2009

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Recommended Citation

Elhashash, Abed; Rothblum, Uriel G.; and Szyld, Daniel B. (2009), "Paths of matrices with the strong Perron-Frobenius property converging to a given matrix with the Perron-Frobenius property", Electronic Journal of Linear Algebra, Volume 19.

DOI: https://doi.org/10.13001/1081-3810.1349

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PATHS OF MATRICES WITH THE STRONG PERRON-FROBENIUS PROPERTY CONVERGING TO A GIVEN MATRIX WITH THE PERRON-FROBENIUS PROPERTY

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Abstract. A matrix is said to have the Perron-Frobenius property (strong Perron-Frobenius property) if its spectral radius is an eigenvalue (a simple positive and strictly dominant eigenvalue) with a corresponding semipositive (positive) eigenvector. It is known that a matrix \( A \) with the Perron-Frobenius property can always be the limit of a sequence of matrices \( A(\varepsilon) \) with the strong Perron-Frobenius property such that \( \| A - A(\varepsilon) \| \leq \varepsilon \). In this note, the form that the parameterized matrices \( A(\varepsilon) \) and their spectral characteristics can take are studied. It is shown to be possible to have \( A(\varepsilon) \) cubic, its spectral radius quadratic and the corresponding positive eigenvector linear (all as functions of \( \varepsilon \)); further, if the spectral radius of \( A \) is simple, positive and strictly dominant, then \( A(\varepsilon) \) can be taken to be quadratic and its spectral radius linear (in \( \varepsilon \)). Two other cases are discussed: when \( A \) is normal it is shown that the sequence of approximating matrices \( A(\varepsilon) \) can be written as a quadratic polynomial in trigonometric functions, and when \( A \) has semipositive left and right Perron-Frobenius eigenvectors and \( \rho(A) \) is simple, the sequence \( A(\varepsilon) \) can be represented as a polynomial in trigonometric functions of degree at most six.

Key words. Perron-Frobenius property, Generalization of nonnegative matrices, Eventually nonnegative matrices, Eventually positive matrices, Perturbation.

AMS subject classifications. 15A48.

1. Introduction. A real matrix \( A \) is called nonnegative (respectively, positive) if it is entry-wise nonnegative (respectively, positive) and we write \( A \geq 0 \) (respectively, \( A > 0 \)). This notation and nomenclature is also used for vectors. A column or a row vector \( v \) is called semipositive if \( v \) is nonzero and nonnegative. Likewise, if \( v \) is nonzero and entry-wise nonpositive then \( v \) is called seminegative. We denote the spectral radius of a matrix \( A \) by \( \rho(A) \). We say that a real square matrix \( A \) has the Perron-Frobenius (P-F) property if \( Av = \rho(A)v \) for some semipositive vector \( v \), called a right P-F eigenvector, or simply a P-F eigenvector. We call a semipositive vector \( w \) a left P-F eigenvector if \( A^T w = \rho(A)w \). Moreover, we say that \( A \) possesses the
strong P-F property if \( A \) has a simple, positive, and strictly dominant eigenvalue with a positive eigenvector. Further, define the sets WPFn (respectively, PFn) of \( n \times n \) real matrices \( A \) such that \( A \) and \( A^T \) have the P-F property (respectively, the strong P-F property); see, e.g., [2], [3], [11], [14], where these concepts are studied and used. The P-F property is historically associated with nonnegative matrices; see the seminal papers by Perron [12] and Frobenius [5] or the classic books [1], [7], [15], for many applications.

In [2, Theorem 6.15], it is shown that for any matrix \( A \) with the P-F property, and any \( \varepsilon > 0 \), there exists a matrix \( A(\varepsilon) \) with the strong P-F property such that \( \|A - A(\varepsilon)\| \leq \varepsilon \). In the same paper it is shown that the closure of PFn is not necessarily WPFn. Nevertheless, there are two situations in which we can identify paths of matrices \( A(\varepsilon) \) in PFn converging to any given matrix \( A \) in WPFn. One of these cases is that of normal matrices in WPFn, and the other is when the spectral radius is a simple eigenvalue.

In this note we address the following question: Can we determine a simple expression for the aforementioned matrices \( A(\varepsilon) \) as a function of \( \varepsilon \)? and in particular, can we write \( A(\varepsilon) \) and its corresponding spectral characteristics (spectral radius and corresponding eigenvector) as a polynomial in \( \varepsilon \) of low degree (with matrix coefficients)? In other words, what can we say about a path of matrices \( A(\varepsilon) \) with the strong P-F property that converges to a given matrix \( A \) possessing the P-F property as \( \varepsilon \to 0 \)? We show that it is possible to have \( A(\varepsilon) \) cubic, \( \rho[A(\varepsilon)] \) quadratic and the corresponding positive P-F eigenvector linear in \( \varepsilon \) (where the constant term of the corresponding polynomials are the true characteristics of \( A \)). In particular, the result demonstrates that it is possible to construct a matrix satisfying the strong P-F property such that the matrix, its spectral radius and its positive P-F eigenvector approximate the unperturbed values with order \( O(\varepsilon) \). For normal matrices in WPFn and those in WPFn for which the spectral radius is simple, we present approximating matrices that satisfy the strong P-F property and are polynomials of small degree in trigonometric functions of \( \varepsilon \); the corresponding positive P-F eigenvectors have a similar expansion whereas the corresponding spectral radii are linear in \( \varepsilon \).

2. Polynomial representation of approximating sequences. To answer the questions posed in the introduction on the form of the sequence \( A(\varepsilon) \) with the strong P-F property converging to \( A \) with the P-F property, we begin with the following result.

**Theorem 2.1.** Let \( A \) be an \( n \times n \) real matrix with the P-F property and let \( v \) be a semipositive eigenvector of \( A \) corresponding to \( \rho(A) \). Then, there exist \( n \times n \) real matrices \( A_1, A_2 \) and \( A_3 \), an \( n \)-vector \( v_1 \) and scalars \( \rho_1 \) and \( \rho_2 \) such that for all
sufficiently small positive scalars $\epsilon$, the matrix

$$A(\epsilon) = A + \epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3$$

has the strong P-F property,

$$\rho[(A(\epsilon))] = \rho(A) + \epsilon \rho_1 + \epsilon^2 \rho_2,$$

and a corresponding positive eigenvector of $A(\epsilon)$ is $v(\epsilon) = v + \epsilon v_1$. Furthermore, if $\rho(A)$ is a simple, positive and strictly dominant eigenvalue of $A$, then $A_3 = 0$ and $\rho_2 = 0$.

Proof. The proof is constructive. Let $v$ be a P-F eigenvector of $A$, that is, $v$ is semi-positive and $Av = \rho(A)v$. Let $P$ be an $n \times n$ nonsingular real matrix such that $P^{-1}AP = J(A)$ is in real Jordan canonical form. We may assume that $v$ is the first column of $P$ and that the first diagonal block of $P^{-1}AP$ corresponds to $\rho(A)$. Consider the vector $w$ given by

$$w^T e_i = \begin{cases} 1 & \text{if } v^T e_i = 0 \\ 0 & \text{if } v^T e_i \neq 0, \end{cases}$$

where $e_i$ ($i = 1, \ldots, n$) denotes the $i^{th}$ canonical vector of $\mathbb{R}^n$, i.e., the $i^{th}$ entry of $e_i$ is 1 while all the other entries are 0’s. The vector $w$ satisfies $w^T v = 0$ and its nonzero coordinates are all ones, in particular, $w = 0$ if and only if $v > 0$. For any $\epsilon > 0$, let $P(\epsilon) = P + \epsilon we_1^T$ and let $J(\epsilon) = J(A) + \delta \epsilon e_1 e_1^T$, where $\delta = 0$ if $\rho(A)$ is a simple positive and strictly dominant eigenvalue of $A$, or else $\delta = 1$. Note that for all scalars $\epsilon > 0$, we have that $\rho[J(\epsilon)] = \rho(A) + \delta \epsilon$ is a simple positive and strictly dominant eigenvalue of $J(\epsilon)$ with a corresponding eigenvector $e_1$. Furthermore, for a sufficiently small $\epsilon > 0$, it holds that $P(\epsilon)$ is nonsingular and for any such $\epsilon$, the matrix $B(\epsilon) = P(\epsilon)J(\epsilon)P(\epsilon)^{-1}$ has $\rho[B(\epsilon)] = \rho[J(\epsilon)] = \rho(A) + \delta \epsilon$ as a simple positive and strictly dominant eigenvalue of $B(\epsilon)$ with a corresponding eigenvector $v(\epsilon) = P(\epsilon)e_1 = v + \delta \epsilon w > 0$. Therefore, $B(\epsilon)$ has the strong P-F property. In order to build $A(\epsilon)$ from $B(\epsilon)$, we observe that $P(\epsilon)^{-1}$ can be expressed explicitly by

$$P(\epsilon)^{-1} = P^{-1} - \frac{\epsilon \rho^{-1} we_1^T P^{-1}}{1 + \epsilon \rho e_1^T P^{-1} w},$$

the above is easy to verify directly (in fact, it is an instance of the Sherman-Morrison-Woodbury formula; see, e.g., [6]). We continue by considering $\epsilon > 0$ sufficiently small so that, in addition to satisfying the aforementioned properties, $1 + \epsilon \rho e_1^T P^{-1} w > 0$. Letting $A(\epsilon) := (1 + \epsilon \rho e_1^T P^{-1} w)B(\epsilon)$, we then have that $A(\epsilon)$ has the strong P-F property. Furthermore,

$$A(\epsilon) = (P + \epsilon we_1^T)[J(A) + \delta \epsilon e_1 e_1^T][(1 + \epsilon \rho e_1^T P^{-1} w)P^{-1} - \epsilon P^{-1} we_1^T P^{-1}]$$

$$= (P + \epsilon we_1^T)[J(A) + \delta \epsilon e_1 e_1^T]P^{-1} + \epsilon [(e_1^T P^{-1} w)P^{-1} - P^{-1} we_1^T P^{-1}].$$
Thus, \( A(\varepsilon) \) has a representation
\[
A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 \quad \text{with} \quad A_0 = PJ(A)P^{-1} = A,
\]
\[
A_3 = \delta w e_1^T \left[ (e_1^T P^{-1} - w) P^{-1} - P^{1} - \varepsilon we_1^T P^{-1} \right] \quad \text{and corresponding} \ A_1 \ \text{and} \ A_2.
\]
Further,\[
\rho(A(\varepsilon)) = (1 + \varepsilon e_1^T P^{-1} w) \rho(A) + \varepsilon \delta = \rho(A) + \varepsilon \rho_1 + \varepsilon^2 \rho_2
\]
with \( \rho_1 = \delta + \rho(A) e_1^T P^{-1} w \) and \( \rho_2 = \delta e_1^T P^{-1} w \), and \( v(\varepsilon) = v + \varepsilon w \) is a corresponding positive eigenvector. In particular, if \( \rho(A) \) is a simple, positive and strictly dominant eigenvalue of \( A \), then \( \delta = 0 \) which implies \( A_3 = 0 \) and \( \rho_2 = 0 \).

**Remark 2.2.** Let \( \tau \equiv e_1^T P^{-1} w \). We note that this quantity may be negative. In this case, in (2.2), \( \rho_1 \) and/or \( \rho_2 \) may take negative values, and \( \rho[A(\varepsilon)] \) may be smaller than \( \rho(A) \) for sufficiently small \( \varepsilon > 0 \); the proof of Theorem 2.1 still verifies (2.2) with the corresponding expressions for \( \rho_1 \) and \( \rho_2 \). We note also that when \( \delta = 1 \), the eigenvalues of \( A(\varepsilon) \) (with the exception of \( \rho[A(\varepsilon)] \)) are multiples of eigenvalues of \( A \) by the scalar \( 1 + \tau \varepsilon \), while \( \rho[A(\varepsilon)] = (1 + \tau \varepsilon)(\rho(A) + \varepsilon) \). Thus, when \( \tau < 0 \), and \( \delta = 1 \), for sufficiently small \( \varepsilon > 0 \), we have that \( 0 < 1 + \tau \varepsilon < 1 \) and thus the order of the eigenvalues of \( A(\varepsilon) \) in absolute value is maintained, i.e., it is the same order (in absolute value) as that of the eigenvalues of \( A \).

**Theorem 2.3.** Let \( A \) be a normal \( n \times n \) real matrix with the P-F property, and let \( v \) be its P-F eigenvector. Then, for all sufficiently small positive scalars \( \varepsilon \), there exists an approximating sequence of normal matrices \( A(\varepsilon) \) in \( \text{PF}^n \) (and hence having the strong P-F property) having the form
\[
A(\varepsilon) = A + \varepsilon A_1 + \sum_{1 \leq j + k \leq 2} \sin j \varepsilon (\cos \varepsilon - 1)^k A_{jk} + \varepsilon \sum_{1 \leq j + k \leq 2} \sin j \varepsilon (\cos \varepsilon - 1)^k B_{jk}
\]
where the matrices \( A_1, A_{jk}, \) and \( B_{jk} \) are real \( n \times n \) matrices, their spectral radius has the form \( \rho[A(\varepsilon)] = \rho(A) + \varepsilon \), and the corresponding eigenvector is \( v(\varepsilon) = (\cos \varepsilon) v + (\sin \varepsilon) u \), where \( u^T v = 0 \). Furthermore, if \( \rho(A) \) is simple positive and strictly dominant eigenvalue, then \( B_{jk} = 0 \) and \( \rho[A(\varepsilon)] = \rho(A) \), and if \( A \) has a positive P-F vector, then \( A_{jk} = B_{jk} = 0 \), and \( v(\varepsilon) = v \).

**Proof.** This proof is also constructive. Let \( A \) be a normal \( n \times n \) real matrix with the P-F property and let \( v \) be a P-F eigenvector of \( A \). Then, \( A = PMP^T \), where \( P \) is an \( n \times n \) real orthogonal matrix and \( M \) is a direct sum of \( 1 \times 1 \) real blocks or positive scalar multiples of \( 2 \times 2 \) real orthogonal blocks; see, e.g., [7, Theorem 2.5.8]. We may assume that the first diagonal block of \( M \) is the \( 1 \times 1 \) block \( [\rho(A)] \) and that \( v \) is the first column of \( P \). Note that \( v \) is in this case a unit vector and that it is both a right and a left P-F eigenvector of \( A \). Let \( \varepsilon \) be any given nonnegative real number. We define a matrix \( Q_\varepsilon \) as follows: If \( v \) is a positive vector then define the matrix \( Q_\varepsilon \) to be the \( n \times n \) identity matrix for all \( \varepsilon \in [0, \infty) \). Otherwise consider the vector \( w \) given by (2.3), then the vector \( u := w/||w|| \) is a semipositive vector of unit length which is
orthogonal to \( v \). For all \( \varepsilon \in [0, \frac{\pi}{2}) \) define the orthogonal matrix
\[
Q_\varepsilon := I + (\cos \varepsilon - 1)(vu^T + uu^T) + \sin \varepsilon (uv^T - vu^T),
\]
and then define the matrix
\[
A(\varepsilon) := Q_\varepsilon P(M + \varepsilon \delta e_1 e_1^T)P^T Q_\varepsilon^T,
\]
where \( \delta = 0 \) if \( \rho(A) \) is a simple positive and strictly dominant eigenvalue of \( A \) or else \( \delta = 1 \). Observe that the spectral radius of the matrix \( A(\varepsilon) \) (which is \( \rho(A) + \varepsilon \delta \)) is a simple positive and strictly dominant eigenvalue of \( A(\varepsilon) \) for all \( \varepsilon \in (0, \frac{\pi}{2}) \) and that the vector \( Q_\varepsilon v = (\cos \varepsilon)v + (\sin \varepsilon)u \) is a positive right and left P-F eigenvector of \( A(\varepsilon) \) for all \( \varepsilon \in (0, \frac{\pi}{2}) \). Hence, \( A(\varepsilon) \) is in PFn and thus \( A(\varepsilon) \) has the strong P-F property. Moreover, it follows from (2.5) that \( A(\varepsilon) \) is unitarily diagonalizable and therefore is normal. Taking into consideration the explicit form of \( Q_\varepsilon \) from (2.4), we can write the matrix \( A(\varepsilon) \) as follows:
\[
A(\varepsilon) = Q_\varepsilon (PM + \varepsilon \delta e_1 e_1^T)P^T Q_\varepsilon^T = A + \varepsilon vu^T + \sum_{1 \leq j+k \leq 2} \sin j \varepsilon (\cos \varepsilon - 1)^k A_{jk} + \varepsilon \delta \sum_{1 \leq j+k \leq 2} \sin j \varepsilon (\cos \varepsilon - 1)^k B_{jk},
\]
where \( A_{jk} \) and \( B_{jk} \) are real \( n \times n \) matrices. Furthermore, it follows from (2.5) that \( A(\varepsilon) \to A \) as \( \varepsilon \to 0 \) and that if \( v \) is a positive vector then \( Q_\varepsilon = I \) and thus \( A(\varepsilon) = A + \varepsilon vu^T \).

A normal \( n \times n \) real matrix \( A \) has the P-F property if and only if \( A \) is in WPFn. Hence, Theorem 2.3 gives the form of a normal approximating sequence of matrices \( A(\varepsilon) \) in PFn that converges to a given normal matrix \( A \) in WPFn as \( \varepsilon \to 0 \) (even though it is not true that WPFn is the closure of PFn; see [2]). However, if we consider matrices \( A \) in WPFn for which \( \rho(A) \) is a simple eigenvalue then we obtain the next result.

**Theorem 2.4.** Let \( A \) be a matrix in WPFn such that \( \rho(A) \) is a simple eigenvalue, and let \( u \) and \( v \) be the corresponding right and left P-F eigenvectors, respectively. Then, there is an approximating sequence of matrices \( A(\varepsilon) \) in PFn of the form:
\[
A(\varepsilon) = \sum_{k,m} (\cos k \varepsilon \sin^m \varepsilon) B_{km} + \sum_{k,m} (\varepsilon \cos k \varepsilon \sin^m \varepsilon) C_{km},
\]
where \( k \) and \( m \) are integers such that \( 0 \leq k \leq 6 \) and \( 0 \leq m \leq 2 \); \( B_{km} \) and \( C_{km} \) are real \( n \times n \) matrices, their spectral radius have the form \( \rho[A(\varepsilon)] = \rho(A) + \varepsilon \),
and the corresponding P-F eigenvectors have the form

\begin{align}
(2.6) & \quad u(\varepsilon) = u + \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) \hat{B}_{km} u, \\
(2.7) & \quad v(\varepsilon) = v + \sum_{k,m} (\cos^k \varepsilon \sin^m \varepsilon) \hat{B}_{km} v, 
\end{align}

where $0 \leq k \leq 3$, $0 \leq m \leq 2$, and $\hat{B}_{km}$ are real $n \times n$ matrices. Thus $A(\varepsilon) \to A$ as $\varepsilon \to 0$. Furthermore, if $\rho(A)$ is a strictly dominant eigenvalue then $C_{km} = 0$ for all $k$ and $m$.

**Proof.** Consider a matrix $A$ in WPFn for which $\rho(A)$ is a simple eigenvalue. Let $A = P \left[ [\rho(A)] \oplus J_2 \right] P^{-1}$ be the real Jordan decomposition of matrix $A$, where $J_2$ is the direct sum of all the real Jordan blocks that correspond to eigenvalues other than $\rho(A)$ and suppose that $u$ and $v$ are respectively the first column of $P$ and the transpose of the first row of $P^{-1}$. Thus, $u$ and $v$ are, respectively, right and left eigenvectors of $A$ corresponding to $\rho(A)$. Moreover, $u^T v = v^T u = 1 > 0$ since $P^{-1} P = I$.

Let a nonnegative scalar $\varepsilon$ be given. We begin by finding an orthogonal matrix $Q_\varepsilon$ that converges to the identity matrix as $\varepsilon \to 0$ and that maps the two semipositive vectors $u$ and $v$ simultaneously to a pair of positive vectors for all sufficiently small positive values of $\varepsilon$. Most of the proof that follows is dedicated to constructing $Q_\varepsilon$.

The orthogonal matrix $Q_\varepsilon$ will be defined as the product of three orthogonal matrices $Q_{\langle j, \varepsilon \rangle}$ ($j = 1, 2, 3$) which are rotations.

Partition the set $\langle n \rangle$ by writing $\langle n \rangle = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ where $\alpha_1 = \{ j \mid u_j = v_j = 0 \}$, $\alpha_2 = \{ j \mid u_j > 0 \text{ and } v_j = 0 \}$, $\alpha_3 = \{ j \mid u_j > 0 \text{ and } v_j > 0 \}$, and $\alpha_4 = \{ j \mid u_j = 0 \text{ and } v_j > 0 \}$. Let $k_j$ denote the cardinality of $\alpha_j$ for $j = 1, 2, 3, 4$, and note that $k_3 \neq 0$ because $u^T v > 0$. We may assume that the elements of $\alpha_1$ are the first $k_1$ integers in $\langle n \rangle$, the elements of $\alpha_2$ are the following $k_2$ integers in $\langle n \rangle$, the elements of $\alpha_3$ are the following $k_3$ integers in $\langle n \rangle$, and the elements of $\alpha_4$ are the last $k_4$ integers in $\langle n \rangle$, i.e., $\alpha_1 = \{ 1, 2, \ldots, k_1 \}$, $\alpha_2 = \{ k_1 + 1, k_1 + 2, \ldots, k_1 + k_2 \}$, $\alpha_3 = \{ k_1 + k_2 + 1, k_1 + k_2 + 2, \ldots, k_1 + k_2 + k_3 \}$, and $\alpha_4 = \{ k_1 + k_2 + k_3 + 1, k_1 + k_2 + k_3 + 2, \ldots, n \}$. Let $w_4$ denote the vector $e_{k_1 + k_2 + 1}$. If the cardinality of $\alpha_2$ is zero, i.e., $k_2 = 0$ then let $Q_{\langle 1, \varepsilon \rangle} = I$ otherwise define the vector

$$w_4[\alpha_j] = \begin{cases} 0 & \text{if } j = 1, 3, 4 \\ \frac{1}{||u[\alpha_2]||} u[\alpha_2] & \text{if } j = 2 \end{cases}$$

and let

$$Q_{\langle 1, \varepsilon \rangle} = I + (\cos \varepsilon - 1) (w_1 w_1^T + w_4 w_4^T) - \sin \varepsilon (w_1 w_4^T - w_4 w_1^T)$$

where $\varepsilon \in [0, \delta_1]$ and $\delta_1$ is a sufficiently small positive scalar. Similarly, if the cardin-
nality of $\alpha_4$ is zero, i.e., $k_4 = 0$ then let $Q_{(2, \epsilon)} = I$ otherwise define the vector
\[ w_3[\alpha_j] = \begin{cases} 0 & \text{if } j = 1, 2, 3 \\ \frac{1}{||v[\alpha_4]||}v[\alpha_4] & \text{if } j = 4 \end{cases} \]
and let
\[ Q_{(2, \epsilon)} = I + (\cos \epsilon - 1)(w_1 w_1^T + w_3 w_3^T - \sin \epsilon(w_1 w_3^T - w_3 w_1^T) \]
where $\epsilon \in [0, \delta_2]$ and $\delta_2$ is a sufficiently small positive scalar. Furthermore, if the cardinality of $\alpha_4$ is zero, i.e., $k_1 = 0$ then let $Q_{(3, \epsilon)} = I$ otherwise define the vector $w_2 = (k_1)^{-1/2} \sum_{j=1}^{k_1} e_j$ and let
\[ Q_{(3, \epsilon)} = I + (\cos \epsilon - 1)(w_1 w_1^T + w_3 w_3^T) - \sin \epsilon(w_1 w_3^T - w_3 w_1^T) \]
where $\epsilon \in [0, \delta_3]$ and $\delta_3$ is any scalar in the open interval $(0, \frac{\pi}{2})$. Define the rotation $Q_{\epsilon} := Q_{(3, \epsilon)} Q_{(2, \epsilon)} Q_{(1, \epsilon)}$ for all $\epsilon \in [0, \delta]$ where $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Thus,
\[ Q_{\epsilon} = I + (\cos^3 \epsilon - 1)w_1 w_1^T - (\sin^2 \epsilon)w_1 w_2^T - (\cos \epsilon \sin \epsilon)w_1 w_3^T - (\cos^2 \epsilon \sin \epsilon)w_1 w_4^T + (\cos^2 \epsilon \sin \epsilon)w_2 w_2^T + (\cos \epsilon - 1)w_2 w_3^T - (\sin^2 \epsilon)w_2 w_4^T - (\cos \epsilon \sin^2 \epsilon)w_3 w_3^T + (\cos \epsilon \sin \epsilon)w_3 w_4^T + (\cos \epsilon - 1)w_3 w_4^T - (\sin^2 \epsilon)w_3 w_4^T + (\sin \epsilon)w_4 w_4^T + (\cos \epsilon - 1)w_4 w_4^T. \]
Define the approximating matrix $A(\epsilon)$ as follows:
\[ A(\epsilon) := Q_{\epsilon} P \left[ [\rho(A) + \epsilon] \oplus J_2 \right] (Q_{\epsilon} P)^{-1} \]
for all $\epsilon$ in $[0, \delta]$. The matrix $A(\epsilon)$ is in PFn for all $\epsilon$ in $(0, \delta]$ and the right and left P-F eigenvectors of $A(\epsilon)$ are $Q_\epsilon u$ and $Q_\epsilon v$ respectively, which have the form (2.6) and (2.7). Moreover, it is clear from the form of $Q_\epsilon$ and (2.8) that $A(\epsilon)$ can be written as follows:
\[ A(\epsilon) = Q_{\epsilon} P \left[ [\rho(A) + \epsilon] \oplus J_2 \right] P^{-1}Q_{\epsilon}^T = \sum_{k,m} (\cos^k \epsilon \sin^m \epsilon)B_{km} + \sum_{k,m} (\epsilon \cos^k \epsilon \sin^m \epsilon)C_{km} \]
where $k$ and $m$ are integers such that $0 \leq k \leq 6$ and $0 \leq m \leq 2$; $B_{km}$ and $C_{km}$ are real $n \times n$ matrices; and $A(\epsilon) \to A$ as $\epsilon \to 0$. Furthermore, if $\rho(A)$ is a strictly dominant eigenvalue then $C_{km} = 0$ for all $k$ and $m$. □

Remark 2.5. We note that Theorem 2.4 holds for more general matrices. The spectral radius in this theorem does not need to be a simple eigenvalue. It suffices that a $1 \times 1$ Jordan block corresponding to the spectral radius exists and that to this block there correspond right and left P-F eigenvectors $u$ and $v$, respectively. Furthermore, the approximating matrices in Theorems 2.3 and 2.4 can be written as power series in $\epsilon$ after replacing $\cos \epsilon$ and $\sin \epsilon$ with their corresponding Taylor series.
Acknowledgments. The authors would like to thank an anonymous referee for helpful comments and remarks. Research leading to this note was commenced during a Workshop on Nonnegative Matrices held at the American Institute of Mathematics (AIM), 1–5 December 2008. Support of AIM to the participants of the workshop and the atmosphere of collaboration that the Institute fosters is greatly appreciated. The work of the second author was also supported in part by a grant for the promotion of research at the Technion. The work of the third author was also supported in part by the U.S. Department of Energy under grant DE-FG02-05ER25672.

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