show that under reasonable conditions, specifically, (%,) & (#) together, the PT question has an affirmative answer.

The $n \times n$ matrix $M$ with entries from $R[x_{i}^{\pm 1}]^{+}$ is said to satisfy (*) if there exists an integer $m$ such that for all vertices $v$ of $K = \text{cvx Log}(M^{m})$, the real matrix obtained by taking the coefficient of $x^{v}$ in $M_{ij}$ is not nilpotent. This is equivalent to the following. Set $P = \sum_{w \in K}Z^{w}x^{w}$; if $\beta : (R^{d})^{++} \to R^{+}$ denotes the large eigenvalue of $M$ as an analytic function, then there exist positive real numbers, $e$ and $d$ so that as a function on $(R^{d})^{++}$, $e < \beta/P < d$.

If $Y$ is a convex set, $\partial_{Y}$ denotes the set of extreme points of $Y$. The following is probably well known.
Lemma 10.1  Let $L$, $V$, and $V'$ be compact convex polyhedra in $\mathbb{R}^d$. If $L + V \subseteq L + V'$, then $V \subseteq V'$.

Proof. Set $W = \text{cvx} \, V \cup V'$. Then $\partial_e W \subseteq \partial_e V \cup \partial_e V'$. We will show that $L + W = L + V'$. Select $x$ in $\partial_e(L + W)$; this entails $x = t + w$ for $t$, $w$ vertices in $L$, $W$ respectively [H5, VIII.5, p. 68]. If $w$ belongs to $\partial_e V'$, then $x$ belongs to $L + V'$; otherwise, $w$ belongs to $\partial_e V$, so $x$ belongs to $L + V \subseteq L + V'$. Hence $L + W \subseteq L + V'$, and the other inequality is trivial. Now the Cancellation Lemma [H5, p. 71] yields $W = V'$, so that $V \subseteq W = V'$.

If $u$ exhibits fractional polynomial behaviour, then we define $\text{cvx} \log u$ to be $\text{cvx} \log J$; this is well defined (by Lemma 3.3 applied to the quotients). There is no obvious way to define $\log u$.

Lemma 10.2 Let $u_j \,(j = 1, 2, \ldots)$ and $u$ exhibit fractional polynomial behaviour, and let $m_j$ and $P$ belong to $A^+_Q$. If the inequality $\sum u_j m_j \leq Pu$ holds on $(\mathbb{R}^d)^+$, then for all $j$,

$$\text{cvx} \log u_j m_j \leq \text{cvx} \log Pu.$$ 

Proof. If not, there exists $w$ in $\partial_e \text{cvx} \{ \cup \text{cvx} \log u_j m_j \}$ that does not belong to $\text{cvx} \log Pu$. We may select $Q$ in $A^+_Q$ with the property that $\text{cvx} \log Q = \text{cvx} \{ (\cup \text{cvx} \log u_j m_j) \cup \text{cvx} \log Pu \}$. We see that the original inequality transforms into an inequality with respect to $R_Q$,

$$\sum u_j m_j \leq \frac{Pu}{Q}.$$ 

Evaluating at the pure state of $R_Q$ corresponding to the vertex $w$ of $\text{cvx} \log Q$, we deduce

$$\sum \gamma_w \left( \frac{u_j m_j}{Q} \right) \leq 0.$$ 

However, we know that $\gamma_w$ is a limit of point evaluation states, so that, as $u_j m_j$ are each strictly positive on $(\mathbb{R}^d)^+$, we deduce for each $j$, $\gamma_w((u_j m_j)/Q)$ must be nonnegative, hence by the equation above, must be zero. However, there exists $j$ such that $w$ belongs to $\text{cvx} \log u_j m_j$, and necessarily to $\partial_e \text{cvx} \log u_j m_j$ (for the same $j$), and so $w$ must belong to $\text{cvx} \log u_j m_j$. Hence for this $j$, $\gamma_w((u_j m_j)/Q)$ must be nonzero, a contradiction.

Proposition 10.3  Let $a$ belong to $A^+_Q$ and $y$ exhibit fractional polynomial behaviour. Then $\text{cvx} \log ay = \text{cvx} \log a + \text{cvx} \log y$.

Proof. Pick $P$ in $A^+_Q$ such that $y/P$ is bounded above and below on $(\mathbb{R}^d)^+$. Let $z''$ be any vertex of $\text{cvx} \log a + \text{cvx} \log y$. By [H5, VIII.5, p. 68], there will exist $z$ in $\partial_e \text{cvx} \log a$ and $z'$ in $\partial_e \text{cvx} \log y$ such that $z'' = z' + z$. There exists $u$ in $\mathbb{R}^d$ that simultaneously exposes $z$, $z'$, and $z''$ with respect to $\text{cvx} \log a$, $\text{cvx} \log y$, and $\text{cvx} \log a + \text{cvx} \log y$, respectively (see the proof of the Cancellation Lemma, [H5, p. 71]). Now define $X(t) = t^u = (t^{u(1)}, \ldots, t^{u(d)})$ and consider $\lim_{t \to \infty} (au/P)(X(t)) = \lim_{t \to \infty} (v/P)(X(t)) > 0$. Thus $\text{cvx} \log a + \text{cvx} \log y \subseteq \text{cvx} \log ay$, and the converse is trivial.

Suppose $M$ is a matrix in $\mathbb{R}[x^{\pm 1}]$ for which $M(1)$ is primitive. There exists a left eigenvector for $\beta$ whose entries are real analytic (that is, with no branch points) and algebraic (or even integral) over $\mathbb{R}[x^{\pm 1}]$, $v$. By the usual Perron-Frobenius theorem, each entry of $v$ is strictly positive (or simultaneously strictly negative) as a function on $(\mathbb{R}^d)^+$. Assume for now that $(\%)$ holds, i.e., we have chosen $v = (v_i)$ such that there exist $P_i$ in $A^+_Q$ such that $v_i/P_i$ are bounded above and below, and pick $P_0$ such that $\beta/P_0$ is also (we already know that such $P_0$ exist).

From $vM = \beta v$, we obtain the equations,

$$\sum_j M_{ij} v_j = \beta v_i \leq CP_0 P_i.$$
(as functions on \((\mathbb{R}^d)^{++}\), and \(C\) is some positive integer) for all \(i\). Fixing \(i\), and applying Proposition 10.3, we deduce

\[(\dagger) \quad \text{cvx Log } M_{ij}v_j \subseteq \text{cvx Log } P_0P_i = \text{cvx Log } P_0 + \text{cvx Log } P_i.
\]

Now set \(\Delta = \text{diag } (P_i)\), and replace \(M\) by \(\overline{M} := (\prod P_i)\Delta^{-1}M\Delta\). (This is the global version of the local trick discussed in section 8.) Its right eigenvector corresponding to the large eigenvalue \(\beta\) is \(\Delta^{-1}v = (v_i/P_i)^T\). Multiply this by \(\prod P_i\) to obtain another eigenvector, \(\overline{v} := (\overline{v}_i)^T = (v_i\prod_{k\neq i} P_k)^T\). The outcome of this is

\[\text{cvx Log } \overline{v}_i = \sum_{k=1}^n \text{cvx Log } P_k.
\]

Hence \(\text{cvx Log } \overline{v}_i\) are all equal to each other (for \(i = 1, 2, \ldots, n\)); we denote this compact polyhedron \(L\). Together with \((\dagger)\) applied to \(M\), we obtain

\[\text{cvx Log } \overline{M}_{ij} \subseteq (\text{cvx Log } P_0) + L.
\]

However, by Proposition 10.3,

\[\text{cvx Log } \overline{M}_{ij} + L \subseteq (\text{cvx Log } P_0) + L.
\]

By Lemma 10.1,

\[\text{cvx Log } \overline{M}_{ij} \subseteq \text{cvx Log } P_0.
\]

This means that the matrix \(\overline{M}\) satisfies condition (*)

We have deduced the following:

**Proposition 10.4** Let \(M\) be an element of \(M_n\mathbb{R}[x_i^{\pm 1}]^{++}\) possessing a left eigenvector whose entries exhibit fractional polynomial behaviour. Then there exist \(P_i\) in \(A_Q^+\) and \(\Delta = \text{diag } (P_i)\) such that the matrix \(\overline{M} := (\prod P_i)\Delta^{-1}M\Delta\) belongs to \(M_nA_Q^+\) and satisfies the condition (*). Moreover, there exists a right eigenvector \(\overline{v} = (\overline{v}_i)^T\) for \(M\) such that for all \(w\) in \(\partial e\text{Log } PP_0\), \(\gamma_w(\overline{v}_i/P) \neq 0\), and \(\gamma_w(\overline{M}/P_0\overline{v})\) has no zero rows, where \(P\) is the product of the \(P_i\).

**Proof.** The only thing yet to be proved is that \(\gamma_w(\overline{M}/P_0)\) has no zero rows. However, we observe that \((\overline{M}/P_0)\overline{v} = (\beta/P_0)\overline{v}\); dividing by \(P\) and applying \(\gamma_w\), we deduce from \(\gamma_w(\overline{v}/P)\) being a strictly positive real vector and from \(\gamma_w(\beta/P_0)\) being greater than zero that the real column \(\gamma_w(\overline{M}/P_0)\gamma_w(\overline{v}/P)\) is strictly positive; this precludes \(\gamma_w(\overline{M}/P_0)\) having a row of zeroes.

A column (or row) \(v = (v_j)^T\) of real meromorphic functions algebraic (over \(\mathbb{R}[x_i^{\pm 1}]^{++}\) on \((\mathbb{R}^d)^{++}\) is said to be very positive if there exist \(P_i\) in \(A_Q^+\) and \(P\) in \(A_Q^+\), such that \(v_j/P_j\) are bounded as functions on \((\mathbb{R}^d)^{++}\), and for all \(L\) in \(\partial e\text{S}(RP)\), \(L(v_j/P_j) > 0\). In particular, this forces each \(v_j\) to exhibit fractional polynomial behaviour (as the set of pure states of \(RP\) is compact and contains the point evaluations). From the definitions, the presence of a left eigenvector that is very positive is equivalent to (\%).

One case in which it is very easy to prove that a matrix has a very positive eigenvector is if \(d = 1\), that is, if the matrix only involves one variable. To see this, we simply recall from [A, p. 290], that if \(p\) is a (real) analytic function (say on \((0, \infty]\)) that satisfies an equation with coefficients from \(\mathbb{R}[x_i^{\pm 1}]\), then its behaviour at 0 is of fractional power order, i.e., \(\lim_{t \to 0} p(t)/t^{a/b}\) exists and is not zero, and similarly, \(\lim_{t \to \infty} p(t)/t^{a/b}\) exists, is not zero, and in both cases, the denominators \(b\) and \(b'\) divide the degree of the polynomial satisfied by \(p\). Then if \(s = (s_i)\) is the right eigenvector, we simply define \(P_i\) to be the element of \(A_Q, \sum x^j, j\) varying over the set \(\frac{1}{n} \mathbb{Z} \cap \left[\frac{a}{b}, \frac{a'}{b'}\right]\).
Let \( \beta_2(r) \) denote the second largest absolute value of eigenvalues of \( M(r) \), for \( r \in (\mathbb{R}^d)^++ \); that is, \( \beta_2(r) = \max \{|z| \mid z \text{ is a complex eigenvalue of } M(r), z \neq \beta(r)\} \). Recall that \( M \) satisfies (#) if there exists a positive real number \( \delta \) such that \( 1 - \frac{\beta_2}{\beta} \geq \delta \) on \((\mathbb{R}^d)^++\).

Suppose that \( M \) admits a left eigenvector \( \mathbf{v} = (v_i)^T \) for \( \beta \) all of whose entries exhibit fractional polynomial growth. The we may form \( \mathbf{M} := (\prod P_i) \Delta M \Delta^{-1} \) with \( \Delta = \text{diag}(P_i) \) for \( P_i \in A^+_Q \) with \( v_i/P_i \) bounded above and below. Then \( \mathbf{M} \) has entries in the ring \( A^+_Q \), and moreover its left eigenvector for the large eigenvalue \( \beta \prod P_i \) is \( \mathbf{v} \Delta^{-1} \). Each entry of the latter is strictly positive, and bounded above and below away from zero—this is precisely the definition of a very positive (left) eigenvector, as above. Note that if \( M \) satisfies (#) so does \( \mathbf{M} \) (since the ratio \( \frac{\beta_2}{\beta} \) is unchanged). We also notice that for every point evaluation pure state, \( L \), \( L \left( \mathbf{M}/(\prod P_i) \right) \) has a has a gap from the spectral radius in its spectrum that is bounded below away from zero; it follows that if \( L \) is any pure trace of \( R_P \) that can be defined coordinatewise, the large eigenvalue of \( L \left( \mathbf{M}/(\prod P_i) \right) \) has multiplicity one and the corresponding block is primitive.

**Proposition 10.5** Condition (#) implies that for all \( \epsilon > 0 \), for all sufficiently large integers \( N \),

\[
1 + \epsilon \geq \frac{\text{tr} M^N(r)}{\beta^N(r)} \geq 1 - \epsilon \quad \text{for all } r \in (\mathbb{R}^d)^++.
\]

**Proof.** As \( \text{tr} M^N(r) \) is the sum of the \( N \)th powers of the eigenvalues of \( M(r) \), we deduce

\[
\beta^N(r) + (n-1)\beta_2^N(r) \geq \text{tr} M^N(r) \geq \beta^N(r) - (n-1)\beta_2^N(r).
\]

For all sufficiently large \( N \), \((1-\delta)^N < \epsilon/(n-1)\). Dividing by \( \beta^N(r) \), the desired inequality comes out. \( \blacksquare \)

This forces \( \text{tr} M^N/\beta^N \) to converge uniformly on \((\mathbb{R}^d)^++\) to the constant function 1. Note that \( \text{tr} M^N/\beta^N \) converges pointwise to 1 merely if \( M(1) \) is primitive (which we assume throughout anyway). Neither inequality in Proposition 10.5 need hold if we drop condition (#).

Starting with the matrix \( M \) and a left eigenvector \( \mathbf{v} \) whose entries exhibit fractional polynomial behaviour, apply the diagonalization trick. We obtain a new matrix, which for the purposes of the next result, we also call \( M \), with various properties. Specifically, \( M \) now satisfies (*) (since we allow polynomials with fractional exponents), and its new left eigenvector satisfies \( (v_i/P_i) \) is bounded above and below away from zero. Now consider the matrix \( \text{Adj}((\beta/P_i) \mathbf{I} - M/P_i) \). We observe that \( M/P \) has entries in \( R_P \), so that the entries of the adjacency matrix are integral over \( R_P \). Let \( \sigma_i \) be the \( i \)-th column of \( \text{Adj}((\beta/P_i) \mathbf{I} - M/P_i) \), and define \( \sigma \) to be the sum \( \sum_i (-1)^{e(i)} \sigma_i \), where \( e(i) = 0 \) if \( \sigma_i(1,1,\ldots,1) \) (\( \sigma_i \) evaluated at 1) exceeds zero, \( e(i) = 1 \) if \( \sigma_i(1,1,\ldots,1) \) (\( \sigma_i \) evaluated at 1) is less than zero, and any \( \sigma_i \) whose value at 1 is zero is discarded. Notice that the entries of \( \sigma \) are bounded on \((\mathbb{R}^d)^+\), so we may apply the states of \( R_P \) to them (see the lemma about limits of bounded algebraic functions).

As in the Appendix, if \( B \) is a nonnegative real matrix with only one eigenvalue of absolute value equaling its spectral radius, then that eigenvalue belongs to a unique block of \( B \); this will be called the principal block.

**Lemma 10.6** Let \( M \) be an element of \( M_n A^+_Q \) with the following properties:

(a) \( M \) satisfies (#);

(b) there exists \( P \) in \( A^+_Q \) such that \( \text{cvx Log } M_{ij} \subseteq \text{cvx Log } P \);

(c) there exists a left nonnegative eigenvector for \( \beta \), \( \mathbf{v} = (v_i)^T \) consisting of algebraic and real analytic functions, such that \( v_i/P \) is bounded on \((\mathbb{R}^d)^++\);

(d) for all vertices \( w \) of \( \text{cvx Log } P \), \( L_w(\sigma) \) (which equals \( \lim_{t \to \infty} \sigma(\exp u \ln t) \) for \( u \) any vector in \( \mathbb{R}^d \) that exposes \( \{w\} \) as a face of \( \text{cvx Log } P \)) is strictly positive.

Then for all pure states \( L \) of \( R_P \), \( L(\sigma) \) is strictly positive.
Proof. We have that \( L (\text{Adj} ((\beta / P) I - M / P)) = \text{Adj} (L(\beta / P) I - L(M / P)) \) for any pure trace \( L \) of \( R_P \). Let \( F \) denote the face of \( K = \text{cvx Log} P \) that contains \( \Lambda_P (L) \), and let \( P_F = \sum_{w \in F} \lambda_w x^w \) (where \( P = \sum \lambda_w x^w \); that is, disregard the terms in \( P \) whose exponents do not lie on \( F \)). Then following [H5, Section VIII], we may define \( R_{P_F} \), and there is a natural map \( \pi_F : R_P \to R_{P_F} \), which simply replaces \( k \)th powers of \( P \) in the denominator by the same powers of \( P_F \), and changes the numerators by discarding the exponents not in \( kF \) (as occurs in going from \( P \) to \( P_F \)). The map \( \pi_F \) extends in a natural way to functions algebraic over \( R_P \), via Lemma 3.1. Let \((\mathbf{R}^d)^{++}\) parameterize the pure faithful traces of \( R_{P_F} \) (that is, there is a change of variables so that elements of \( R_{P_F} \) are rational functions in \( d' \) variables, etc.). Now, \( L \) induces a pure state on \( R_{P_F}, \overline{T} \), which corresponds to a point evaluation (since \( \Lambda_P (L) \) is in the relative interior of \( F \) [H5, IV]) on this ring of rational functions (effectively in fewer variables than \( R_P \)).

Let \( M \) denote the \( n \times n \) matrix (with entries from \( R_{P_F} \)) \( \pi_F (M / P) \), and let \( w \) be an vertex of \( F \). We have that \( L(M) \) has a strictly positive eigenvector, namely the image of that obtained from the adjacency matrix as described in the paragraph preceding this lemma. As \( L_w \) is a limit of point evaluation states on \( R_{P_F} \), it follows that for at least at one point, \( r' \) (of \( \mathbf{R}^d \)), \( L_{r'} (\pi_F (\sigma)) \) is a strictly positive row. If we now multiply \( M \) by a suitable power of \( P_F \) so as to remove the denominators, we obtain a matrix, call it \( N \), with entries in \( \mathbf{R}[x_i^{\pm 1}]^+ \), with the property that for at least one point, \( r' \) of \( \mathbf{R}^d \), \( N(r') \) has a strictly positive eigenvector. Inasmuch as the entries of \( N \) are Laurent polynomials, the \( 0 - 1 \) “skeleton” matrix of \( N(r) \) obtained by replacing every nonzero entry of \( N(r) \) by \( 1 \), is independent of the choice of \( r \). A real matrix with nonnegative entries and having a largest eigenvalue of multiplicity just one has a strictly positive left eigenvector (see the Appendix) if and only if there exists an index \( i \) belonging to the irreducible block with the largest eigenvalue such that for all sufficiently large powers of the matrix, the \( i \)th row is strictly positive (if \( B \) is such a real matrix with large eigenvalue \( \rho \), consider the asymptotic behaviour of \( \{ (B/\rho)^m \} \) as \( m \to \infty \)). Although the skeleton matrix of \( N \) is independent of the choice of \( r \), the equivalence class of indices corresponding to the largest eigenvalue could change, as \( r \) varies.

We show in fact that if for all \( r, N(r) \) has its largest eigenvalue of multiplicity \( 1 \), then even the equivalence classes are independent of the choice of \( r \). Suppose for some index \( i \), there exist \( r^1 \) and \( r^2 \) so that \( i \) belongs to the block with the largest eigenvalue for \( N(r^1) \), but not for \( N(r^2) \). Let \( Y : [0, 1] \to (\mathbf{R}^d)^{++} \) be any (continuous) path with \( Y(0) = N(r^1) \) and \( Y(1) = r^2 \). We note that \( N(Y(t)) \) has its largest eigenvalue of multiplicity \( 1 \) for any value of \( t \). Define

\[
\begin{align*}
O &= \{ t \in [0, 1] \mid i \ \text{belongs to the principal block } N(Y(t)) \} \\
O' &= \{ t \in [0, 1] \mid i \ \text{belongs to a block of } N(Y(t)) \ \text{other than the principal one} \}.
\end{align*}
\]

Clearly, both \( O \) and \( O' \) are open, but because \( N(Y(t)) \) has its largest eigenvalue of multiplicity \( 1 \) for all choices of \( t \), we have that \( O \cup O' = [0, 1] \). This forces one of \( O \) or \( O' \) to be empty, a contradiction.

Thus it makes sense to talk of an index \( i \) that belongs to the irreducible block of \( N \) (i.e., of \( N(r) \) for all \( r \)) with largest eigenvalue. As \( N(r) \) has a strictly positive eigenvector, there exists an integer \( m_0 \) such that for all \( m \geq m_0 \), the \( i \)th column of \( N^m(r) \) is strictly positive. Since the skeleton of \( N \) is independent of the choice of \( r \), it follows that for all such \( m \), and for all \( r \), the \( i \)th column of \( N^m(r) \) is also strictly positive. As \( i \) belongs to the block with largest eigenvalue, \( N(r) \) has a strictly positive eigenvector, and moreover, this is unique (up to positive scalar multiple). Now \( L_{r'} (\pi_F (\sigma)) \) is a nonzero eigenvector for the large eigenvalue of \( M \) and thus of \( N \), so it is a positive scalar multiple of the unique one, hence it is strictly positive. However, the original \( L \) factors through \( L_r \) for some \( r \) in \( \mathbf{R}^d \), that is, \( L(\sigma) = L_r (\pi_F (\sigma)) \), and we are done.

Theorem 10.7 Let \( M \) and \( M' \) be square matrices with entries from \( \mathbf{R}[x_i^{\pm 1}]^+ \) satisfying the following properties:

(a) Both \( M \) and \( M' \) satisfy (#);
(b) \( \beta_M = \beta_{M'}; \)
(c) both \( M^T \) and \( M' \) satisfy (%).
Then there exists a nonzero rectangular matrix $X$ with entries from $\mathbb{R}[x_i^{\pm 1}]^+$, such that $XM = M'X$.

**Proof.** Since the process in going from $M$ to $\overline{M}$ described earlier is a very special finite equivalence together with multiplication by a polynomial with no negative coefficients, it is sufficient to work in the context of $\overline{M}$.

The matrix $\overline{M}$ satisfies (#), and also (*) by Proposition 10.4; for any vertex $w$ of $\text{cvx} \log P$, $L_w(M/P_0)$ is a positive matrix with no zero rows, having $L_w(b/P_0)$ as its large eigenvalue, and this has multiplicity one. If we take the canonical right eigenvector (for $M$) to initiate the process of getting from $M$ to $\overline{M}$, (i.e., using the right eigenvector instead of the left to construct $\Delta$), the corresponding left eigenvector (on the left, $\sigma$), has the property that for all $w$ in $\partial_e \text{cvx} \log P$, the row $L_w(\sigma)$ is strictly positive (the hypothesis of Lemma 10.6 applies to $M^T$; this strict positivity is the whole point of using $\Delta$) and for any $L$ in $\partial_e S(R_P, 1)$, $L(\sigma)$ is nonnegative and nonzero. By Lemma 10.1, this (right) eigenvector, obtained from $\text{Adj}(\beta I - M/P)$, is very positive. Now from the definition of adjoint, each entry of $\text{Adj}(\beta I - \overline{M}/P)$ can be written uniquely as a sum of the form $\sum_{i=0}^{e-1} n_i \beta_i$, where $e$ is the degree of the characteristic polynomial of $\beta_0 = \beta/P$ over the function field, $\mathbb{R}_i[x_i]$ (the field of fractions of $\mathbb{R}[x_i^{\pm 1}]$), and $n_i$ all belong to $R_P$.

Next, we do the same process with $M'$, and so obtain a right eigenvector $\tau$ for $\overline{M}'/P$ (note that the $P$ can be chosen to be the same for $M$ as for $M'$, since the corresponding beta functions are equal). Following the prescription in [PT], we form the matrix $X_1 = \tau \sigma$; if $M$ is $n \times n$ and $M'$ is $m \times m$, then $X_1$ is $m \times n$. Obviously $X_1$ intertwines $\overline{M}$ and $\overline{M}'$. From the remark in the previous paragraph, we may write (uniquely) $X_1 = \sum_{i=0}^{e-1} N_i(\beta/P)^i$, where the $N_i$ are $m \times n$ matrices with entries in the ring $R_P$ (which is contained in $\mathbb{R}_i[x_i]$). Now $\beta_0$ is integral of degree $e$ over $R_P$ and satisfies no algebraic equation of smaller degree, and it easily follows that $\{ \beta_i \}_{i=0}^{e-1}$ is a basis for the ring $R_P[\beta_0]$ over $R_P$. Thus each $N_i$ intertwines the matrices. We obtain our preliminary candidate intertwining matrix, $N = \sum_i N_i(\overline{M}/P)^i$. Certainly $N$ is a matrix with entries in $R_P$ that intertwines the relevant matrices. We shall show that there exists $k$ such that each entry of $N(\overline{M}/P)^k$ is positive in $R_P^+$.

Let $L$ be a pure trace of $R_P$. We have guaranteed that $L(M/P)$ has a strictly positive left eigenvector for the large eigenvalue $L(\beta_0)$, and no other eigenvalues have this as their absolute value. By Lemma A1.2, there exists a unique nonnegative right eigenvector, necessarily for the same eigenvalue, call it $w_L$.

Consider the matrix product

$$L(N)w_L = \sum L(N_i)L(\overline{M}/P)^i w_L$$

$$= \sum L(N_i)L(\beta/P)^i w_L$$

$$= L(\tau \sigma) w_L$$

$$= L(\tau)(L(\sigma) \cdot w_L).$$

Now $L(\sigma) \cdot w_L$ is the inner product of a strictly positive vector with a nonzero nonnegative one, so is a positive real number. As $L(\tau)$ is strictly positive, this means that for each row, $u$ of $N$, $L(u)w_L$ is strictly positive. As $w_L$ is the unique nonnegative right eigenvector of $L(M/P)$, there exists $K = K(L)$ such that $L(u)L(M/P)^K$ is strictly positive. Hence $L(N(M/P)^k)$ is strictly positive for some sufficiently large (and hence all larger) $K$. Hence there exists a neighbourhood, $W$, of $L$ in the pure trace space of $R_P$ such that for all $L_0$ in $W$, $L_0(N(M/P)^k)$ is also strictly positive. The pure trace space being compact, we obtain a finite open covering and a fixed integer $k$ such that for all $L$ in the pure trace space, $L(N(M/P)^k)$ is strictly positive. By e.g., [HM, I.1 & I.2], every entry of $N(M/P)^k$ is a positive element of $R_P$. Hence there
exists $k'$ such that $X^1 := N(\mathcal{M}/P)^k P^{k'}$ is a matrix with entries in $\mathbb{R}[x_i^{+1}]^+$, and this clearly intertwines $\mathcal{M}$ and $\mathcal{M}'$. Now we can unravel the $\Delta$ and obtain the $X$ that intertwines the original $M$ and $M'$.

Corollary 10.8 Let $M$ and $M'$ be square matrices with entries from $\mathbb{R}[x_i^{+1}]^+$. If $M$ satisfies:

(a) $(#)$;

(c) $(\%)$,

then there exists nonnegative rectangular $Y$ with entries in $A^+$ such that $YM = M^T Y$ and $M^T$ satisfies (c). If $M'$ satisfies (a) and (c) and $\beta_M = \beta_{M'}$, then there exist rectangular $X$ and $Z$ with entries in $A^+$ such that $XM = M^t X$ and $MZ = Z M'$.

Proof: Setting $M'' = M^T$ in Theorem 10.7, we see that (c) therein is satisfied; hence there exists nonnegative rectangular $X$ such that $YM = M'Y$. If $w_M$ is a right nonnegative eigenvector for $M$, then $Xw_M$ is a right eigenvector for $M''$—hence $M''$ satisfies (c) of the hypotheses above. Thus $M$ admits a good left nonnegative eigenvector.

If $M'$ satisfies (a) and (c) here, then the result of previous paragraph implies that its transpose does as well. Hence the transposes of $M$ and $M'$ are intertwined by an “$X$”, and the existence of the $Z$ follows by applying the transpose to the resulting equation.

Still open: Is it true that if $M$ merely satisfies Corollary 10.8(c), does $M^T$? (That is, drop $(#)$; a counter-example would exhibit a matrix not finitely equivalent to its transpose.)

The proof can be modified (as indicated below) to work under weaker hypotheses.

- $M'$ has a left nonnegative eigenvector for $\beta$ all of whose entries exhibit fractional polynomial behaviour, and all of whose entries lie in $\mathbb{R}[x_i^{+1}][\beta]$ (the latter is important—it is not enough that the entries be integral, since the integral closure of $\mathbb{R}[x_i^{+1}]$ in $\mathbb{R}(x_i)(\beta)$ can be strictly bigger than $\mathbb{R}[x_i^{+1}][\beta]$);

- this permits us to obtain a very positive right eigenvector $\tau$ for $\mathcal{M}'$, except that this time, we use a different $\Delta$ to define the latter; here, use $\Delta'$ whose diagonal entries are determined by entries of the right eigenvector. Correspondingly, we obtain $P'$, which now need not be the same as $P$.

- $M$ admits a right nonnegative eigenvector $v = (v_i)$ for $\beta$ with the following properties:

  (i) there exist $P_i \in A_Q$ such that $v_i/P_i$ is bounded above—by itself, this forces the matrix $\mathcal{M} := P \text{diag} (P_i) \mathcal{M} \text{diag} (P_i)^{-1}$ to satisfy $(\%)$;

  (ii) for all pure traces $L$ of $R_P$, $L(\mathcal{M}/P)$ admits nonnegative right eigenvector(s) only for its large eigenvalue (which must be $L(\beta/P)$), and not for any smaller ones, and moreover, $L(\sigma)$ must be in the interior of the positive left eigenspace for the eigenvalue $L(\beta/P)$;

  (iii) $v_i$ lie in $\mathbb{R}[x_i^{+1}][\beta]$.

We show that that these conditions are sufficient. We note that $L(\beta/P) \neq 0$ implicit in (ii) entails that $\text{cvx Log} P = \text{cvx Log} P'$. To make sure that $\mathcal{M}$ and $\mathcal{M}'$ have the same large eigenvalue ($\beta PP'$), we multiply the first by $P'$ and the second by $P$, and so the new “$P''$ is $PP''$' and it is easy to check that the $\tau$ and $\sigma$ we obtain as in the proof still have entries in $R_P[\beta_0]$. The argument will now go through, on noting that although there may be several choices for $w_L$, each one will have nonzero inner product with $L(\sigma)$, by the interior hypotheses.

In the original proof, the reason for making a construction based on the adjoint matrix was to guarantee (iii) and its analogue for $M'$; and also, we could deduce some properties of $\tau$ and $\sigma$ directly from their construction, that a more arbitrary choice may not satisfy. If $(#)$ holds, then of course (ii) is automatic. If $(#)$ does not hold, in fact a direct construction of $\tau$ or $\sigma$ could not use the adjoint matrix, since the latter will vanish at some $L$. (However, it might be reasonable to assume only that $M'$ satisfies $(#)$.)

If we assume for example, that $\beta$ is rational, then we may take an eigenvector with polynomial entries, and the problem then boils down to showing that the canonical choice (we note that the kernel of the map $\beta I - M : A^n \to A^n$ is free of rank one over $A$, so the generator is defined up to multiplication by a monomial) can be made eventually positive by multiplication by some polynomial; this amounts to showing
(in this context), that \( L(v_i/P_i) \neq 0 \) for every \( L \) in \( R_F \). After factoring out the ideal \( \ker \pi_F \) (see Lemma 3.6), this boils down to checking what happens at a point evaluation of the reduced polynomial obtained by restricting to a face. Of course, the usual problem rears its head—the restricted matrix need not be primitive, so we cannot conclude the entry must be nonzero.

In case the extreme points of \( K(M) \) are not lattice points, we may add a polynomial with fractional exponents (i.e., \( Q \) could come from \( A_n^\mathbb{Q} \)) instead. Since the finite equivalence problem is insensitive to adding scalars, the main theorem of this section applies just as well with weak (#) replacing (#).

It might be possible to delete the (#) hypothesis altogether from results in this section. It seems plausible that if \( M \) and \( M' \) are two primitive matrices with entries from \( A^+ \), with the same beta function, and \( M^T \) and \( M' \) satisfy (%), then there should exist a polynomial \( q \) in \( A[Z] \) (\( Z \) being another, dummy, variable) such that both \( q(M) \) and \( q(M') \) are primitive matrices and satisfy (#) (the polynomial \( q \) could be replaced by a rational function). Obviously, both \( q(M)^T \) and \( q(M') \) satisfy (%) since their eigenvectors may be taken as the eigenvectors of \( M^T \) and \( M' \) respectively, so they could be intertwined by a positive rectangular matrix, by the main theorem of this section. If we could also arrange that there is a rational function \( p \) in \( A[Z] \) such that \( p(q(M)) = M \) and \( p(q(M')) = M' \) (or some suitable weakening of this; this is very often the case), then we would deduce that \( M^T \) and \( M' \) are finitely equivalent. (In fact, we could dispense with the polynomial \( q \), and just replace \( q(M) \) and its counterpart by a matrix commuting with \( M \) whose commutant is the same as that of \( M \) and another one for \( M' \) so that the two matrices satisfy (#) and have the same beta functions as each other.)

For instance, the matrix

\[
M = \begin{bmatrix}
x + y & 0 & 1 \\
1 & x & 0 \\
0 & 1 & y 
\end{bmatrix}
\]

satisfies (%) (an easy exercise), but fails to satisfy (#) because at the singleton faces corresponding to \( x \) and \( y \), the matrix is a direct sum of two equal scalars and the zero matrix. The form of \( M \) (and the facts that it is \( \text{GL}(3, \mathbb{Z}) \)-conjugate to its companion matrix, and its characteristic polynomial is irreducible over \( A := \mathbb{R}[x, y] \)) suggest nothing commuting with it can satisfy (#). (Since the characteristic polynomial is irreducible, any nonscalar matrix that commutes with \( M \) is a rational function thereof, and since \( M \) is conjugate to a companion matrix, anything that commutes with it is of the form \( p_0 I + p_1 M + p_2 M^2 \) where \( p_i \) belong to \( M \)—this can be seen directly by exploiting any row of \( M \)—it has a zero and a one; moreover, the commutant of any nonscalar that commutes with it is the same as that of \( M \).) Very unexpectedly, there is a matrix commuting with it satisfying (#):

\[
N = \begin{bmatrix}
xy & 1 & y \\
x & 0 & 1 \\
1 & 0 & 0 
\end{bmatrix}
\]

The matrices \( M \) and \( N \) satisfy \( M = N^2 - xyN \) and \( N = M^2 - (x + y)M + xyI \), and \( N \) is shift equivalent to its companion matrix (the characteristic polynomial is \( \lambda^3 - xy\lambda^2 - (x + y)\lambda - 1 \)). So the proposal of the previous paragraph is at least plausible.

11. Structural similarity
Let \( M \) and \( M' \) be two matrices in \( M_n \mathbb{A}_q \). We say they have the same shape (or are shape similar), if for each \( i \) and \( j \),

\[
\text{cvx} \log M_{ij} = \text{cvx} \log M'_{ij}.
\]

If for some \( k \), \( M^k \) and \((M')^k \) have the same shape, then we say that they eventually have the same shape. This notion will be used to extend our so far rather feeble results on finite equivalence. The two matrices \( M \) and \( M' \) will be structurally similar if for all \( i \) and \( j \), \( \log M_{ij} = \log M'_{ij} \).
We first note that (\#) is not an invariant of shape similarity or structural similarity; take the constant $2 \times 2$ matrices $I$ and $\text{diag}(2, 1)$ (nonconstant primitive examples can easily be built up from this). It is also not true that structural similarity of primitive matrices forces the corresponding left nonnegative eigenvectors to be bounded with respect to each other (after making some normalization). For example, it is an easy exercise to see that the respective nonnegative left eigenvectors of

\[
\begin{bmatrix} 1 + x & 1 + x \\ x & 2 + x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 + x & 1 + x \\ x & 1 + x \end{bmatrix}
\]

are not comparable—see what happens as $x \to 0$. A minor variation on this leads to an example showing that (\%) is also not an invariant of structural similarity.

The matrices

\[
M = \begin{bmatrix} 1 + x & 1 \\ y & x + y \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} 1 + 2x & 1 \\ y & x + y \end{bmatrix}
\]

are clearly structurally similar. The matrix $M$ admits the constant vector $(1, 1)$ as its left eigenvector for the large eigenvalue function, so $M$ obviously satisfies (\%). It is not difficult to check that $M'$ does not satisfy (\%) on either side; see section 12 for details.

On the other hand, $T(M)$ is an invariant of structural similarity, as is immediate from the definitions. Also $T(M)$ is an invariant of shape similarity, and the necessary condition (Proposition 7.5) in order that all order ideals of $E_b(G_M)$ possess an order unit is an invariant of structural similarity. More generally, shape similarity also implies a bijection on the lattice of order ideals.

If $M$ and $M'$ eventually have the same shape, then $K(M) = K(M')$; to see this, notice that $\text{cvx Log tr } M^{kn!} = \text{cvx Log tr } (M')^{kn!}$, and these are respectively equal to $kn!K(M)$ and $kn!K(M')$.

The condition (\**\**) is just eventual structural similarity to a (necessarily rank one) matrix all of whose entries are the same. Of some interest are those matrices that are shape similar to a matrix whose order unit is a polynomial. These have rather pleasant properties, as we shall see.

**Proposition 11.1** Let $M$ and $M'$ be primitive eventually shape similar matrices in $M_n A^+_Q$, both satisfying (\#). If $M$ satisfies (\%), then so do $M'$ and $M^T$.

**Proof.** Since the eigenvectors are unchanged on replacing the matrices by their powers, we may assume the matrices are already shape similar; we may incorporate a factor of $n!$ into the power, so assume that $\text{tr } M/\beta$ and $\text{tr } M'/\beta'$ are bounded away from zero and infinity on $(\mathbb{R}^d)^{++}$. Let $v = (v_i)^T$ be the hypothesized left eigenvector for $\beta$, and apply the now standard conjugation with $\Delta = \text{diag } (P_i)$ where $P_i$ belong to $A^+_Q$, and $v_i/P_i$ are bounded away from zero and infinity on $(\mathbb{R}^d)^{++}$. Set $P_0 = \prod P_i$, and form the new matrices $\overline{M} := P_0^2 \Delta M \Delta^{-1}$ and $\overline{M}' := P_0^2 \Delta M' \Delta^{-1}$, both in $M_n A^+_Q$.

We note that $\overline{M}$ and $\overline{M}'$ are still shape similar (that is, to each other), and obviously both satisfy (\#). Since $\overline{M}$ satisfies (\*) (with respect to some $Q$ in $A_Q$; see section 10), so does $\overline{M}'$. We notice that $w := (v_i/P_i)$ is a left nonnegative eigenvector for $\overline{M}$, and it is bounded, positive, and bounded away from zero; as a consequence, for each pure state $L$ of $R^P$, the row $L(w) \gg 0$, and $L(w)$ is thus a strictly positive left eigenvector for $L(\overline{M}/Q)$, for the large eigenvalue, $L(\beta P_0/Q)$. As a consequence of (\#), the multiplicity of $L(\beta P_0/Q)$ as an eigenvalue of $L(\overline{M}/Q)$ is one; so the real matrix $L(\overline{M}/Q)$ is nonnegative, has its spectral radius of multiplicity one and no other eigenvalues of the same multiplicity, and admits a strictly positive left eigenvector for that eigenvalue. Moreover, since $\overline{M}$ and $\overline{M}'$ are structurally similar, $L(\overline{M}/Q)$ and $L(\overline{M}'/Q)$ have nonzero entries in exactly the same position. On replacing $N'_L := L(\overline{M}'/Q)$ by some power of itself and applying Lemma A1.2, we deduce that $N'_L$ has a strictly positive (real) left eigenvector.

Now suppose that $M'$ has no nonnegative left eigenvector for the large eigenvalue function which is both algebraic and has fractional polynomial growth; then neither would $\overline{M}'$, and thus neither would $\overline{M}'/Q$. 

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There exists a left eigenvector $U := (U_i)^T$ for the large eigenvalue function of $\overline{M}/Q$ with algebraic entries. Form $U' := U_i / \sum U_j$, so that the sum of the entries of $U'$ is the constant with value 1; in particular, the entries of $U'$ are bounded algebraic functions which are strictly positive on $(R^d)^+$. 

Pick any pure trace $L$ on $R_P$; then $L(U')$ (a real vector) makes sense as follows. If $L$ is a point evaluation at $r$ in $(R^d)^+$, $L(U') = (U_i(r))^T$. Otherwise, $L$ is obtained as a limit along the exponential of a ray in $R^d$, and it follows from Lemma 3.1, that the limit of $U'_i$ exists along this path. This gives the definition of $L(U')$. Clearly, $L(U')$ is a nonzero (entries add to 1) nonnegative left eigenvector of the real matrix $N_L$ for the spectral radius. Since the multiplicity of the spectral radius is 1, $L(U')$ is strictly positive. Since this is true for any $L$, and the set of such $L$ is a compactification of $(R^d)^+$, it follows that each entry of $U'$ is bounded above and below away from 0 on $(R^d)^+$. Of course, this means each entry of $U'$ satisfies fractional polynomial growth (with the polynomial 1), and when we translate back to $M'$ by undoing the process, it is immediate that the corresponding left eigenvector of the latter satisfies fractional polynomial growth.

That $M^T$ has a left eigenvector with the desired properties is a consequence of Corollary 10.8.

A much easier result occurs when $M$ is structurally similar to a rank one matrix. To avoid pathologies such as that occurring in Example 5.12, we call a matrix $M$ of positive rank one if there exist a row and column $v$ and $w$ respectively, with entries from $A^+$ such that $M = wv$. Obviously, any matrix satisfying (**) is shape equivalent to a positive rank one matrix (which in turn is strongly shift equivalent to a size one matrix), but the pathological example is a rank one matrix that is not even shift equivalent to a positive rank one matrix.

**Proposition 11.2** Let $M$ and $M'$ be primitive matrices with entries from $A^+$, and that $M'$ is of positive rank one. Suppose $M$ is shape similar to $M'$.

(a) There exists $Q$ in $A^+$ such that some power of $MQ$ is structurally similar to $M'Q$.

(b) There exists $Q'$ in $A^+$ such that some power of $MQ'$ is shift equivalent to a matrix satisfying (**).

(c) $M$ satisfies (#) and both $M$ and $M$ Tr satisfy (%).

**Proof.** (a) (We may replace the matrices by their powers in order to accommodate the definition of “rank one”.) The hypotheses guarantee that there exist $P_i$ and $Q_j$ in $A^+$ such that $cvx \text{Log } M_{ij} = cvx \text{Log } P_i + cvx \text{Log } Q_j$. By the Absorption lemma (5.7), there exists $R_{(ij)}$ in $A^+$ such that $\text{Log } R_{(ij)}^T M_{ij} = \text{Log } P_i Q_j R_{(ij)}$. Set $Q = \prod R_{(ij)}$.

(b) After replacing the matrices by suitable powers of themselves, and applying (a), we have that there exist $P_i$ and $Q_j$ in $A^+$ such that $\text{Log } P_i Q_j = \text{Log } M_{ij}$. Let $W = (P_i)^T$ and $V = (Q_j)$. Define $\Delta = \text{diag } (P_i)$, and set $P_0 \prod P_i$. Then $P_0 \Delta^{-1}$ has entries in $A^+$ and obviously $P_0 \Delta^{-1} M \Delta$ is strongly shift equivalent to $M P_0$. The $(ij)$ entry of the former is $M_{ij} P_j \prod_{k \neq i} P_k$. Hence its Log set is $\text{Log } P_j + \text{Log } Q_j + \sum \text{Log } P_k$—which depends only on $i$.

Now consider the elementary row operations $i \to i - \epsilon k$ (which replace row $i$ by itself minus a small real multiple of row $k$) applied to $P_0 M$. The $ij$ entry before such a transformation has Log set $\text{Log } P_j Q_j + \sum \text{Log } P_k$ and we are subtracting from it a small multiple of a polynomial with the same Log set. Hence for sufficiently small real $\epsilon$, we can be sure that the outcome has exactly the same Log set and no negative coefficients. The inverse operation adds a small multiple of column $i$ to column $k$—this increases the Log set in entry $jk$ from $\text{Log } P_j Q_j + \text{Log } P_0$ to $(\text{Log } P_i Q_i \cup \text{Log } P_j Q_j) + \text{Log } P_0$. To avoid confusion, we do $n^2 - n$ row operations (one for each pair $(i, k)$, choosing various small $\epsilon$s along the way; none of the row operations changes the Log sets or the signs of the coefficients). Applying the corresponding inverse operations, we see that every entry of the final matrix has as its Log set $\text{Log } P_0 \cup \text{Log } P_i Q_i$, which clearly satisfies (**). This sequence of row and column operations constitutes a strong shift equivalence.

(c) Both (#) and polynomial behaviour of the nonnegative left/right eigenvectors are preserved by shift equivalences, and we already know (***) implies both conditions, so (b) yields the result.
If a matrix were shape similar to a rank one matrix, then it is plausible that some power of it is structurally similar to a rank one matrix. This fails for the following example,

$$M = \begin{bmatrix} x^2 & x^2 \\
1 + x + x^3 & 1 + x^3 \end{bmatrix}.$$ 

It is still possible that such a matrix would be shift equivalent to a matrix satisfying (**).

Call a matrix in $M_n A^+$ full, if there exists $k$ such that $x^w M^{-kn!}$ belongs to $E_b(G_M)$ for all $w$ in $\log \text{tr} M^{kn!}$. For example, all matrices satisfying (***) are full (with $k = 1$), as is any matrix shift equivalent to a full one. Another way to say this is that some power of $PM^{-n!}$ is in $E_b(G_M)$, where $P = \text{tr} M^{n!}$.

Proposition 11.3 Let $M$ be a primitive matrix in $M_n A^+$. If $M$ is shape similar to a matrix of positive rank one, then it is full.

Proof. This follows from the fact that $T(M)$ is an invariant of shape similarity, as is $\log M^k$ for any $k$. 


12. Two by Two Matrices

In this section, we deduce a simple necessary and sufficient condition on $2 \times 2$ matrices with entries in $A^+$ in order that they satisfy (%): the discriminant must divide a polynomial with no negative coefficients. (This condition is easy to decide in this context, because the discriminant has a special form.)

Let $M$ be a square matrix with entries from $A^+$. We say $M$ is “stripped” if $\cap_i \text{Log} M_{ii}$ is empty. In other words, if $x^w$ appears in a diagonal entry, then there is another diagonal entry where it does not appear. Obviously, if $M$ is size 2, then it is stripped if and only if $\log M_{11} \cap \log M_{22} = \emptyset$.

If $M$ is an arbitrary element of $M_n A^+$, we may reduce it to a stripped form by subtracting off the following multiple of the identity matrix:

$$\sum_{w \in \cap_i \log M_{ii}} \left( \inf_{j \leq n} (M_{ij}, x^w) \right) x^w,$$

where $(P, x^w)$ denotes the coefficient of $x^w$ appearing in $P$. Obviously, the stripped form still belongs to $M_n A^+$, it has the same eigenvectors as the original form, and some other properties are preserved by this process. In particular, stripping does not affect property (%).

It turns out that for size 2 stripped matrices, (%%) is equivalent to the weak version of (#) discussed briefly in section 10. Adding a multiple of the identity matrix can ruin weak (#); it can also improve weak (#) to (#); stripping will preserve weak (#), but may reduce (#) to weak (#); some criteria for (%%) are most easily dealt with for stripped matrices.

The discriminant, $D(M)$, of a (square) matrix $M$ will be defined as the discriminant of its characteristic polynomial (it may be useful at times to define it to be the discriminant of the minimal polynomial of $\beta$ when $M$ belongs to $M_n A^+$; we shall not do so). In particular, $D(M)$ is an element of $A$. For a size 2 matrix $M = \begin{bmatrix} P & Q \\
R & S \end{bmatrix}$, the discriminant is $D(M) = (S - P)^2 + 4QR$, which is all we need to know about discriminants for this section. It is easy to check that the polynomial $\Delta^+ := (S + P)^2 + 4QR$ in $A^+$ satisfies $(\beta^2/\Delta^+)^{+1}$ is bounded on $(R^d)^{++}$, so that $2K(M) = \text{cvx} \log \Delta^+$.

Now assume that $M$ is primitive and in $M_n A^+$. Let $F$ be a non-trivial face of $M$. Then $M_F$ is either irreducible or is diagonal. If $M_F$ reducible (i.e., diagonal), the it is of the form $\text{diag}(P_F, S_F)$ where $P_F$ has the obvious meaning, i.e., $P_F = \sum_{w \in E} \frac{1}{2} (P, x^w) x^w$, etc. Suppose that condition (%%) holds. We know from section 8 that either $P_F = S_F$ or $P_F - S_F$ does not vanish on $(R^d)^{++}$, this for every face $F$ for which $M_F$ is reducible. If $M_F$ is reducible, then obviously it is reducible for every subface of $F$, and we quickly deduce (using e.g., results of [H5]), that if $P_F \neq S_F$, then either $P_F - S_F \geq \epsilon P_F$ or $S_F - P_F \geq \epsilon S_F$ as functions on $(R^d)^{++}$ for some epsilon. If $M$ were a stripped matrix, then of course $P_F \neq S_F$ (P and $S$
have no monomials in common, and at least one of $P_F$ and $S_F$ is not zero). Notice that if $M_F$ is irreducible, (%)) tells us nothing new about its entries.

If $F$ is a (closed) face of a polyhedron, Rel int($F$) is the empty set if $F$ is a singleton, and the relative interior of $F$ otherwise; in the latter case, it is an open set in $\mathbb{R}^{\dim F}$.

Theorem 12.1 Let $M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ in $M_n A^+$ be primitive and stripped, with discriminant $D := D(M) = (S - P)^2 + 4QR$, and $\Delta^+ = (S + P)^2 + 4QR$. The following are equivalent.

(i) There exists a nonzero polynomial $T$ in $A$ such that $TD$ has no negative coefficients.

(ii) There exists an integer $m$ such that $(\Delta^+)^m D$ not only has no negative coefficients, but satisfies Log $(\Delta^+)^m D = \log(\Delta^+)^{m+1}$.

(iii) The condition (%) is satisfied by $M$.

(iv) For all faces $F$ of $K(M)$ such that $M_F$ is reducible, $P_F - S_F$ has no zeroes in $(\mathbb{R}^d)^{++}$.

Remarks. 1. The property of $D$ in part (i) is called “Handelman” in [MT]. Since $D$ is a square plus a nonzero polynomial with no negative coefficients, it automatically is positive on $(\mathbb{R}^d)^{++}$, so the criterion for (i) to hold [H2] reduces to $D_G$ being strictly positive for every proper face $G$ of $D$. This is usually easy to check (see remark 3).

2. Property (ii) and an observation about the Log sets of $\Delta^+$ and $D$ imply that the element $D/\Delta^+$ of $R_{\Delta^+}$ is an order unit, and indeed this is how it is proved (via results in [H2]).

3. We already knew that (%) is not a shape invariant (section 11). Here is a very simple family of examples, using the theorem.

Let $\alpha \geq 0$ be a variable constant. Define the family of matrices in $M_n \mathbb{R}[x, y]^+$,

$$M_\alpha = \begin{bmatrix} \alpha x & 1 \\ y & 1 + x^2 \end{bmatrix}.$$ 

When $\alpha = 2$, this becomes the matrix appearing in [MT2], as part of a counter-example to the finite equivalence conjecture. The discriminant of $M_\alpha$ is $(x - \alpha/2)^2 + 1 - \alpha^2/4 + 4y$. The convex hull of its Log set is cvx $\{(0, 0), (2, 0), (0, \frac{1}{2})\}$, and since the polynomial has no positive zeroes, it is sufficient (to check whether (i) holds) to test the three boundary polynomials (one for each edge of the triangle). We find immediately that the discriminant satisfies (i) if and only if $\alpha < 2$. So $M_\alpha$ satisfies (%) precisely when $\alpha < 2$.

4. Since (%) is quite a strong property, it is natural to enquire whether it implies other pleasant properties, e.g., drawn, which is a consequence of noetherianness of $E_0(G_M)$. In fact, these two properties are independent. Take $M = M_0$ of remark 3; for no positive integers $a < b$ it is true that $M^a < M^b$, so that $M$ is not drawn. Since being drawn is an invariant of structural similarity, it is obvious that the same is true for any value of $\alpha > 0$ (for $\alpha = 0$, $M_0$ is drawn; however, $M_0$ is not structurally similar to the others— it is a companion matrix, so satisfies considerably stronger properties).

5. Both (i) and (iii) are unaffected by the addition of a scalar matrix to $M$; hence the theorem holds if we drop (ii), (iv), and the stripped hypothesis.

Proof. (i) implies (ii). Clearly Log $D \subseteq \Delta^+$. We first notice that since $D$ is a square plus a nonzero element of $A^+$, $D$ is strictly positive on $(\mathbb{R}^d)^{++}$. Since $M$ is stripped, Log $P \cap$ Log $S$ is empty, so their respective sets of extreme points are disjoint. Suppose that $w$ is an extreme point of cvx Log $(S + P)^2$. Then $w = 2v$ for $v$ an extreme point of at least one of cvx Log $P$ or cvx Log $S$. Disjointness ensures that $v$ belongs to exactly one of Log $P$ or Log $S$, and this forces the coefficient of $x^w$ in $(S - P)^2$ to be nonzero, in fact positive. Hence cvx Log $(S + P)^2 \subseteq$ cvx Log $(S - P)^2$. The other inclusion is trivial, so cvx Log $(S + P)^2 \subseteq$ cvx Log $(S - P)^2$, and the coefficient of $x^w$ in $(S - P)^2$ is positive for any extreme point $w$. 

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Next, \( \text{cvx Log } \Delta^+ = \text{cvx (cvx Log } QR \cup \text{Log } (S + P)^2) \), and the previous equality together with the positivity of the coefficients at extreme points ensures that \( \text{cvx Log } \Delta^+ = \text{cvx Log } D \). Now (ii) follows from (i), the fact that \( D(1,1,\ldots) > 0 \), and \( \text{cvx Log } \Delta^+ = \text{cvx Log } D \), by [H2, V.3A].

(ii) implies (iv). We know from the proof of (i) that \( \text{cvx Log } \Delta^+ = \text{cvx Log } D \) in any case, and it follows from (ii) and for example, [H2, V.4(b)] that \( D/\Delta^+ \) is an order unit of \( R_{\Delta^+} \). This means that for any face \( G \) of \( \text{cvx Log } \Delta^+ \), the facial polynomial \( D_G \) (obtained by throwing away all the terms in monomials whose exponents do not belong to \( G \)) is strictly positive on \( (\mathbb{R}^d)^++ \). Now let \( F \) be a face of \( K(M) \) such that \( M_F \) is reducible. Its two diagonal entries are \( P_F \) and \( S_F \), and since \( M_F \) is reducible, \( \text{Log } (QR) \subseteq \text{Rel int cvx (Log } S^2 \cup \text{Log } P^2)_{2F} \). Then \( G := 2F \) is a face of \( \text{cvx Log } D \), and \( [(S - P)^2]_G \) does not vanish at any point of \( (\mathbb{R}^d)^++ \). However, this is just the square of \( (S - P)F \), so the latter does not vanish at any positive \( d \)-tuple.

(iv) implies (i). At those faces \( G \) of \( K(M) \) for which \( M_G \) is irreducible, necessarily \( \text{Log } (P - S)_G \subseteq \frac{1}{2} \text{cvx Log } (QR)_{2G} \), and obviously \( (QR + (P - S)^2)_{2G} \) is strictly positive as a function. At the faces at which \( M \) is reducible, the hypotheses guarantee strict positivity of the corresponding facial function. As \( \text{cvx Log } (P - S)^2 + 4QR = 2K(M) \), it follows from [H2, V.6, p.53] that \( D \) divides a polynomial with no negative coefficients.

(i, ii) imply (iii). It is sufficient to show that \( (\beta - S)^2 \) is bounded above and below away from zero by a rational function whose numerator and denominator belong to \( A^+ \). First, we notice that for any point \( r \) in \( (\mathbb{R}^d)^++ \), \( M(r) \) a primitive matrix with spectral radius \( \beta(r) \) and \( S(r) \) is a principal block therein. Hence \( \beta(r) > S(r) \) for all \( r \) in \( (\mathbb{R}^d)^++ \).

We pick the candidate \( f/g \) as follows. Call a face \( F \) of \( K(M) \) of type (a) if \( M_F \) is irreducible, otherwise of type (b) if \( P_F - S_F \) is positive as a function, or of type (c) if \( S_F - P_F \) is positive as a function at least one point. By hypothesis, in the latter two cases it must be positive at any point, and moreover, this persists on taking subfaces. It easily follows from [H2, V.4], that in case (b), \( (P - S)_F/(P + S)_F \) is an order unit of \( R_{(P+S)_F} \), and thus there exists \( \epsilon > 0 \) such that \( (P - S)_F \geq \epsilon (P_F + S_F) \) as functions on \( (\mathbb{R}^d)^++ \). Similarly, if case (c) applies to \( F \), \( (S - P)_F \geq \epsilon (P_F + S_F) \); we may obviously arrange this to hold with the same \( \epsilon \) for all faces \( F \) of types (b) and (c). We will repeatedly use the following elementary lemma.

**Proposition 12.4** Let \( r \) and \( s \) be real numbers, the latter positive.

(a) If \( r \) is constrained to be negative, the function \( r + (r^2 + s^2)^{1/2} \) is bounded above and below away from zero (i.e., uniformly as \( r \) and \( s \) vary) by \( s^2/(|r| + s) \). In particular, \( (r + (r^2 + s^2)^{1/2})^2 \) is bounded above and below away from zero by \( s^4/(r^2 + s^2) \).

(b) If \( r \) is constrained to be positive, the function \( r + (s^2 + r^2)^{1/2} \) is bounded above and below away from zero by \( r + s \).

**Proof.** (a) Set \( r = -ts \) (with \( t > 0 \)); it is easy to verify (by differentiating with respect to \( t \)) that \( -t + (1 + t^2)^{1/2} \) is bounded above and below by what it’s supposed to. Part (b) is even more trivial.

For this portion of the proof only, we define the following relations. If \( U \) is a subset of \( (\mathbb{R}^d)^++ \) and \( f, g : (\mathbb{R}^d)^++ \to \mathbb{R} \) are everywhere positive functions, we write \( f \leq g \) relative to \( U \) if \( f/g \) is bounded above as a function on \( U \) (i.e., \( |f|/|g| \leq k \cdot |U| \) for some positive real number \( k \)). We write \( f \sim g \) relative to \( U \) if both \( f \leq g \) and \( g \leq f \) hold relative to \( U \). If no \( U \) is referred to in these definitions, it is assumed to be all of \( (\mathbb{R}^d)^++ \).

Let \( Z \) denote the collection of extreme points \( w \) of \( K = K(M) \) such that the singleton face \( G := \{ w \} \) is of type (b), i.e., \( M_G \) is diagonal with nonzero upper left entry. As \( M \) is stripped, and \( G \) consists of a single point, the lower right entry of \( M_G \) is zero. Now set \( P' = \sum_{w \in Z} x^w \), and then define

\[
H = \frac{(QR)^2 + (P')^4}{\Delta^+}.
\]
Obviously \( H \sim (QR + (P')^2)((QR + (P')^2)/\Delta_+) \), and since \( \log P' \subseteq \log P \), it follows that \( H \lesssim QR + (P')^2 \). We will show that \( (\beta - S)^2 \sim H \), by showing that this holds relative to each of four subsets \( \{ U_i \}_{i=1}^4 \) whose union is \((\mathbb{R}^d)^{++}\).

If \( F \) and \( G \) are faces of types (b) and (c) respectively, then every face is of the same type; more importantly, any path of edges (one dimensional faces) joining \( F \) to \( G \) must contain at least one edge that is of type (a), as is evident from (iv). This implies that if \( v \) in \( \mathbb{R}^d \) \( \setminus \{ 0 \} \) is a vector exposing a type (c) face with respect to \( K(M) \), then \( \max v \cdot \log QR \geq 2 \max v \cdot \log P' \). Now we are almost ready to define \( U_1 \) (this is the most difficult of the four regions with which to deal).

Suppose that \( \{ r_n \} \in \mathbb{N} \) is a sequence of points in \((\mathbb{R}^d)^{++}\) with the property that \( (P/S)(r_n) \to 0 \). We show that \( \lim \inf(QR/(P')^2)(r_n) > 0 \).

To this end, set \( p = (P')^2 + QR \), \( a = (P')^2/p \), \( b = 1 - a \), and \( c = (P^2 + 4QR)/\Delta_+ \). Then the ordered rings \( R_P \Delta_+ \) and \( R_{\Delta_+} \) are defined, the former contains the latter, each of \( a \) and \( b \) belong to \( R_P \Delta_+ \), and \( c \) belongs to \( R_{\Delta_+} \). Suppose that

\[
\begin{align*}
(1) & \lim_{n \to \infty} P(r_n)/S(r_n) = 0, \\
(2) & \lim_{n \to \infty} QR(r_n)/(P')^2(r_n) = 0.
\end{align*}
\]

(Obviously, if \( \lim \inf(QR/(P')^2)(r_n) = 0 \), we may refine the sequence \( \{ r_n \} \) so that (2) holds.) For each \( n \), let \( \tau_n : R_{\Delta_+} \to \mathbb{R} \) be the pure trace, evaluation at \( r_n \), and let \( \tau \) be a limit point of \( \{ \tau_n \} \) in the pure trace space of \( R_{\Delta_+} \). (The pure trace space of such rings is compact, e.g., [H5].) From the fact that \( \log P' \subseteq \log P \), it follows from (1) and (2) that \( \tau_n(c) \to 0 \), so \( \tau(c) = 0 \). Now (2) implies that \( \tau(b) = 1 \), so \( \tau(a) = 0 \).

As \( a \) and \( c \) are nonzero and positive elements of \( R_{\Delta_+} \), \( \tau \) is a non-faithful pure trace. By [H5, Theorem IV.1, p. 42], there exist \( B \) in \((\mathbb{R}^d)^{++}\) and \( v \) in \( \mathbb{R}^d \) \( \setminus \{ 0 \} \) such that if \( X_{B,v} : \mathbb{R}^+ \to (\mathbb{R}^d)^{++} \) is the path defined by \( X_{B,v}(t) = (B_1v(1), \ldots, B_dv(d)) \), then for all \( y \) in \( R_{\Delta_+} \),

\[ \tau(y) = \lim_{t \to -\infty} y(X_{B,v}(t)). \]

(Existence of each of these limits is a simple consequence of L'Hôpital’s rule.) We have already seen that for a polynomial \( g \) in \( A^+ \), \( g(X_{B,v}(t)) = f(B)t^{\max v \cdot \log g} + o(t^{\max v \cdot \log g}) \), with \( f(B) > 0 \). From \( \tau(b) = 0 \), we deduce

\[
\text{(s)} \quad \max v \cdot \log QR < 2 \max v \cdot \log P'.
\]

From \( \tau(c) = 0 \), it follows that

\[
2 \max v \cdot \log S > \max v \cdot \log QR, \quad 2 \max v \cdot \log P.
\]

As \( \log A_+ = \log \{ \log P^2 \cup \log S^2 \cup \log QR \} \), we deduce that \( \max v \cdot \log A_+ = 2 \max v \cdot \log S \). As \( \log A_+ \) is \( 2K(M) \), we infer that the face exposed by \( v \) is of type (c). However, the earlier remark (essentially saying that a line that exposes a type (c) face hits \( \frac{1}{2} \log QR \) at least as soon as it hits \( \log P' \)) gives \( \max v \cdot \log QR \geq 2 \max v \cdot \log P' \). This contradicts \( \text{(s)} \).

Thus there exist \( \delta \) and \( \epsilon \) exceeding zero such that for \( r \) in \((\mathbb{R}^d)^{++}\), \( P(r) < \delta S(r) \) implies \( \epsilon(P')^2(r) < QR(r) \). We may assume \( \delta < 1 \). Set \( U_1 = \{ r \in (\mathbb{R}^d)^{++} \mid P(r) \leq (\delta/2)S(r) \} \). Then \( U_1 \) is a closed subset of \((\mathbb{R}^d)^{++}\), and \( (P')^2 \lesssim QR \) relative to \( U_1 \). It is convenient to reformulate the definition of \( U_1 \) in terms of the relationship of \( S - P \) to \( S + P \). We simply notice that for positive numbers \( y \) and \( z \), \( y \leq (\delta/2)z \) (for \( \delta/2 < 1 \)) is equivalent to \( z - y \geq \kappa(z + y) \), where \( \kappa = (2 - \delta)/(2 + \delta) \). With this choice for \( \kappa \), we have

\[
U_1 = \{ r \in (\mathbb{R}^d)^{++} \mid (S - P)(r) \geq \kappa(S + P)(r) \}.
\]
Now we compare $H$ and $(\beta - S)^2$ on $U_1$. Since $S - P$ is positive on $U_1$, we deduce from Proposition 12.4 above that
\[(\beta - S)^2 \sim \frac{(QR)^2}{D} \text{ relative to } U_1\]
Since $D \sim \Delta_+$ (relative to $(\mathbb{R}^d)^{++}$) and $(P')^2 \lesssim QR$ relative to $U_1$, it follows immediately that $(\beta - S)^2 \sim H$ relative to $U_1$.

Now set $U_2 = \{ r \in (\mathbb{R}^d)^{++} \mid |(P - S)(r)| \leq \epsilon_2(P + S)(r) \}$, where $\epsilon_2$ is to be determined. Let $\theta = \inf_{r \in (\mathbb{R}^d)^{++}} D(r)/\Delta_+(r)$; we know $\theta > 0$ by (ii). We have that $(P - S)^2/\Delta_+ \leq ((P - S)/(P + S))^2 \lesssim \epsilon_2^2$ on $U_2$. So, restricted to $U_2$, \[
\frac{4QR}{\Delta_+} = \frac{D - (P - S)^2}{\Delta_+} \geq \frac{D}{\Delta_+} - \epsilon_2^2 \geq \theta - \epsilon_2^2.
\]
Choose $\epsilon_2 < \theta^{1/2}/2$. As functions on $U_2$, $QR \geq \frac{\theta - \epsilon_2^2}{4} \Delta_+$, whence $\Delta_+ \lesssim QR$ relative to $U_2$. Since $QR \lesssim \Delta_+$ (relative to $(\mathbb{R}^d)^{++}$), it follows that $QR \sim \Delta_+$ relative to $U_2$. Hence relative to $U_2$, $H \sim QR$.

Relative to the subregion of $U_3$ where $P(r) \geq S(r)$, $(\beta - S)^2 \sim (QR)^2/D \sim (QR)^2/\Delta_+ \sim QR \sim H$. Relative to the subregion where $S(r) \geq P(r)$, $(\beta - S)^2 \sim (P - S)^2 + QR \sim QR \sim H$. Hence $(\beta - S)^2 \sim H$ relative to $U_2$.

Now set $U_3 = \{ r \in (\mathbb{R}^d)^{++} \mid \epsilon_2 \leq \frac{(S - P)(r)}{(S + P)(r)} \leq \kappa \}$. (If this turns out to be empty, ignore this part.) On $U_3$, $S(r) > P(r)$, so $(\beta - S)^2 \sim (QR)^2/D$ relative to $U_3$. Since $D \sim \Delta_+$, we obtain immediately that $(\beta - S)^2 \lesssim H$ relative to $U_3$.

To show $(\beta - S)^2 \sim H$ relative to $U_3$, it is thus sufficient to show that $(P')^2 \lesssim QR$ relative to $U_3$ (this is similar to the situation with $U_1$, but considerably easier). If this condition fails, there would exist a sequence $\{r_n\}$ of elements of $U_3$ such that $\{(P')^2/(QR)(r_n)\} \to 0$. Set $C = QR/\Delta_+$, a positive element of $R_{\Delta_+}$. As $\log P' \subset \log P$ and $P \sim S$ relative to $U_3$, it follows that $C(r_n) \to 0$. As before, let $r_n$ be the point evaluation trace at $r_n$ (for $R_{\Delta_+}$ this time), and let $\tau$ be a limit point; then $\tau(C) = 0$, so $\tau$ is a pure non-faithful trace, and in the argument with $U_1$ above, there exists a path $X_{B,v}$ such that $\tau$ is given as the limit as $t \to \infty$ of the point evaluations along the path.

Let $F$ be the face exposed by the vector $v$. Set $b'' = (S - P)^2/\Delta_+$; this is an element of $R_{\Delta_+}$ and $4C + b'' = D/\Delta_+$; we see that $\{b''(r_n)\}$ is bounded below away from zero, so $\tau(b'') > 0$; it is also clear that $\tau(b'') < 1$. The latter implies that $F$ is not of type (c) (if it were, it would force $\tau(b'') = 1$), and the former that it is not a face of type (b). Hence it must be a face of type (a), which would force $\tau(C) = 1/4$, a contradiction.

Hence $(P')^2 \lesssim QR$ relative to $U_3$, and so $(\beta - S)^2 \sim H$ relative to $U_3$.

Finally, define $U_4 = \{ r \in (\mathbb{R}^d)^{++} \mid \epsilon_2 \leq \frac{(P - S)(r)}{(S + P)(r)} \}$. Clearly $(\beta - S)^2 \sim P^2 + QR$ relative to $U_4$. It is thus sufficient to show that $(P')^2 + QR \sim P^2 + QR$ relative to $U_4$ (notice that the $S$ terms in $\Delta_+$ can be “absorbed” into their $P$ counterparts, relative to $U_4$). If this fails, then there would exist a sequence $\{r_n\}$ in $U_4$ such that $\{(P')^2 + QR)/(P^2 + QR)\} \to 0$ along $r_n$. Define $f = P^2 + QR$, $c = ((P')^2 + QR)/f$ and $b' = (P^2 - S^2)/\Delta_+$ in $R_{\Delta_+}$. Then $c(r_n) \to 0$ and $b'(r_n)$ is bounded below away from zero. Hence if $\tau$ is a limit point of the point evaluation traces at $r_n$, we have that $\tau(c) = 0$ but $\tau(b') > 0$.

Once more, $\tau$ is determined by a path $X_{B,v}$, and $\tau(b') \neq 0$ entails that the face $F$ exposed by $v$ is not of type (c) (if it were on a type (c) face, then $\tau(b') < 0$, since the top entry has to be strictly less than the bottom entry at all traces). At a type (a) face, $\tau(c) > 0$ (arising from the $QR$ term), so $F$ must be type
(b). However, this forces all of its extreme points to be type (b) as well, hence they all belong to \( \text{Log} \ P' \), which would again force \( \tau(c) > 0 \). This contradiction yields \( (P')^2 + QR \sim P^2 + QR \) relative to \( U_4 \), and so \( (\beta - S)^2 \sim H \) relative to each of the \( U_i \), and thus relative to their union, \((\mathbb{R}^d)^{++}\).

(iii) implies (iv) follows from the remark just prior to the statement of this theorem.

Corollary 12.3 Suppose \( M \) is a primitive size 2 matrix with entries from \( A^+ \).

(i) If \( M \) satisfies weak (#), then it satisfies (%).

(ii) If \( M \) is stripped and satisfies (%), then it satisfies weak (#).

**Proof.** (i) Stripping preserves the weak (#) property, so we may already assume \( M \) is stripped. Then there is only one block per face that ever hits the spectral radius, as any face is of one of the types (a), (b), (c) occurring in the proof of Theorem 12.1; condition Theorem 12.1(iv) is thus satisfied.

(ii) Obvious for size 2 matrices.

Each of the following matrices satisfies (%).

\[
\begin{bmatrix}
1 & x + y \\
x + y & xy
\end{bmatrix}
\quad \quad
\begin{bmatrix}
1 & x + y \\
x + y & (xy)^3
\end{bmatrix}.
\]

In the first case, \( K(M) \) is the standard unit square in the plane and the function \( H \) (from the proof above) can be chosen to be \( ((x + y)^4 + 1)/(x^2 + y^2 + x^2y^2 + 1) \); here the discriminant is \( 4(x + y)^2 + 1 - 2xy + x^2y^2 \) which already has positive coefficients. In the second example, the discriminant is \( (1 - x^3y^3)^2 + 4(x + y)^2 \), \( K(M) \) is the quadrilateral with vertices at \((0, 0), (1, 0), (0, 1), (3, 3)\), and it is easy to check that at each edge, the discriminant has only positive coefficients, which is sufficient for the theorem to apply. Note that in the first case, \( \beta - S \) is bounded on \((\mathbb{R}^d)^{++}\), while in the second case it is not. This shows that we cannot simply replace the complicated expression for \( H \) by something simpler, such as \( (P^2) + (QR)^2/D \) (one of my earlier attempts). In either case, \( P' = 1 \).

In contrast, consider the one parameter family of matrices \((0 \leq \alpha)\) appearing in Remark 3,

\[
M_\alpha = \begin{bmatrix}
\alpha x & y \\
1 & 1 + x^2
\end{bmatrix}.
\]

Then \( 2K(M_\alpha) \) has vertices \( \{(0, 0), (4, 0), (0, 1)\} \), and the discriminant of \( M_\alpha \) is \( 1 + x^2 - \alpha x x^2 + 4y \). Along the bottom edge, the resulting polynomial is \( (1 + x^2 - \alpha x)^2 \), and this will be strictly positive on \((\mathbb{R}^2)^{++}\) if and only if \( \alpha < 2 \), and it follows that \( D \) divides a polynomial with no negative coefficients precisely when this holds. So \( M_\alpha \) satisfies (% if and only if \( \alpha < 2 \). (One could also treat \( \alpha \) as another variable, in which case the resulting matrix fails to satisfy (%).)

It seems likely that (weak) (#) implies (%); for \( n = 2 \), this is Corollary 12.3; for \( n = 3 \), it is true provided \( \text{tr} M = 0 \) (in which case, (#) holds automatically); and we also know that (%) holds if either \( M \) or \( M^{\text{tr}} \) is in companion matrix form.

Let \( \rho : (\mathbb{R}^d)^{++} \rightarrow \mathbb{R}^+ \) be a nowhere vanishing real analytic algebraic function that is nonnegative. Let \( B = (B_1, \ldots, B_d) \) be a strictly positive \( d \)-tuple, and let \( v = (v(i)) \) be a nonzero real \( d \)-tuple. Define the path in \((\mathbb{R}^d)^{++}, X_{B,v}(t) = (B_1^{v(i)}, \ldots, B_d^{v(d)}). \) As we have seen, \( \rho(X_{B,v}(t)) \) grows asymptotically like \( t^a \) for some real \( a \) (we have only seen this when \( v \) has only rational entries; we can however use the same argument with formal polynomials, although a continuity argument to go from the rationals to the reals cannot work). In other words, we can define a function, \( \psipsilon : (\mathbb{R}^d)^{++} \times (\mathbb{R}^d \setminus \{0\}) \rightarrow \mathbb{R} \)

\[
\psiepsilon(B,v) = \lim_{t \rightarrow \infty} \frac{\log \rho(X_{B,v})(t)}{\log t}
\]

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If $\rho'$ is another such nonnegative function such that $\rho'/\rho$ is bounded above and below away from zero on $(\mathbb{R}^d)^{++}$, then obviously $\psi_\rho = \psi_{\rho'}$. More generally (i.e., without the boundedness assumptions on the quotient), $\psi_{\rho'/\rho'} = \psi_\rho - \psi_{\rho'}$.

If $p$ is in $A^+_Q$, then $\psi_p(B, v) = \max v \cdot N(p)$ (recall that $N(p) = \text{cvx Log } p$, the Newton polyhedron of $p$, and for a convex set $K$, $\max v \cdot K$ is supposed to be $\max \{ v \cdot k \mid k \in K \}$; this maximum is attained at an extreme point, so $\max v \cdot N(p) = v \cdot u$ for some $u$ in $\text{Log } p$). Thus if $p$ and $q$ are in $A^+_Q$, $\psi_{p/q}(B, v) = \max v \cdot N(p) - \max v \cdot N(q)$. In particular, this is independent of the choice of $B$, as we had observed earlier. However, it is also continuous as a function of $v$ (that is, $v \mapsto \phi_{p/q,B}(v)$ is continuous); in fact, it is piecewise linear, although at the moment, I do not see how to exploit this. It follows from the comments above about bounded ratios, that if $\rho$ has fractional power asymptotic behaviour, then $\psi_{\rho,B}$, defined via $\psi_{\rho,B}(v) = \psi_{\rho}(B, v)$ is continuous in $v$ and independent of $B$. We may also view it as a function on $S^{d-1} = (\mathbb{R}^d \setminus \{0\})/\mathbb{R}^++$.

For more general $\rho$, for almost all $B$, $\psi_\rho(B, v) = \max_{B'} \psi_\rho(B', v)$, i.e., the maximum value is the generic one. (This is easy to see from the polynomial satisfied by $\rho$; the set of anomalous $B$ is a finite union of varieties.)

Is it true for $\rho$ an algebraic and analytic positive valued function on $(\mathbb{R}^d)^{++}$, that $\psi_\rho$ being independent of $B$ (for all $v$) forces $\rho$ to have fractional rational asymptotic behaviour? This would be a useful step toward proving that $(\#)$ implies $(\%)$.

13. Eventual weak similarity to matrices satisfying $(**)$

In this section, we show that certain matrices can be transformed so as to satisfy $(**)$.

Select $(B, \nu)$ from the product space $\mathbb{R}^d \times \mathbb{R}^d$. Write $B = (B(1), \ldots, B(d))$ and $\nu = (\nu(1), \ldots, \nu(d))$, and as usual define the path, $X_{B,\nu} : (0, \infty) \to (\mathbb{R}^d)^{++}$, via

$$X_{B,\nu}(t) = \left( B(1)^{\nu(1)}, \ldots, B(d)^{\nu(d)} \right)$$

In other words, $X_{B,\nu}$ is just the exponential of the ray in the direction $\nu$, starting at $\log B$. Note that if $\nu$ is zero, then the path is constant. If $\lambda : (\mathbb{R}^d)^{++} \to \mathbb{R}$ is real analytic and is algebraic over $A$, then $\lambda (X_{B,\nu})$ is an algebraic and real analytic function of one variable; thus if $\lambda$ is additionally bounded,

$$\tau_{B,\nu}(\lambda) = \lim_{t \to \infty} \lambda (X_{B,\nu}(t))$$

exists for all choices of $(B, \nu)$. If $B - B' \in \mathbb{R} \cdot \nu$, then the two paths, $X_{B,\nu}$ and $X_{B',\nu}$, will differ only by an initial segment, so that the limits $\tau_{B,\nu}$ and $\tau_{B',\nu}$ will be equal for all choices of $\lambda$. This suggests that we put an equivalence relation $\sim$ on $\mathbb{R}^d \times \mathbb{R}^d$, via

$$(B, \nu) \sim (B', \nu') \quad \text{if } \nu = s\nu' \text{ for some real } s > 0 \text{ and } B - B' \in \mathbb{R} \cdot \nu$$

Let $\mathcal{P}$ denote the set of equivalence classes. Note that $\mathcal{P} \setminus \{ \mathbb{R}^d \times \{0\} \}$ is a copy of the tangent space of the $d$-sphere (Erhard Neher pointed this out to me). Now we show that if $[B, \nu]$ and $[B', \nu']$ are two distinct equivalence classes (i.e., elements of $\mathcal{P}$), then there exist bounded rational (and analytic) functions that separate the linear functionals $\tau_{B,\nu}$ and $\tau_{B',\nu'}$ corresponding to $[B, \nu]$ and $[B', \nu']$, except if $\nu$ and $\nu'$ are positive scalar multiples of each other and not all the ratios among the entries of $b$ are rational. In the latter exceptional case, even if we permit algebraic functions, we still could not separate them, although fractions involving $x^{\mu}$ where $\mu$ is an arbitrary point of $\mathbb{R}^d$ would suffice.

If both $\nu$ and $\nu'$ happen to be zero, then at least one of $\{ (\sum i \mu_i x_i) / (\sum x_i + 1) \}$ will have different values at $B, B'$, for a selection of positive integers $\{ \mu_i \}$. If $\nu$ is zero while $\nu'$ is not, then $x_1/(1 + x_1 + x_2^2)$ will vanish under $\tau_{B',\nu'}$ but not at $B$. So we may assume that neither $b$ nor $\nu'$ are zero. If $\nu$ is not a positive scalar multiple of $\nu'$, then we may find a lattice polytope $K$ in $\mathbb{R}^d$ with the following property. There exist two disjoint faces
Lemma 13.1 Suppose that $R$ is a matrix with entries from $A$ with characteristic polynomial $\chi$, that satisfies the following properties.

(a) $R$ admits an eigenvalue $\beta$ satisfying (1) with respect to $\chi$.

(b) There exist left and right eigenvectors of $R$ for $\beta$, $\Lambda$ and $\Theta$ respectively that are strictly positive.

(c) All the entries of $R^l$ are bounded below by integer multiples of $\beta^l$, for some integer $l$ (i.e., for all $i,j$, $(R^l)_{i,j}/\beta^l(r) \geq -B$ for some integer $B$).

(d) For all sufficiently large integers $J$, there exists nonzero $Q$ in $A^+$ (possibly depending on $J$) such that $Q \cdot \text{tr} \chi^{(J)}(r)$ itself has no negative coefficients.

Then there exists an integer $k$ together with an element $P$ of $A^+$ such that

$$(PR)^k \in M_n A^+$$

and satisfies (**)  

\[ \text{tr} \chi^{(N)}(r) \] is just the sum of the $N$th powers of the roots of $\chi$, it follows from (1) that 

\[ |\text{tr} \chi^{(N)}(r) - \beta^N(r)| \leq (n-1)(1-\delta)^N \beta^N(r). \]

Select $N$ so that $(1-\delta)^N < 1/3(n-1)$; thus
\[ |(\text{tr } \chi^{(N)})/\beta^N - 1| < 1/3, \text{ and so } |\beta^N/\beta^N - 1| < 1/2. \text{ If } N \text{ is increased, this inequality will be preserved.} \]

For \( N \) sufficiently large, select \( Q \) in \( A^+ \) such that \( Q \cdot \chi^{(N)} \) also belongs to \( A^+ \). Set \( P_N = \text{tr } \chi^{(N)} Q^N \). We may replace \( \beta^N \) by \( \beta^N Q^N, R \) by \( QR, \chi = X^n + \sum p_i X^i \) by \( X^n + \sum p_i Q^i X^i \), etc. With this renaming, \( \|\beta^N/P_N - 1\| < 1/2 \).

Define
\[ P = \sum w^u \text{ the sum over } w \in \mathbb{Z}^d \cap \text{cvx Log } P_N^d. \]

By e.g., [HM, V.4], \( P^d_N/P \) is bounded below away from zero on \((\mathbb{R}^d)^{++}\). Clearly for all \( a \) and \( b, \tau_{B,v}(\beta^N/P_N) > 0 \). Thus the same is true of \( \beta^N d/P \).

Consider the matrix \( R' = R^{N d}/P \) in \( M_n A [P^{-1}] \). We shall show that every entry of \( R' \) lies in the algebra \( R_P \). Since \( \text{cvx Log } P \cap \mathbb{Z}^d = \text{Log } P \) and \( \text{cvx Log } P d \) is \( d \) times a lattice polytope (this is the reason for raising \( P_N \) to the \( d \)th power), it is sufficient to show that every entry of \( R' \) is bounded as a function on \((\mathbb{R}^d)^{++}\) [HSLN, III.2 & III.4].

Reselect \( N \) so that in addition it is a multiple of \( l \). Then it follows from \( (\text{tr } \beta^N/P_N) \pm 1 \) being bounded that all of the entries of \( R' \) are bounded below. The equation \( \Lambda R = \beta \Lambda \) yields \( \Lambda R' = \beta^N d \Lambda \). Fix \( j \); we deduce
\[ \sum \lambda_i(R')_{ij} = \beta^N d \Lambda / P \lambda_j. \]

Evaluating at \( r \) in \((\mathbb{R}^d)^{++}\), the boundedness below hypothesis yields that the entries of \( R' \) are bounded above, and thus bounded. So, \( R' \) belongs to \( M_n R_P \). Set \( \beta^j \) to be the large eigenvalue function of \( R' \), which we regard as a continuous functions on the pure state space of \( R_P \).

Let \( \gamma \) be a pure state of \( R_P \). Restricted to \( R_P \), it is of the form \( \tau_{B,v} \) for some \((B,v) \) in \( \mathbb{R}^d \times \mathbb{R}^d \) [HSLN, Section 3]. It is routine to verify that \( \tau_{B,v}(\beta^j) \) is the large eigenvalue of the non-negative real matrix \( \gamma(R') \), and the corresponding eigenvectors are \( \tau_{B,v}(\Lambda) \) and \( \tau_{B,v}(\Theta) \). Since \( \gamma \) is a limit of point evaluation states, \( (\#) \) implies that \( \tau_{B,v}(\beta^j) \) is larger than the modulus of all the other eigenvalues of \( \gamma(R') \). By [H1, 2.1], there exists an integer \( D \) so that for all \( D' \geq D, \gamma \left( (R')^{D'} \right) \) is strictly positive. (A real nonnegative matrix with a largest eigenvalue of multiplicity one whose corresponding right and left eigenvectors are strictly positive, is primitive--consider the limit of powers of the matrix normalized so that spectral radius is 1.) Hence there exists a neighbourhood \( U_\gamma \) of \( \gamma \) in the pure state space of \( R_P \) such that for all \( \rho \) in \( U_\gamma, \rho \left( (R')^{D'} \right) \) and the same expression with \( D \) replaced by \( D + 1 \) are strictly positive—it follows that all sufficiently high powers are strictly positive; denote the minimum power, \( E_\gamma \).

Compactness of the pure state space yields a finite open covering \( \{U_i\} \) with corresponding \( E_i \). Set \( E = \max \{ E_i \}; \) then \( \gamma \left( (R')^{E+k} \right) \) is strictly positive for all nonnegative integers \( k \). If \( a \) is an element of \( R_P, \gamma(a) > 0 \) for all pure states \( \gamma \), then not only is \( a \) positive in \( R_P \), but it is actually an order unit. Thus all the entries of \( (R')^{E+k} \) are order units. There thus exists an integer \( F \) such that all the entries of \( P^F R^{E+k} \) belong to \( A^+ \), for \( k = 0, 1, 1 \). This means that all sufficiently large powers of \( P^F R \) have entries from \( A^+ \). To check that \( (P^F R)^E \) satisfies (**) , we simply note that this follows from the entries of \( (R')^E \) being order units.

The intricate hypotheses make the lemma quite easy to prove. However, they really are necessary. Concerning (c) for example, start with any reasonable choice for \( R_0 \) with strictly positive eigenvectors \( \Lambda \) and \( \Theta \) and eigenvalue \( \beta \) etc., and find a matrix \( S \) whose entries are bounded rational functions such that \( S^2 = 0, \Lambda S = 0, \) and \( S \Theta = 0 \) (‘reasonable’ means in particular that the irreducible polynomial satisfied by \( \beta \) is of degree (over \( A \)) less than the size of \( R_0 \), or else no such \( S \) can be found). Pick a lattice point \( w \) such that the growth of \( x^w \) is not dominated by that of \( \beta \). Set \( R = R_0 + x^w S \). Then \( \Lambda \) and \( \Theta \) are also eigenvectors for \( R \) with eigenvalue \( \beta \). If we assume (for example) that the determinant of \( R_0 \) is not zero.
(almost everywhere on \((R^d)^{++}\), for which it is enough that it not vanish at one point), then the argument of the proof will fail, and it is easy to construct an example where the conclusion of the Lemma fails.

Set \(\Lambda = (3, 1, 2), \Theta = (1, 1, 1)^T\) (no variables!), and \(R_0 = \Theta \Lambda + I \in M_3\mathbb{Z}^+\). Now put

\[
S = \begin{pmatrix}
0 & 1 & -1 \\
0 & -1 & 1 \\
0 & -1 & 1
\end{pmatrix},
\]

so \(S^2 = 0\), and \(\Lambda S\) and \(S \Theta\) are both zero. Also \(R_0 S = SR_0 = S\); finally, \(R = R_0 + x_1 S\) admits \(\Lambda\) and \(\Theta\) as strictly positive eigenvectors, but no power can be multiplied by a polynomial without negative coefficients to a nonnegative matrix.

As we pointed out earlier, the hypothesis concerning dividing a polynomial with no negative coefficients is necessary. Moreover, necessary and sufficient conditions are known and easy to deal with [H2, V.6]; in fact, it is sufficient that the \(Q\) of Lemma 13.1(d) belong to \(A\) and be nonzero.

To use these results, especially when \(R\) is the companion matrix of a polynomial (or a power of a companion matrix), we require some elementary algebraic results about manipulating pairs of vectors.

**Lemma 13.2** Let \(A\) be a commutative ring such that all finitely generated projective modules of rank \(n - 1\) are free. Let \(v\) and \(v'\) be rows of size \(n\) with entries from \(A\), and let \(w\) and \(w'\) be columns of size \(n\) with entries from \(A\). Suppose that \(v \cdot w = v' \cdot w'\), and the latter is invertible in \(A\). Then there exists \(P\) in \(\text{GL}(n,A)\) such that \(vP = v'\) and \(P^{-1}w = w'\).

**Proof.** Obviously, we may assume \(v \cdot w = 1\); it is enough to prove the result in the case that \(v' = (1, 0, \ldots, 0) = (w')^T\). In particular, each of \(v\) and \(v'\) is unimodular; as projectives are free, there exists \(P_0\) in \(\text{GL}(n,A)\) such that \(vP_0 = (1, 0, \ldots, 0)\). Relabel \(vP_0\) and \(P_0^{-1}w\) as \(v\) and \(w\) respectively, so that \(w = (1, *, \ldots, *)^T\). If we add any \(A\)-multiple of the first entry of the current \(w\) to any other, the inverse elementary operation subtracts a zero from the first entry of \(v\)—so \(v\) is left unchanged. We may thus assume that \(w = (1, -1, *, \ldots, *)^T\). Let \(W\) denote the column of size \(n - 1\) obtained by removing the first entry of \(w\).

There exists \(Q\) in \(\text{GL}(n-1,A)\) such that \(Q^{-1}W = (-1, 0, \ldots, 0)^T\) (size \(n - 1\)). Applying \(P_1 = 1 \oplus Q\) to our current \(v\) and \(w\), we may assume

\[
v = (1, 0, \ldots, 0) \quad \text{and} \quad w = (1, -1, 0, \ldots, 0).
\]

Now subtract the second entry of the current \(v\) from the first entry; the inverse operation adds the first entry of \(w\) to the second, and we are done.

Let \(P\) be an element of \(A^+\), let \(K = cvx \log P\), and form \(R_K\) as in [HSLN]. Then \(R_K\) embeds as a dense ring of continuous real-valued functions on the pure state space of \(K\) with the additional property that all elements of \(R_K\) that are invertible as continuous functions are already invertible in \(R_K\). By [Sw], any invertible matrix in the (real-valued) continuous function ring can be uniformly approximated by invertible matrices with entries in \(R_K\) (and via determinants, we see that the inverses have entries in \(R_K\) as well). By [HSLN, Section 4], the pure state space of \(R\) is homeomorphic to the compact convex set \(K\), so is contractible. Thus all projectives over the continuous function ring are free—we take the real-valued continuous function ring as our choice for \(A\) in Lemma 13.2.

Now let \(M\) be a square matrix with entries from \(R_P\) (or \(R_K\)). Suppose that \(M\) admits a largest real eigenfunction \(\beta\) (with values in the pure \(1\) state space of \(R_P\)) and there exist right and left eigenvectors \(v\) and \(w\) for \(\beta\) whose entries are integral over \(R_P\) (or \(R_K\)) such that \(\gamma(v \cdot w) > 0\) for all pure states \(\gamma\). As a continuous function on the pure state space, \(v \cdot w\) is invertible. By the lemma above, there exists a strictly positive pair—that is, \(v'\) and \(w'\), whose entries are strictly positive continuous functions on the pure state space—into which \(v\) and \(w\) are simultaneously transformed by a single invertible matrix. Since the invertible
matrix can be approximated (uniformly) by matrices from $R_K$, the usual compactness argument yields a matrix $B$ in $GL(R_K)$ such that both $vB$ and $B^{-1}w$ are strictly positive. Applying the earlier lemma, we deduce the following:

Proposition 13.3 Let $R$ be a matrix with large eigenvalue $\beta$, such that for some integer $k$, there exists $P$ in $A^+$ satisfying:

(i) $\log R^k \subseteq \log P$;

(ii) $\beta$ satisfies $(\#)$ with respect to $\chi$.

(iii) For all sufficiently large integers $J$, $\text{tr} \chi^{(J)}$ has no negative coefficients.

Then there exists $Z$ in $GL(n, R_K)$ and an integer $N$ so that for all $M \geq N$, the matrix $(P'^{\dagger}ZRZ^{-1})^M$ has all of its entries in $A^+$, and satisfies (**); here $P'$ is an element of $A^+$ with $\log P' = \log \text{tr} \beta^k$ for some integer $k$.

This litany of conditions is actually satisfied in a number of situations. For example, if $\chi$ satisfies (iii) above and $\beta$ is a root of a $\chi$ satisfying $(\#)$ (with respect to $\chi$), then we choose $R$ to be the companion matrix for $\chi$. To start, suppose that $(\beta/P)^{\pm1}$ is bounded, where $P$ belongs to $A^+$. For example, relabel $\beta^{n^1}$ as $\beta$, so that the trace of $\chi$ will do. We may also assume that $R_P$ is integrally closed (by altering $P$ somewhat, but not enough to affect the convex hull of its Log set). The companion matrix of $\beta/P$ has entries in $R_P$, so each will belong to $R_P$. Then there exists a power of $P$ so that $P^N R'$ satisfies the conclusion in the proposition above.

If instead we begin with $R$ in $M_n A^+$, then the characteristic polynomial $\chi$ will satisfy (iii) automatically; if we assume that the large eigenvalue satisfies $(\#)$, then the preceding paragraph applies to the companion matrix of the characteristic polynomial. There is a much more difficult argument to show that a power of $R$ can be replaced by a matrix in $M_n A^+$ for which condition (i) holds. So some power of $R$ is conjugate to a matrix which after multiplication by a suitable polynomial, satisfies (**).

Appendix

Lemma A1.1 Let $D$ be a commutative domain, and let $M$ and $N$ be $n \times n$ matrices over $D$. Suppose that $s$ is an integer relatively prime to $n!$, and that $M^n N = NM^n$. If the field of fractions of $D$ contains no non-trivial $s$th roots of unity, then $M^n N$ commutes with $M$.

Proof: We may assume $D$ is already a field. Let $K$ be the splitting field of the characteristic polynomial of $M$. The relative primeness condition guarantees that $K$ contains no new $s$th roots of unity, so contains no non-trivial ones at all. We may thus assume that $M$ is in Jordan normal form. The centralizer of $M$ in $M_n K$ is obtained by fixing an eigenvalue $\lambda$, noting how the number of blocks of a given size occur in the generalized eigenspace attached to $\lambda$, and then taking the corresponding matrix ring over the polynomials in the block. For $M^n$, and $\lambda \neq 0$, the block sizes are the same: If $\lambda^s = \lambda'^s$, then the ratio $\lambda/\lambda'$ is an $s$th root of unity. Since the 0 block of $M^n$ is just the zero matrix, we see immediately that if $M^n N = NM^n$, then $(M^n N) M = M (M^n N)$.

If $M$ is a nonnegative real matrix with only one eigenvalue of absolute value equaling its spectral radius, then that eigenvalue belongs to a unique block of $M$ (when $M$ is written in block triangular form); this will be called the principal block. The following is probably well-known; certainly, portions of it are special cases of results in [Ga]. Compare this with [Tu2, Theorem 2, p. 293], which characterizes the existence of a positive left eigenvector.

Lemma A1.2 Let $M$ be a nonnegative real matrix of size $n$. Suppose $M$ has its spectral radius as an unique eigenvalue of multiplicity one, and all other eigenvalues have strictly smaller absolute value. The following are equivalent:

(a) $M$ has a left strictly positive eigenvector for its large eigenvalue;
(b) in some power of $M$, all rows whose diagonal entry lies in the principal block of $M$ are strictly positive;

(c) $M$ admits only one nonnegative right eigenvector (up to scalar multiple).

When (b) occurs, it will occur for all powers exceeding $n^2 - n + 2$, and no other rows will be strictly positive.

**Proof.** Without loss of generality, we may assume the spectral radius is 1. Let $v$ and $w$ be the left and right nonnegative eigenvectors for 1 respectively, normalized so that the scalar product $vw$ is 1. Then $\{M^k\}$ converges (any reasonable norm) to the matrix $uw$.

Hence, if $v$ is strictly positive, then for all sufficiently large $k$, the rows of $M^k$ corresponding to the positions of $w$ that are not zero are strictly positive. Of course, these can only belong to the principal block, since the support of $w$ must include the principal block.

Suppose (b) holds. Eventually, the rows involving the principal block are all strictly positive, which entails that no other rows are ever strictly positive (otherwise they would belong to the principal block). Moreover, it also follows that all the columns corresponding to the principal block must be zero below that block. Write $M = \begin{bmatrix} A & X \\ 0 & C \end{bmatrix}$, where $A$ is the principal block, and on replacing by a sufficiently high power, we may assume $A$ and $X$ are strictly positive, and $C$ is merely nonnegative. Let $v = (v_A, s)$ be the partitioned left eigenvector for 1. Then $v_A$ is the left eigenvector for $A$ so is strictly positive; as $v_A X + sC = s$, $s$ is strictly positive, so that so is $v$. Hence (b) implies (a).

Assume (a). If $u$ is a right eigenvector for any eigenvalue other than 1, we must have $vu = 0$; as $v$ is strictly positive, $u$ cannot be nonnegative. So (a) implies (c).

Suppose (c) holds; write $M$ in block upper triangular form. The right nonnegative eigenvector for the upper left block extends to an eigenvector for $M$ by simply adjoining zeros; since $M$ always has a nonnegative eigenvector whose eigenvalue is the spectral radius, uniqueness forces the upper left block, $A$, to be the principal block, and the unique nonnegative right eigenvector is simply $w = (w_A, 0)^T$.

Now a routine induction on the number blocks can proceed.

Theorem A1.3 Let $M$ be a square nonnegative real matrix of size $n$ and let $f$ be a real column of size $n$. Set $S = \{i | (M^n f)_j \neq 0 \text{ for some } j \}$. Then $S$ is the lefthand eigenvector for $M^n$.

(a) There exists an integer $m$ such that the column $M^m f$ is strictly positive if and only if $M$ has no zero rows and $v \cdot f > 0$ for all nonnegative left eigenvectors $v$ of $M$ corresponding to nonzero eigenvalues.

(b) There exists an integer $m$ such that the column $M^m f$ is nonnegative if and only if $v_S \cdot f_S > 0$ for all nonnegative left eigenvectors $v_S$ of $M_S$ that correspond to nonzero eigenvalues.

**Proof:** (a). Define $G_M = \lim M : \mathbb{R}^n \to \mathbb{R}^n$ equipped with the limit ordering. Being a finite dimensional vector space, it admits an order unit. By [HCM, 1.3], the pure states are all obtained from left nonnegative eigenvectors of $M$, in the sense that if $v$ is one such, we define a state $V$ on $G_M$ via $[f, k] \mapsto (v \cdot f) / s^{k-1}$ where $s$ is the corresponding positive eigenvalue of $M$. Notice that no left nonnegative eigenvector can correspond to a zero eigenvalue, as $M$ has no rows consisting entirely of zeroes. All pure states are of this form. The stated conditions on $f$ and $M$ are clearly necessary. Suppose they hold. Obviously $V([f, 1]) > 0$ for all pure states $V$ follows from the hypothesis. As $G_M$ admits an order unit, and is obviously unperforated (even a dimension group), [HM, 1.1] guarantees that $[f, 1]$ is an order unit of $G_M$. Set $y = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. As $f$ is an order unit, there exists a positive integer $J$ such that $J[f, 1] - [y, 1] \geq 0$ in $G_M$. From the way the limit ordering is defined, there exists an integer $N$ such that $M^N(Jf - y)$ has only nonnegative entries; thus every entry of $M^N f$ is at least as large as $1 / J$ times that of $M^N y$, which is of course strictly positive.

(b) By (a), that there will exist an integer $N$ (which we may take to be greater than $n$) such that $(M_S)^N f_S$ is strictly positive if and only if for all (extreme) left nonnegative eigenvectors $v_S$ of $M_S$, $v_S \cdot f_S > 0$, and thus $M^N f \geq 0$. 

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Conversely, assume that $M^N f \geq 0$ for some $N$, and $M^N f \neq 0$ we may assume $N > n$. Suppose $[i] \leq [j]$, and $(M^N f)[j] \neq 0$. Applying a sufficiently large further power of $M$ (in fact we only need to use the primitive matrix $M[j]$), we can even obtain that each entry of $(M^N f)[j]$ is strictly positive. It follows immediately that $(M^N f)[i]$ is not zero, and so applying more powers of $M$ (actually $M[i]$) if necessary, we deduce that $(M^N f)[i]$ is also strictly positive. We easily conclude that for some sufficiently large $N$, $(M^S)^N f_S$ is strictly positive. Hence, there will be affirmative answer to the question if and only if either $M^N f = 0$, or for all nonnegative left eigenvectors $v_S$ of the cut-down matrix $M_S$, $v_S \cdot f_S > 0$.

Suppose at the outset, we replace $f$ by $f_0 = M^b f$, where $b$ is max \{#([i]) | i \in \{1, 2, \ldots, n\}\}. Then we eliminate the indices belonging to a minimal $[i]$ if $f_0[i]$ is the zero column, and a simplification occurs.

References


