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REACHABILITY MATRICES AND CYCLIC MATRICES

AUGUSTO FERRANTE† AND HARALD K. WIMMER‡

Abstract. We study reachability matrices $R(A, b) = [b, Ab, \ldots, A^{n-1}b]$, where $A$ is an $n \times n$ matrix over a field $K$ and $b$ is in $K^n$. We characterize those matrices that are reachability matrices for some pair $(A, b)$. In the case of a cyclic matrix $A$ and an $n$-vector of indeterminates $x$, we derive a factorization of the polynomial $\det(R(A, x))$.

Key words. Reachability matrix, Krylov matrix, cyclic matrix, nonderogatory matrix, companion matrix, Vandermonde matrix, Hautus test.

AMS subject classifications. 15A03, 15A15, 93B05.

1. Introduction. Let $K$ be a field, and $A \in K^{n \times n}$, $b \in K^n$. The matrix

$$R(A, b) = [b, Ab, \ldots, A^{n-1}b] \in K^{n \times n}$$

is the reachability matrix of the pair $(A, b)$. A matrix $A$ is called cyclic (e.g. in [3], [4]) or nonderogatory (e.g. in [2], [9]), if there exists a vector $b \in K^n$ such that

$$\text{span}\{b, Ab, A^2b, \ldots, A^{n-1}b\} = K^n. \quad (1.1)$$

In that case the pair $(A, b)$ is said to be reachable. Let

$$a(z) = z^n - (a_{n-1}z^{n-1} + \cdots + a_1z + a_0) \quad (1.2)$$

be the characteristic polynomial of $A$. The matrix

$$F_a = \begin{bmatrix} 0 & a_0 \\ 1 & a_1 \\ \vdots & \vdots \\ 0 & a_{n-1} \end{bmatrix} \quad (1.3)$$

is a companion matrix of $a(z)$.
is the companion matrix of the second type [1] associated with (1.2). It is well known (see e.g. [4, p. 299]) that \( A \) is cyclic if and only if \( A \) is similar to the companion matrix \( F_a \). Or equivalently, if \( x_0, \ldots, x_{n-1} \) are indeterminates over \( K \) and \( x := [x_0, \ldots, x_{n-1}]^\top \), then \( A \) is cyclic if and only if the polynomial \( \det R(A, x) \) is not the zero polynomial.

In this note we are concerned with the following questions. When is a given matrix \( M \in K^{n \times n} \) a reachability matrix? How can one factorize the polynomial \( \det R(A, x) \)?

2. Companion and reachability matrices. In this section we characterize those matrices that are reachability matrices for some pair \((A, b)\). We first show that each nonsingular matrix \( M \) is a reachability matrix. Let

\[ e_0 = [1, 0, \ldots, 0]^\top, \ldots, e_{n-1} = [0, \ldots, 0, 1]^\top, \]

be the unit vectors of \( K^n \).

**Theorem 2.1.** Let \( M = [v_0, v_1, \ldots, v_{n-1}] \in K^{n \times n} \) be nonsingular. Then \( M = R(A, b) \) if and only if \( b = v_0 \) and \( A = MF_aM^{-1} \) for some nonsingular companion matrix \( F_a \). In particular, \( M = R(A, v_0) \) with

\[ A = [v_1, \ldots, v_{n-1}, v_0]M^{-1}. \] (2.1)

**Proof.** We have \( e_0 = M^{-1}v_0 \). Hence, if \( b = v_0 \) and \( A = MF_aM^{-1} \) then \( A^ib = MF_a^ie_0 = Me_i = v_i \), and thus \( M = R(A, b) \). We obtain (2.1) if we choose \( F_a = (e_1, e_2, \ldots, e_{n-1}, e_0) \). Conversely, if \( M = R(A, b) \), then \( b = v_0 \), and \( AM = MF_a \) for some companion matrix \( F_a \). Hence if \( M \) is nonsingular then \( A = MF_aM^{-1} \). \( \Box \)

**Theorem 2.2.** Let \( M = [v_0, v_1, \ldots, v_{n-1}] \in K^{n \times n} \) and \( \text{rank} M = r \). The following statements are equivalent.

(i) \( M \) is a reachability matrix.
(ii) Either \( \text{rank} M = n \), i.e. the matrix \( M \) is nonsingular, or

\[ \text{rank} M = \text{rank} [v_0, v_1, \ldots, v_{r-1}] = r < n \] (2.2)

and

\[ \text{Ker}[v_{k-1}, v_k, \ldots, v_{k+r-1}] \subseteq \text{Ker}[v_k, v_{k+1}, \ldots, v_{k+r}], \]

\[ k = 1, 2, \ldots, n - 1 - r. \] (2.3)
Proof. If (2.2) holds then (2.3) means that there exist $c_i \in K$, $i = 0, \ldots, r - 1$, such that
\[ v_{r+k} = \sum_{i=0}^{r-1} c_i v_{i+k}, \quad k = 0, 1, \ldots, n - r - 1. \]  
(2.4)

(i) \Rightarrow (ii) If $M$ is a reachability matrix and rank $M = r < n$ then it is obvious that the conditions (2.2) and (2.4) are satisfied.

(ii) \Rightarrow (i) If rank $M = n$ then it follows from Theorem 2.1 that $M$ is a reachability matrix. Now assume (2.2) and (2.4). Let $Q \in K^{n \times n}$ be nonsingular such that
\[ Q \begin{bmatrix} v_0, \ldots, v_{r-1} \end{bmatrix} = \begin{bmatrix} I_r \\ 0 \end{bmatrix}. \]

Let $\hat{e}_0, \ldots, \hat{e}_{r-1}$ be the canonical unit vectors of $K^r$. Then (2.4) implies
\[ QM = \begin{bmatrix} w_0 & \cdots & w_{r-1} & w_r & \cdots & w_{n-1} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \hat{e}_0 & \cdots & \hat{e}_{r-1} & w_r & \cdots & w_{n-1} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, \]  
(2.5)

and the vectors $w_i$ satisfy
\[ w_{r+k} = \sum_{i=0}^{r-1} c_i w_{i+k}, \quad k = 0, 1, \ldots, n - r - 1. \]

Set
\[ \hat{A} = \begin{bmatrix} 0 & 0 & c_0 \\ 1 & 0 & c_1 \\ & \ddots & \ddots \\ 0 & 0 & 1 & c_{r-1} \end{bmatrix}. \]  
(2.6)

Then
\[ w_r = [c_0, c_1, \ldots, c_{r-1}]^T = \hat{A} \hat{e}_{r-1} = \hat{A} \hat{A}^{r-1} \hat{e}_0 = \hat{A}^r \hat{e}_0, \]

and
\[ [\hat{e}_0, \hat{A} \hat{e}_0, \ldots, \hat{A}^{r-1} \hat{e}_0, \hat{A}^r \hat{e}_0, \ldots, \hat{A}^{n-1} \hat{e}_0] = [\hat{e}_0, \ldots, \hat{e}_{r-1}, w_r, \ldots, w_{n-1}]. \]

Hence the matrix $QM$ in (2.5) can be written as
\[ QM = R(A, b) \quad \text{with} \quad A = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} \in K^{n \times n}, \quad b = \begin{bmatrix} \hat{e}_0 \\ 0 \end{bmatrix} \in K^n, \]

and we obtain $M = R(Q^{-1} A Q, Q^{-1} b)$. \[ \square \]
3. A factorization theorem. We first characterize companion matrices in terms of reachability matrices. Let \( b = (b_0, b_1, \ldots, b_{n-1})^\top \in K^n \). We call \( b(z) = (1, z, \ldots, z^{n-1})b = b_0 + b_1z + \cdots + b_{n-1}z^{n-1} \) (3.1) the polynomial associated to \( b \).

**Proposition 3.1.** Let \( a(z) = z^n - \sum a_i z^i \) be the characteristic polynomial of \( A \in K^{n \times n} \), and let \( F_a \) be the companion matrix in (1.3). Then \( A = F_a \) if and only if
\[
R(A, b) = b(A) \quad \text{for all} \quad b \in K^n. \tag{3.2}
\]

**Proof.** It is obvious that \( A \) is a companion matrix of the form (1.3) if and only if
\[
A[e_0, e_1, \ldots, e_{n-2}] = [e_1, e_2, \ldots, e_{n-1}] \tag{3.3}
\]
Assume now that (3.2) is satisfied. Choose \( b = e_0 \). Then \( b(z) = 1 \) and \( b(A) = I \). Therefore
\[
R(A, e_0) = [e_0, Ae_0, \ldots, A^{n-1}e_0] = I = [e_0, e_1, \ldots, e_{n-1}].
\]
Hence we obtain (3.3), and we conclude that \( A = F_a \). To prove the converse we have to show that
\[
R(F_a, b) = b(F_a) \tag{3.4}
\]
holds for all \( b = \sum_{i=0}^{n-1} b_i e_i \). We have \( e_i = F_a^i e_0 \), \( i = 0, \ldots, n-1 \). From \( R(F_a, e_0) = I \) and \( R(F_a, e_i) = F_a^i R(F_a, e_0) \) follows \( R(F_a, e_i) = F_a^i \). Therefore
\[
R(F_a, b) = \sum_{i=0}^{n-1} b_i R(F_a, e_i) = \sum_{i=0}^{n-1} b_i F_a^i = b(F_a). \quad \blacksquare
\]

Suppose \( A \) is cyclic. Let \( S \) be nonsingular such that \( SAS^{-1} = F_a \), and let the polynomial \((Sb)(z)\) be defined in analogy to (3.1). Then \( SR(A, b) = R(F_a, Sb) \). From (3.2) we obtain
\[
R(A, b) = S^{-1} (Sb)(F_a). \tag{3.5}
\]
Note that for all \( b \in K^n \) we have \( A R(A, b) = R(A, b) F_a \). Hence, if the pair \((A, b)\) is reachable then the matrix \( S = R(A, b)^{-1} \) satisfies
\[
SAS^{-1} = F_a. \tag{3.6}
\]
The identity (3.6) can be found in [6, Section 6.1].
For the following observation we are indebted to a referee. Suppose $A$ is a matrix with distinct eigenvalues, and $X^{-1}AX = D$ is a Jordan form. Then the corresponding companion matrix $F_a$ is similar to the diagonal matrix $D$. The similarity transformation $VF_aV^{-1} = D$ is accomplished by a Vandermonde matrix $V$ whose nodes are the eigenvalues of $A$ (see e.g. [12, Section 1.11]). One can write $V$ as a reachability matrix, that is, $V = R(D,e)$, where $e = (1,1,\ldots,1)^\top$.

Let $a_j(z)$, $j = 1,\ldots,r$, be the monic irreducible factors of the polynomial $a(z)$ in (1.2). Suppose $\deg a_j(z) = \ell_j$ and
\begin{equation}
    a(z) = a_1(z)^{m_1} \cdots a_r(z)^{m_r},
\end{equation}
such that $\sum_{j=1}^r m_j \ell_j = n$. Let $F_{a_j} \in K^{\ell_j \times \ell_j}$ be the corresponding companion matrices. The main result of this section is the following.

**Theorem 3.2.** Let $A$ be cyclic with characteristic polynomial $a(z)$, and let (3.7) be the prime factorization of $a(z)$. Suppose $SAS^{-1} = F_a$ and $\det S = 1$. \( (3.8) \)

Set $g_j(x) = \det(Sx)(F_{a_j})$, $j = 1,\ldots,r$. Then
\begin{equation}
    \det R(A,x) = (g_1(x))^{m_1} \cdots (g_r(x))^{m_r}. \quad (3.9)
\end{equation}
The polynomials $g_1(x),\ldots,g_r(x)$ are irreducible, and homogeneous of degree $\ell_1,\ldots,\ell_r$, respectively.

**Proof.** Define
\[
    C(a_j, m_j) = \begin{pmatrix}
        F_{a_j} & I_{\ell_j} & 0 & \cdots & 0 \\
        0 & F_{a_j} & I_{\ell_j} & \cdots & 0 \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        \vdots & \vdots & \vdots & \ddots & F_{a_j} \\
        \vdots & \vdots & \vdots & \ddots & F_{a_j}
    \end{pmatrix}_{m_j \ell_j \times m_j \ell_j}
\]

Then
\begin{equation}
    TF_aT^{-1} = \text{diag} \left( C(a_1,m_1), \ldots, C(a_r,m_r) \right) \quad (3.10)
\end{equation}
for some $T \in K^{n \times n}$. The right hand side of (3.10) is the rational canonical form (the Frobenius canonical form) of $F_a$. Let $\hat{b} \in K^n$. From (3.4) be obtain
\[
    R(F_a,\hat{b}) = \hat{b}(F_a) = T^{-1} \hat{b} \left[ \text{diag} \left( C(a_1,m_1), \ldots, C(a_r,m_r) \right) \right] T.
\]
Hence
\begin{equation}
    \det R(F_a,\hat{b}) = \det \hat{b}(C(a_1,m_1)) \cdots \det \hat{b}(C(a_r,m_r)) = \left( \det \hat{b}(F_{a_1}) \right)^{m_1} \cdots \left( \det \hat{b}(F_{a_r}) \right)^{m_r}. \quad (3.11)
\end{equation}
Suppose \( SAS^{-1} = F_a \) and \( \det S = 1 \). Then (3.5) implies \( \det R(A, x) = \det R(F_a, Sx) \). Taking \( \hat{b} = Sx \) in (3.11) we obtain (3.9).

Suppose one of the polynomials \( g_j(x) \) is reducible. E.g. let \( g_1(x) = p(x)q(x) \) and \( \deg p \geq 1, \deg q \geq 1 \). Then \( x = S^{-1}(-1, z, 0, \ldots, 0)^\top \) yields \( (Sx)(F_{a_1}) = -F_{a_1} + zI \). Hence \( g_1(x) = \det (-F_{a_1} + zI) = a_1(z) \). On the other hand \( g_1(x) = \tilde{p}(z)\tilde{q}(z) \), and \( \deg \tilde{p} \geq 1, \deg \tilde{q} \geq 1 \). This is a contradiction to the irreducibility of \( a_1(z) \).

Since \( F_{a_j} \) is of size \( \ell_j \times \ell_j \) we obtain \( g_j(\lambda x) = \lambda^\ell_j g_j(x) \). Thus, \( g_j(x) \) is homogeneous of degree \( \ell_j \).

We now assume that the characteristic polynomial \( a(z) \) of \( A \) splits over \( K \). If \( \lambda_1, \ldots, \lambda_r \) are the different eigenvalues of \( A \) then

\[
a(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_r)^{m_r}.
\] (3.12)

In that case \( a_j(z) = (z - \lambda_j) \) and \( F_{a_j} = (\lambda_j) \), and

\[
g_j(x) = [1, \lambda_j, \lambda_j^2, \ldots, \lambda_j^{n-1}]S \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}, \quad j = 1, \ldots, r.
\] (3.13)

The factors \( g_j(x) \) in (3.13) are related to the Popov-Belevitch-Hautus controllability test (see e.g. [11, p. 93]). It is known that - up multiplicative constants - the row vectors

\[
w_j^\top = [1, \lambda_j, \ldots, \lambda_j^{n-1}], \quad j = 1, \ldots, r,
\]

are the left eigenvectors of \( F_{a_j} \). Then \( SAS^{-1} = F_a \) implies that \( v_j^\top = w_j^\top S \) are the left eigenvectors of \( A \). Hence

\[
v_j^\top b = g_j(b), \quad j = 1, \ldots, r.
\] (3.14)

Therefore we obtain the PBH criterion in the special case of cyclic matrices.

**Corollary 3.3.** Let \( A \) be cyclic. The following statements are equivalent. (i) The pair \((A, b)\) is reachable. (ii) If

\[
v^\top (A - \lambda I) = 0, \quad v \in K^n, \quad v \neq 0,
\]

then \( v^\top b \neq 0 \).

**Proof.** Because of (3.14) we can rewrite (ii) in the form

\[
g_j(b) \neq 0, \quad j = 1, \ldots, r.
\] (3.15)
From (3.9) follows that (3.15) is equivalent to \( \det R(A, b) \neq 0. \]

We illustrate Theorem 3.2 with an example. Consider

\[
A = \begin{bmatrix}
-6 & -38 & 6 & -4 & 281 \\
-11 & -131 & 10 & -5 & 928 \\
11 & -155 & -6 & -16 & 1191 \\
1 & -170 & 1 & -11 & 1253 \\
-1 & -21 & 1 & -1 & 151
\end{bmatrix}, \quad b = \begin{bmatrix}
3 \\
0 \\
4 \\
0 \\
0
\end{bmatrix}.
\]

Then

\[
R(A, b) = \begin{bmatrix}
3 & 6 & 5 & 2 & 6 \\
0 & 7 & 0 & 0 & 8 \\
4 & 9 & 6 & 3 & 4 \\
0 & 7 & 1 & 0 & 5 \\
0 & 1 & 0 & 0 & 1
\end{bmatrix}, \quad \det R(A, b) = 1,
\]

and the pair \((A, b)\) is reachable. Set \(S = R(A, b)^{-1}\). Then

\[
S = \begin{bmatrix}
3 & -16 & -2 & -3 & 133 \\
0 & -1 & 0 & 0 & 8 \\
0 & 2 & 0 & 1 & -21 \\
-4 & 19 & 3 & 2 & -150 \\
0 & 1 & 0 & 0 & -7
\end{bmatrix},
\]

and \(SAS^{-1} = F_a\). We have

\[
\det(zI - A) = a(z) = z^5 + 3z^4 - 6z^3 - 10z^2 - 21z - 9 = (z - 1)^4(z + 3)^2 = (a_1(z))^3(a_2(z))^2.
\]

Hence \(\det R(A, x) = g_1(x)^3 g_2(x)^2\) with

\[
g_1(x) = [1, 1, 1, 1, 1]Sx = -x_0 + 5x_1 + x_2 - 37x_4
\]

and

\[
g_2(x) = [1, -3, 9, -27, 81]Sx = 111x_0 - 427x_1 - 83x_2 - 48x_3 + 3403x_4.
\]

We conclude with some remarks which place our study into a larger context. Matrices of the form \(K_r(A, b) = [b, Ab, \ldots, A^{r-1}b]\), \(1 \leq r \leq n\), are known as Krylov matrices (see e.g. [9, p. 646]). Thus \(R(A, b) = K_n(A, b)\). We refer to [8] for an investigation of numerical aspects of Krylov and reachability matrices. The concept of Faddeev reachability matrix was introduced in [5] and further elaborated in [10]. A “spectral factorization” of \(R(A, b)\) is due to [7] (see also [13]).
REFERENCES