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KANTOROVICH TYPE INEQUALITIES FOR ORDERED LINEAR SPACES

MAREK NIEZGODA†

Abstract. In this paper Kantorovich type inequalities are derived for linear spaces endowed with bilinear operations $\circ_1$ and $\circ_2$. Sufficient conditions are found for vector-valued maps $\Phi$ and $\Psi$ and vectors $x$ and $y$ under which the inequality

$$\Phi(x) \circ_2 \Phi(y) \leq \frac{C+c}{2\sqrt{Cc}} \Psi(x \circ_1 y)$$


Key words. Kantorovich type inequality, Linear space, Bilinear operation, Preorder, $C^*$-algebra, Unital positive map, Matrix.

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1. Introduction. Let $A$ be an $n \times n$ positive definite matrix such that $0 < mI_n \leq A \leq MI_n$ for some scalars $0 < m < M$. The Kantorovich inequality asserts that (cf. [16, pp. 89-90], [20, p. 28])

$$z^*Az \cdot z^*A^{-1}z \leq \frac{(M+m)^2}{4Mm} (z^*z)^2,$$

where $z \in \mathbb{C}^n$ is a column vector and $^*$ means conjugate transpose. The constant $\kappa = \frac{(M+m)^2}{4Mm}$ is called Kantorovich constant [21, p. 688]. Note that $\sqrt{\kappa} = \frac{M+m}{2\sqrt{Mm}}$ is the ratio of the arithmetic to geometric mean of $M$ and $m$.

Let $V$ be a linear space over $\mathbb{C}$ or $\mathbb{R}$ equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. Dragomir [11, Theorem 2.2] proved the following Kantorovich type inequality:

$$\|x\| \|y\| \leq \frac{|C+c|}{2\sqrt{\text{Re}(Cc)}} |\langle x, y \rangle| \text{ for } x, y \in V,$$

provided scalars $c, C$ satisfy $\text{Re}(Cc) > 0$ and

$$0 \leq \text{Re} \langle x - cy, Cy - x \rangle$$

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Inequality (1.2) is a reverse of Schwarz’s inequality
\[ |\langle x, y \rangle| \leq \|x\|\|y\| \quad \text{for } x, y \in V. \]  

A consequence of (1.4) and (1.2) is the following result of Bourin [5, Theorem 2.9]:
\[ \sum_{j=1}^{n} a_j b_j \leq \frac{M+m}{2\sqrt{Mm}} \sum_{j=1}^{n} a_j b_j, \]
where \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) are \( n \)-tuples of positive numbers with \( 0 < m \leq \frac{a_j}{b_j} \leq M, j = 1, \ldots, n \), and, in addition, \( a_1 \geq \ldots \geq a_n \) and \( b_1 \geq \ldots \geq b_n \) are the entries of \( a \) and \( b \), respectively, arranged in nonincreasing order.

For other Kantorovich type inequalities, the reader is referred to [2, 5, 6, 7, 16, 18, 20, 21].

In this paper we study Kantorovich type inequalities in the framework of linear spaces equipped with binary operations \( \circ_1 \) and \( \circ_2 \). We provide conditions on two (vector-valued) maps \( \Phi \) and \( \Psi \) and vectors \( x \) and \( y \) implying the validity of the inequality
\[ \Phi(x) \circ_2 \Phi(y) \leq \frac{C+c}{2\sqrt{Cc}} \Psi(x \circ_1 y). \]

Complementary inequalities are also derived.

2. Results. Throughout this paper, unless otherwise stated, for \( i = 1, 2 \),
\[ V_i \text{ and } X_i \text{ are linear spaces over } \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}, \]
and
\[ \circ_i : V_i \times V_i \to X_i \text{ is an } \mathbb{F}\text{-bilinear binary operation.} \]

For example, \( \circ_2 \) can be interpreted as a real inner product if \( X_i = \mathbb{R} \), or as an algebra multiplication if \( V_i = X_i \) is a distributive algebra.

In addition, we assume that \( L_i \subset X_i \) is a convex cone inducing cone preorder \( \leq_i \) on \( X_i \) by
\[ y \leq_i x \text{ iff } x - y \in L_i. \]

We also assume that
\[ 0 \leq_i x \circ_i x, \quad \text{i.e., } x^2 = x \circ_i x \in L_i, \quad \text{for } x \in V_i. \]
We denote
\begin{equation}
\text{Sym} (u, w) = \frac{1}{2} (u \circ_2 w + w \circ_2 u) \quad \text{for } u, w \in V_2.
\end{equation}

The following theorem is inspired by [11, Theorem 2.2] (cf. [12, Theorem 1]).

**Theorem 2.1.** Under the above notation and assumptions, let \( \Phi : A \to V_2 \) and \( \Psi : B \to X_2 \) be maps, where \( A \subset V_1 \) and \( B \subset X_1 \) are nonempty sets. Let \( x, y \in A \) and \( C, c \in \mathbb{F} \) with \( Cc > 0 \) and \( C + c > 0 \) be such that
\begin{enumerate}[(i)]
  \item \begin{equation}
  0 \leq_1 (x - cy) \circ_1 (Cy - x),
  \end{equation}
  \item \begin{equation}
  x \circ_1 y = y \circ_1 x,
  \end{equation}
  \item \( L_1 \subset B \) and \( \alpha x \circ_1 y \in B \) for \( \alpha \in \{1, C + c\} \).
\end{enumerate}

Assume that
\begin{enumerate}[(i)]
  \item \begin{equation}
  \Phi(v) \circ_2 \Phi(v) \leq_2 \Psi(v \circ_1 v) \quad \text{for } v \in \{x, y\},
  \end{equation}
  \item \begin{equation}
  b \leq_1 a \quad \text{implies} \quad \Psi(b) \leq_2 \Psi(a) \quad \text{for } a, b \in L_1,
  \end{equation}
  \item \begin{equation}
  \Psi(\alpha a) = \alpha \Psi(a) \quad \text{for } \alpha = C + c \text{ and } a = x \circ_1 y,
  \end{equation}
  \item \begin{equation}
  \Psi(x \circ_1 x + \alpha \Psi(y \circ_1 y) \leq_2 \Psi(x \circ_1 x + \alpha y \circ_1 y) \quad \text{for } \alpha = Cc.
  \end{equation}
\end{enumerate}

Then the following Kantorovich type inequality holds:
\begin{equation}
\text{Sym} \left[ \Phi(x), \Phi(y) \right] \leq_2 \frac{C + c}{2\sqrt{Cc}} \Psi(x \circ_1 y).
\end{equation}

In particular, if \( \Phi(x) \) and \( \Phi(y) \) commute with respect to \( \circ_2 \), then
\begin{equation}
\Phi(x) \circ_2 \Phi(y) \leq_2 \frac{C + c}{2\sqrt{Cc}} \Psi(x \circ_1 y).
\end{equation}

**Remark 2.2.** In some cases Theorem 2.1 can be simplified.

(a) If \( \Psi \) is linear then conditions (2.6)-(2.7) hold automatically and are superfluous in the statement of Theorem 2.1. If in addition \( \Psi \) is positive (i.e. \( \Psi(L_1) \subset L_2 \)) then (2.5) can be dropped out.
If $\Phi = \Psi$ then condition (2.4) represents a Kadison type inequality (see (2.19)). On the other hand, if $\Phi(x) = [\Psi(x^2)]^{1/2}$ then (2.4) holds automatically (cf. Corollary 2.5 and Theorem 2.7, part II).

Condition (2.4) is necessary for (2.8) and (2.9) to hold. In fact, if $x = y$ then (2.3) is met for $c = C = 1$. In this case, each of (2.8) and (2.9) reduces to (2.4).

**Proof of Theorem 2.1.** Since the operation $\circ_1$ is bilinear, (2.3) gives

$$0 \leq Cx \circ_1 y - x \circ_1 x - Cy \circ_1 y + cy \circ_1 x,$$

which is equivalent to

$$x \circ_1 x + Cy \circ_1 y \leq Cx \circ_1 y + cy \circ_1 x,$$

because $\leq_1$ is a cone preorder. Now, (ii) implies

$$x \circ_1 x + Cy \circ_1 y \leq (C + c)x \circ_1 y. \quad (2.10)$$

By (2.1), $x \circ_1 x + Cy \circ_1 y \in L_1$, because $Cc > 0$ and $L_1$ is a convex cone. Therefore (2.10) yields $(C + c)x \circ_1 y \in L_1$. Using (2.7), (2.10), (2.5) and (2.6), we derive

$$\Psi(x \circ_1 x) + Cy \Psi(y \circ_1 y) \leq \Psi(x \circ_1 x + Cy \circ_1 y) \leq (C + c)\Psi(x \circ_1 y).$$

Consequently, by (2.4), we obtain

$$\Phi(x) \circ_2 \Phi(x) + Cy \circ_2 \Phi(y) \leq (C + c)\Phi(x \circ_1 y). \quad (2.11)$$

Hence, by $Cc > 0$,

$$\frac{1}{\sqrt{Cc}} \Phi(x) \circ_2 \Phi(x) + \sqrt{Cc} \Phi(y) \circ_2 \Phi(y) \leq \frac{C + c}{\sqrt{Cc}} \Psi(x \circ_1 y). \quad (2.12)$$

On the other hand, by (2.1),

$$0 \leq \left(\frac{1}{\sqrt{Cc}} \Phi(x) - \sqrt{Cc} \Phi(y)\right) \circ_2 \left(\frac{1}{\sqrt{Cc}} \Phi(x) - \sqrt{Cc} \Phi(y)\right).$$

In consequence, by the bilinearity of $\circ_2$,

$$0 \leq \frac{1}{\sqrt{Cc}} \Phi(x) \circ_2 \Phi(x) - \Phi(x) \circ_2 \Phi(y) + \sqrt{Cc} \Phi(y) \circ_2 \Phi(x) + \sqrt{Cc} \Phi(y) \circ_2 \Phi(y).$$

Hence

$$\Phi(x) \circ_2 \Phi(y) + \Phi(y) \circ_2 \Phi(x) \leq \frac{1}{\sqrt{Cc}} \Phi(x) \circ_2 \Phi(x) + \sqrt{Cc} \Phi(y) \circ_2 \Phi(y),$$
because $\leq_2$ is induced by a convex cone. Simultaneously, by (2.2),

$$2\text{Sym} [\Phi(x), \Phi(y)] = \Phi(x) \circ_2 \Phi(y) + \Phi(y) \circ_2 \Phi(x).$$

Therefore we get

$$2\text{Sym} [\Phi(x), \Phi(y)] \leq_2 \frac{1}{\sqrt{Cc}} \Phi(x) \circ_2 \Phi(x) + \sqrt{Cc} \Phi(y) \circ_2 \Phi(y). \tag{2.13}$$

Combining (2.12) and (2.13), we obtain the required inequality (2.8). □

**Remark 2.3.** Let $H$ be a real linear space with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. It is not hard to verify that Dragomir’s result (1.2) (with $F = \mathbb{R}$ and $C, c > 0$) can be obtained from Theorem 2.1 by setting $V_1 = H$, $V_2 = X_1 = X_2 = \mathbb{R}$, $L_1 = L_2 = \mathbb{R}_+$,

$$x \circ_1 y = \langle x, y \rangle \quad \text{for} \ x, y \in H, \quad \text{and} \quad \alpha \circ_2 \beta = \alpha \beta \quad \text{for} \ \alpha, \beta \in \mathbb{R},$$

$$\Phi(x) = \|x\| \quad \text{for} \ x \in H, \quad \text{and} \quad \Psi(\alpha) = |\alpha| \quad \text{for} \ \alpha \in \mathbb{R}.$$  

In this case, (2.11) takes the form of inequality from [12, Lemma 1].

If $X_i$ is an algebra with unity $e_i$ and convex cone $L_i \subset X_i \ (i = 1, 2)$, then a linear map $\Psi : X_1 \rightarrow X_2$ is said to be a **unital positive map** if $\Psi(e_1) = e_2$ and $\Psi L_1 \subset L_2$.

**Theorem 2.4.** Under the assumptions before Theorem 2.1, let $V_i = X_i$ and let $(V_i, \circ_i)$ be algebra with unity $e_i \ (i = 1, 2)$.

Let $x \in V_1$ be such that

$$0 \leq_1 (x - ce_1) \circ_1 (Ce_1 - x) \tag{2.14}$$

for some scalars $C, c \in F$ with $Cc > 0$ and $C + c > 0$.

Assume that $\Psi : V_1 \rightarrow V_2$ is a positive linear map (i.e., $\Psi L_1 \subset L_2$) and $\Phi : V_1 \rightarrow V_2$ is a unital map (i.e., $\Phi(e_1) = e_2$) satisfying

$$\Phi(x) \circ_2 \Phi(x) \leq_2 \Psi(x \circ_1 x) \quad \text{and} \quad e_2 \leq_2 \Psi(e_1). \tag{2.15}$$

Then we have the inequality

$$\Phi(x) \leq_2 \frac{C + c}{2\sqrt{Cc}} \Psi(x). \tag{2.16}$$
Proof. Set \( y = e_1 \). Conditions (2.5)-(2.7) are fulfilled, because \( \Psi \) is a positive linear map. Moreover, (2.15) gives (2.4). According to Theorem 2.1, we get (2.9) with \( y = e_1 \) and \( \Phi(y) = e_2 \). This proves (2.16).

Corollary 2.5. Under the assumptions of Theorem 2.4 for \( V_i, X_i, L_i, \circ_i \) and \( x \), suppose that for each \( a \in L_2 \) there exists unique vector \( b = a^{1/2} \in L_2 \) such that \( b^2 = b \circ_2 b = a \).

Assume \( \Psi : V_1 \to V_2 \) is a unital positive map. If (2.14) is met then we have the inequality

\[
[\Psi(x^2)]^{1/2} \leq \frac{C + c}{2\sqrt{Cc}} \Psi(x).
\]

(2.17)

Proof. Define

\[
\Phi(v) = [\Psi(v^2)]^{1/2} \quad \text{for } v \in V_1.
\]

(2.18)

Then \( \Phi \) is unital, since \( \Psi \) is so. It follows from (2.18) that (2.15) holds. Now, by using (2.16), we get (2.17). \( \square \)

By \( \mathbb{M}_p \) and \( \mathbb{H}_p \) we denote the linear spaces, respectively, of \( p \times p \) complex matrices, and of \( p \times p \) Hermitian matrices. The Loewner cone of all \( p \times p \) positive semidefinite matrices is denoted by \( \mathbb{L}_p \). For matrices \( A, B \in \mathbb{M}_p \) we write \( B \leq A \) if \( A - B \in \mathbb{L}_p \). The symbol \( I_p \) stands for the \( p \times p \) identity matrix.

Remind that a linear map \( \Psi : \mathbb{M}_n \to \mathbb{M}_k \) is said to be a unital positive map if \( \Psi(I_n) = I_k \) and \( \Psi \mathbb{L}_n \subset \mathbb{L}_k \) (see [4, 14]). It is known that

\[
[\Psi(A)]^2 \leq \Psi(A^2) \quad \text{for } A \in \mathbb{L}_n
\]

(Kadison’s inequality; see [1], [4, p. 2], [8]).

Remark 2.6. (a) In the matrix setting, (2.17) reduces to a result of Ando [1]. Cf. also [6, Corollaries 2.5 and 2.9] and [17, Corollary 2.6, part (ii), \( p = 2 \)].

(b) Inequality (2.17) generalizes a result of Liu and Neudecker [15, Proposition 5] (see also [6, Lemma 1.1]):

\[
(U^*X^2U)^{1/2} \leq \frac{M + m}{2\sqrt{Mm}} U^*XU,
\]

where \( U \) is an \( n \times k \) matrix such that \( U^*U = I_k \), and \( X \) is an \( n \times n \) positive definite matrix satisfying

\[
0 < m \leq \lambda_j(X) \leq M, \ j = 1, \ldots, n, \quad \text{for some scalars } m, M.
\]

(2.20)
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To see this, consider

\[ V_1 = X_1 = \mathbb{M}_n, \quad V_2 = X_2 = \mathbb{M}_k, \quad L_1 = \mathbb{L}_n, \quad L_2 = \mathbb{L}_k, \]

with the usual matrix multiplication, and

\[ \Psi(A) = U^*AU \quad \text{for} \quad A \in \mathbb{M}_n, \]

where \( U \) is an \( n \times k \) matrix such that \( U^*U = I_k \).

We now interpret Theorem 2.1 in the framework of \( C^* \)-algebras \( V_i, i = 1, 2 \), and unital positive maps. Here, for given \( x, y \in V_i \), \( y \leq x \) means \( x - y = a^*a \) for some \( a \in V_i \).

**Theorem 2.7.** For \( i = 1, 2 \), let \( V_i = X_i \) be a \( C^* \)-algebra with unity \( e_i \) and convex cone \( L_i = \{ a^*a : a \in V_i \} \) of all nonnegative elements of \( V_i \).

Let \( x, y \in V_1 \) be two elements such that \( x^*y = y^*x \) and

\[ (x - cy)^*(Cy - x) \geq 0 \quad \text{for some positive scalars} \ C, c. \]

Assume that \( \Psi : V_1 \to V_2 \) is a unital positive map.

(I). If

\[ (\Psi(v))^*\Psi(v) \leq \Psi(v^*v) \quad \text{for} \quad v \in \{x, y\}, \]

then we have the inequality

\[ \frac{1}{2} |(\Psi(x))^*\Psi(y) + (\Psi(y))^*\Psi(x)| \leq \frac{C + c}{2\sqrt{Cc}} \Psi(x^*y). \]

If, in addition, \( \Psi(x) \) and \( \Psi(y) \) are two commuting self-adjoint elements of \( V_2 \), then (2.24) becomes

\[ \Psi(x)\Psi(y) \leq \frac{C + c}{2\sqrt{Cc}} \Psi(x^*y). \]

(II). We have the inequality

\[ \frac{1}{2} \left( [\Psi(x^*x)]^{1/2} [\Psi(y^*y)]^{1/2} + [\Psi(y^*y)]^{1/2} [\Psi(x^*x)]^{1/2} \right) \]

\[ \leq \frac{C + c}{2\sqrt{Cc}} \Psi(x^*y). \]

If, in addition, \([\Psi(x^*x)]^{1/2}\) and \([\Psi(y^*y)]^{1/2}\) are two commuting elements of \( V_2 \), then we have the inequality

\[ [\Psi(x^*x)]^{1/2} [\Psi(y^*y)]^{1/2} \leq \frac{C + c}{2\sqrt{Cc}} \Psi(x^*y). \]
Proof. Put
\[ u \circ_i v = u^i v \quad \text{for } u, v \in V_i, \ i = 1, 2. \]

Then \( \circ_i \) is bilinear over \( F = \mathbb{R} \), and (2.1) is satisfied. Since \( \Psi \) is a unital positive map, conditions (2.5)-(2.7) are fulfilled.

(I). Take \( \Phi = \Psi \). Then (2.4) is met by (2.23). In consequence, by Theorem 2.1, inequalities (2.8) and (2.9) hold with \( \Phi = \Psi \). Therefore (2.24) and (2.25) are valid.

(II). Choose \( \Phi(v) = [\Psi((v^*v)]^{1/2} \) for \( v \in V_1 \). Then (2.4) holds automatically, and (2.26) and (2.27) follow directly from (2.8) and (2.9), respectively. \[ \square \]

In the matrix setting if \( \Phi = \Psi \) is a unital positive map, then condition (2.23) of Theorem 2.7 reduces to Kadison’s inequality (2.19). In general, \( \Psi \) and \( \Phi \) need not be linear maps (see Remark 2.3).

We now discuss inequalities (2.14) and (2.22) which are crucial conditions for Theorems 2.4 and 2.7, respectively, to hold.

**Lemma 2.8.** Let \( V_1 \) be a \( C^* \)-algebra with unity \( e_1 \) and convex cone \( L_1 = \{ a^*a : a \in V_1 \} \). Suppose that for each hermitian element \( x \in V_1 \) there exist real scalars \( \lambda_j = \lambda_{j,x} \) and nonzero hermitian elements \( a_j = a_{j,x} \in L_1 \ j = 1, \ldots, n \), such that

(i) \( x = \lambda_1 a_1 + \ldots + \lambda_n a_n \),
(ii) \( e_1 = a_1 + \ldots + a_n \),
(iii) \( a_j a_l = a_j \) if \( j = l \), and \( a_j a_l = 0 \) if \( j \neq l \),
(iv) \( x \in L_1 \) implies \( \lambda_1, \ldots, \lambda_n \geq 0 \).

Let \( c, C \in \mathbb{R} \) and let \( x, y \in V_1 \) be two commuting hermitian elements with invertible \( y \).

Consider conditions

\[ (2.28) \quad ce_1 \leq xy^{-1} \leq Ce_1, \]
\[ (2.29) \quad c \leq \lambda_{j,xy^{-1}} \leq C \quad \text{for } j = 1, \ldots, n, \]
\[ (2.30) \quad (xy^{-1} - ce_1)(Ce_1 - xy^{-1}) \geq 0, \]
\[ (2.31) \quad (x - cy)(Cy - x) \geq 0. \]

Then \( (2.28) \Rightarrow (2.29) \Rightarrow (2.30) \Rightarrow (2.31) \).
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Proof. By (i) and (ii) applied to hermitian element $xy^{-1}$ we have

\begin{equation}
xy^{-1} - ce_1 = (\lambda_1 - c)a_1 + \ldots + (\lambda_n - c)a_n,
\end{equation}

(2.32)

\begin{equation}
Ce_1 - xy^{-1} = (C - \lambda_1)a_1 + \ldots + (C - \lambda_n)a_n.
\end{equation}

(2.33)

If (2.28) holds, then $xy^{-1} - ce_1 \in L_1$ and $Ce_1 - xy^{-1} \in L_1$. So, using (iv) and (2.32)-(2.33), we obtain

$$\lambda_j - c \geq 0 \text{ and } C - \lambda_j \geq 0 \text{ for } j = 1, \ldots, n,$$

where $\lambda_j = \lambda_{j,xy^{-1}}$. This gives (2.29).

On the other hand, by (2.32)-(2.33) and (iii), we have

$$(xy^{-1} - ce_1)(Ce_1 - xy^{-1}) = (\lambda_1 - c)(C - \lambda_1)a_1 + \ldots + (\lambda_n - c)(C - \lambda_n)a_n.$$\]

In consequence, (2.29) forces (2.30) by $a_j \in L_1, j = 1, \ldots, n$.

To see the implication (2.30) $\Rightarrow$ (2.31), it is sufficient to pre- and post-multiply (2.30) by $y^* = y$, and use the commutativity of $x$ and $y$. \qed

Clearly, employing Lemma 2.8 for $y = e_1$, we obtain the implications

\begin{equation}
(2.34)
ce_1 \leq x \leq Ce_1 \Rightarrow c \leq \lambda_j(x) \leq C \Rightarrow 0 \leq (x - ce_1)(Ce_1 - x).
\end{equation}

Lemma 2.8 gives possibility to produce Kantorovich type inequalities with various variants of assumptions on $x$ and $y$ (see [7, Theorems 2.1 and 2.4, Corollaries 2.2 and 2.3]).

We now return to Theorem 2.7 and inequality (2.27).

COROLLARY 2.9. For $i = 1, 2$, let $V_i, X_i, L_i$ and $e_i$ be as in Theorem 2.7.

Let $x \in L_1$ be an invertible element such that

\begin{equation}
(2.35)
(x - ce_1)(Ce_1 - x) \geq 0 \text{ for some positive scalars } C, c.
\end{equation}

Assume that $\Psi : V_1 \rightarrow V_2$ is a unital positive map. For any integer $p$, if $\Psi(x^{p+1})$ and $\Psi(x^{p-1})$ are two commuting elements of $V_2$, then we have the inequality

\begin{equation}
(2.36)[\Psi(x^{p+1})]^{1/2} [\Psi(x^{p-1})]^{1/2} \leq \frac{C + c}{2\sqrt{Cc}} \Psi(x^p).
\end{equation}

Proof. It follows from Lemma 2.8 that (2.35) implies

$$(x^{p+1} - cx^{-1}) (Cx^{p-1} - x^{p+1}) \geq 0.$$
That is (2.22) holds for $x^{p+1}$ and $x^{p-1}$. Applying (2.27), we obtain (2.36). \[ \square \]

**Example 2.10.** The Kantorovich inequality (1.1) can be derived from Corollary 2.9 applied to the map

$$
\Psi(A) = z^* A z \quad \text{for } A \in \mathbb{M}_n,
$$

where $z \in \mathbb{C}^n$ with $z^* z = 1$. Indeed, $\Psi$ is a unital positive map from $\mathbb{M}_n$ to $\mathbb{C}$. Here

$$
V_1 = X_1 = \mathbb{M}_n, \quad L_1 = \mathbb{L}_n, \quad V_2 = X_2 = \mathbb{C}, \quad L_2 = \mathbb{R}_+.
$$

For $A > 0$, let $0 < c < C$ be scalars such that the spectrum of $A$ lies in the interval $[c, C]$. Then (2.36) with $x = A$ and $p = 0$ becomes (1.1).

In a similar way, from (2.36) one can obtain the Schopf's inequality [20, p. 31]:

$$
z^* A^{p+1} z \cdot z^* A^{p-1} z \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n} (z^* A^p z)^2,
$$

where $p$ is an integer, and $\lambda_1$ and $\lambda_n$ are the largest and smallest eigenvalues of an $n \times n$ positive definite matrix $A$.

In the proof of Theorem 2.1, a key fact leading to (2.8) and (2.9) is inequality (2.10). (2.10) is a consequence of the bilinearity of the operation $\circ_1$. So, in order to get (2.9), it is possible to use (2.10) instead of the bilinearity of $\circ_1$. In fact, in the literature there are inequalities of types (2.10) and (2.9) with non-bilinear $\circ_1$.

**Example 2.11.** Consider the following spaces and cones

$$
V_1 = X_1 = \mathbb{M}_n, \quad L_1 = \mathbb{L}_n, \quad V_2 = X_2 = \mathbb{R}, \quad L_2 = \mathbb{R}_+.
$$

Define maps as follows

$$
\Phi(A) = (z^* A z)^{1/2} \quad \text{for } A \in \mathcal{A} = \mathbb{L}_n,
$$

and

$$
\Psi(A) = z^* A z \quad \text{for } A \in \mathcal{B} = \mathbb{L}_n,
$$

where $z \in \mathbb{C}^n$ with $z^* z = 1$.

Take $\circ_2$ to be the usual multiplication on $\mathbb{R}$. Let $\circ_1$ be the binary operation of geometric mean [21, p. 689]:

$$
A \circ_1 B = G(A, B) = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2} \quad \text{for } 0 < A, B \in \mathbb{L}_n.
$$
With the aid of the version of Theorem 2.1 based on (2.10), we shall show how to obtain the inequality [21, Theorem 2.2]:

\begin{equation}
(z^* A z)^{1/2} (z^* B z)^{1/2} \leq \frac{C + c}{2 \sqrt{CCc}} z^* G(A, B) z
\end{equation}

for $0 < A, B \in L_n$ with $0 < c I_n \leq A, B \leq C I_n$ and $0 < c < C$.

To do this, we use the result [13, 21]:

\begin{equation}
\frac{1}{2} (A + B) \leq \frac{C + c}{2 \sqrt{CCc}} G(A, B)
\end{equation}

for $0 < A, B \in L_n$ with $0 < c I_n \leq A, B \leq C I_n$ and $0 < c < C$. Because $G(A, \alpha B) = \alpha^{1/2} G(A, B)$ for $\alpha > 0$ [21, p. 689], substituting $CcB$ instead of $B$ leads to

$$A + CcB \leq (C + c) G(A, B),$$

which is of the form (2.10).

Furthermore, $G(A, B) = G(B, A)$ [21, p. 689]. Clearly, conditions (2.5)-(2.7) are satisfied. Since $G(A, A) = A$ [21, p. 689], it is readily seen that (2.4) is met.

By the discussion before this example, we get (2.9). It is not hard to check that (2.9), with $\Phi$ and $\Psi$ defined by (2.37) and (2.38), can be rewritten as (2.39).

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REFERENCES


