Some subspaces of the projective space $\text{PG}(\Lambda^kV)$ related to regular spreads of $\text{PG}(V)$

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SOME SUBSPACES OF THE PROJECTIVE SPACE $\text{PG}(\wedge^K V)$
RELATED TO REGULAR SPREADS OF $\text{PG}(V)^*$

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Abstract. Let $V$ be a $2m$-dimensional vector space over a field $\mathbb{F}$ ($m \geq 2$) and let $k \in \{1, \ldots, 2m - 1\}$. Let $A_{2m-1,k}$ denote the Grassmannian of the $(k-1)$-dimensional subspaces of $\text{PG}(V)$ and let $e_{gr}$ denote the Grassmann embedding of $A_{2m-1,k}$ into $\text{PG}(\wedge^k V)$. Let $S$ be a regular spread of $\text{PG}(V)$ and let $X_S$ denote the set of all $(k-1)$-dimensional subspaces of $\text{PG}(V)$ which contain at least one line of $S$. Then we show that there exists a subspace $\Sigma$ of $\text{PG}(\wedge^k V)$ for which the following holds: (1) the projective dimension of $\Sigma$ is equal to $(2m^2 - 2m - k)$; (2) a $(k-1)$-dimensional subspace $\alpha$ of $\text{PG}(V)$ belongs to $X_S$ if and only if $e_{gr}(\alpha) \in \Sigma$; (3) $\Sigma$ is generated by all points $e_{gr}(p)$, where $p$ is some point of $X_S$.

Key words. Regular spread, Grassmannian, Grassmann embedding, Klein correspondence.

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1. The main result. Let $V$ be a $2m$-dimensional vector space over a field $\mathbb{F}$ ($m \geq 2$) and let $\text{PG}(V)$ denote the projective space associated to $V$. For every $k \in \{1, \ldots, 2m - 1\}$, let $A_{2m-1,k}$ denote the following point-line geometry.

- The points of $A_{2m-1,k}$ are the $(k-1)$-dimensional subspaces of $\text{PG}(V)$.
- The lines of $A_{2m-1,k}$ are the sets $L(\pi_1, \pi_2)$ of $(k-1)$-dimensional subspaces of $\text{PG}(V)$ which contain a given $(k-2)$-dimensional subspace $\pi_1$ and are contained in a given $k$-dimensional subspace $\pi_2$ ($\pi_1 \subseteq \pi_2$).
- Incidence is containment.

The geometry $A_{2m-1,k}$ is called the Grassmannian of the $(k-1)$-dimensional subspaces of $\text{PG}(V)$. Obviously, $A_{2m-1,k} \cong A_{2m-1,2m-k}$ and the geometry $A_{2m-1,1} \cong A_{2m-1,2m-1}$ is isomorphic to the (point-line system of) the projective space $\text{PG}(2m-1,\mathbb{F})$.

For every point $p = (\bar{v}_1, \ldots, \bar{v}_k)$ of $A_{2m-1,k}$, let $e_{gr}(p)$ denote the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k \rangle$ of $\text{PG}(\wedge^k V)$. The map $e_{gr}$ defines an embedding of the geometry $A_{2m-1,k}$ into the projective space $\text{PG}(\wedge^k V)$ which is called the Grassmann embedding of $A_{2m-1,k}$. The image of $e_{gr}$ is a so-called Grassmann variety $\mathcal{G}_{2m-1,k}$ of $\text{PG}(\wedge^k V)$.

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A spread of PG(V) is a set of lines of PG(V) partitioning the point-set of PG(V). In Section 2, we will define a nice class of spreads of PG(V) which are called regular spreads.

The following is the main result of this note.

**Theorem 1.1.** Let $S$ be a regular spread of the projective space PG(V). Let $k \in \{1, \ldots, 2m - 1\}$. Let $X_S$ denote the set of all $(k - 1)$-dimensional subspaces of PG(V) which contain at least one line of $S$. Then there exists a subspace $\Sigma$ of PG($\bigwedge^k V$) for which the following holds:

1. The projective dimension of $\Sigma$ is equal to $\binom{2m}{k} - 2 \cdot \binom{m}{k} - 1$.
2. A $(k - 1)$-dimensional subspace $\alpha$ of PG(V) belongs to $X_S$ if and only if $e_{gr}(\alpha) \in \Sigma$.
3. $\Sigma$ is generated by all points $e_{gr}(p)$, where $p$ is some element of $X_S$.

In Theorem 1.1 and elsewhere in this paper, we take the convention that $\binom{n}{z} = 0$ for every $n \in \mathbb{N}$ and every $z \in \mathbb{Z} \setminus \{0, \ldots, n\}$.

**Some special cases.**
1. If $k = 1$, then by Theorem 1.1(1), $\Sigma = \emptyset$. Indeed, in this case we have $X_S = \emptyset$.
2. If $k = 2$, then by Theorem 1.1, $\dim(\Sigma) = m^2 - 1$ and $X_S = S$ consists of all lines $L$ of PG(V) for which $e_{gr}(L) \in \Sigma \cap G_{2m-1,2}$. For a discussion of the special case $k = m = 2$, see Section 4.
3. If $k = m$, then by Theorem 1.1(1), $\Sigma$ has co-dimension 2 in PG($\bigwedge^m V$).
4. If $k \in \{m + 1, \ldots, 2m - 1\}$, then by Theorem 1.1, $\Sigma = PG(\bigwedge^k V)$ and $X_S$ consists of all $(k - 1)$-dimensional subspaces of PG(V).

2. Regular spreads.

**2.1. Definition.** Let PG(3,F) be a 3-dimensional projective space over a field F. A regulus of PG(3,F) is a set $\mathcal{R}$ of mutually disjoint lines of PG(3,F) satisfying the following two properties:

- If a line $L$ of PG(3,F) meets three distinct lines of $\mathcal{R}$, then $L$ meets every line of $\mathcal{R}$;
- If a line $L$ of PG(3,F) meets three distinct lines of $\mathcal{R}$, then every point of $L$ is incident with (exactly) one line of $\mathcal{R}$.

Any three mutually disjoint lines $L_1, L_2, L_3$ of PG(3,F) are contained in a unique regulus which we will denote by $\mathcal{R}(L_1, L_2, L_3)$.
Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$ and $F$ a field. Recall that a spread of the projective space $PG(n, F)$ is a set of lines which determines a partition of the point set of $PG(n, F)$. A spread $S$ is called regular if the following two conditions are satisfied:

(R1) If $\pi$ is a $3$-dimensional subspace of $PG(n, F)$ containing two distinct elements of $S$, then the elements of $S$ contained in $\pi$ determine a spread of $\pi$;

(R2) If $L_1$, $L_2$ and $L_3$ are three distinct lines of $S$ which are contained in some $3$-dimensional subspace, then $R(L_1, L_2, L_3) \subseteq S$.

2.2. Classification of regular spreads. Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $F, F'$ be fields such that $F'$ is a quadratic extension of $F$. Let $V'$ be an $n$-dimensional vector space over $F'$ with basis $\{\overline{e}_1, \ldots, \overline{e}_n\}$. We denote by $V$ the set of all $F$-linear combinations of the elements of $\{\overline{e}_1, \ldots, \overline{e}_n\}$. Then $V$ can be regarded as an $n$-dimensional vector space over $F$. We denote the projective spaces associated with $V$ and $V'$ by $PG(V)$ and $PG(V')$, respectively. Since every $1$-dimensional subspace of $V$ is contained in a unique $1$-dimensional subspace of $V'$, we can regard the points of $PG(V)$ as points of $PG(V')$. So, $PG(V)$ can be regarded as a sub-(projective)-geometry of $PG(V')$. Any subgeometry of $PG(V')$ which can be obtained in this way is called a Baer-$F$-subgeometry of $PG(V')$. Notice also that every subspace $\pi$ of $PG(V)$ generates a subspace $\pi'$ of $PG(V')$ of the same dimension as $\pi$.

The following lemma is known (and easy to prove).

**Lemma 2.1.** Every point $p$ of $PG(V')$ not contained in $PG(V)$ is contained in a unique line of $PG(V')$ which intersects $PG(V)$ in a line of $PG(V)$, i.e. there exists a unique line $L$ of $PG(V)$ for which $p \in L'$.

The line $L$ in Lemma 2.1 is called the line of $PG(V)$ induced by $p$.

Suppose now that $F'$ is a separable (and hence also Galois) extension of $F$ and let $\psi$ denote the unique nontrivial element in $Gal(F'/F)$. For every vector $\overline{x} = \sum_{i=1}^{n} k_i \overline{e}_i$ of $V'$, we define $\overline{x}^\psi := \sum_{i=1}^{n} k_i \overline{e}_i$. For every point $p = \langle \overline{x} \rangle$ of $PG(V')$, we define $p^\psi := \langle \overline{x}^\psi \rangle$ and for every subspace $\pi$ of $PG(V')$ we define $\pi^\psi := \{p^\psi \mid p \in \pi\}$. The subspace $\pi^\psi$ is called conjugate to $\pi$ with respect to $\psi$. Notice that if $\pi$ is a subspace of $PG(V)$, then $\pi^\psi = \pi'$.

The following proposition is taken from Beutelspacher and Ueberberg [1, Theorem 1.2] and generalizes a result from Bruck [2]. See also the discussion in Section 4.

**Proposition 2.2 ([1]).**

(a) Let $t \in \mathbb{N} \setminus \{0, 1\}$ and let $F, F'$ be fields such that $F'$ is a quadratic extension of $F$. Regard $PG(2t-1, F)$ as a Baer-$F$-subgeometry of $PG(2t-1, F')$. Let $\pi$ be
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A $(t - 1)$-dimensional subspace of PG($2t - 1, \mathbb{F}'$) disjoint from PG($2t - 1, \mathbb{F}$). Then the set $S_\pi$ of all lines of PG($2t - 1, \mathbb{F}$) which are induced by the points of $\pi$ is a regular spread of PG($2t - 1, \mathbb{F}$).

(b) Suppose $t \in \mathbb{N} \setminus \{0, 1\}$ and that $\mathbb{F}$ is a field. If $S$ is a regular spread of the projective space PG($2t - 1, \mathbb{F}$), then there exists a quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ such that the following holds if we regard PG($2t - 1, \mathbb{F}$) as a Baer-$\mathbb{F}$-subgeometry of PG($2t - 1, \mathbb{F}'$):

(i) If $\mathbb{F}'$ is a separable field extension of $\mathbb{F}$, then there are precisely two $(t - 1)$-dimensional subspaces $\pi$ of PG($2t - 1, \mathbb{F}'$) disjoint from PG($2t - 1, \mathbb{F}$) for which $S = S_\pi$.

(ii) If $\mathbb{F}'$ is a non-separable field extension of $\mathbb{F}$, then there is exactly one $(t - 1)$-dimensional subspace $\pi$ of PG($2t - 1, \mathbb{F}'$) disjoint from PG($2t - 1, \mathbb{F}$) for which $S = S_\pi$.

Remark 2.3. In Proposition 2.2(bi), the two $(t - 1)$-dimensional subspaces $\pi_1$ and $\pi_2$ of PG($2t - 1, \mathbb{F}'$) disjoint from PG($2t - 1, \mathbb{F}$) for which $S = S_{\pi_1} = S_{\pi_2}$ are conjugate with respect to the unique nontrivial element $\psi$ of $Gal(\mathbb{F}'/\mathbb{F})$. For, a line $L$ of PG($2t - 1, \mathbb{F}$) belongs to $S_{\pi_1}$ if and only if $L'$ intersects $\pi_1$, i.e., if and only if $L' = L^\psi$ intersects $\pi_1^\psi$.

3. Proof of the Main Theorem.

3.1. An inequality. Let $\mathbb{F}$ and $\mathbb{F}'$ be two fields such that $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$. Let $\delta$ be an arbitrary element of $\mathbb{F}' \setminus \mathbb{F}$ and let $\mu_1, \mu_2$ be the unique elements of $\mathbb{F}$ such that $\delta^2 = \mu_1 \delta + \mu_2$. Then $\mu_2 \neq 0$. Let $m \geq 1$ and let $V'$ be a 2$m$-dimensional vector space over $\mathbb{F}'$ with basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2m}\}$. We denote by $V$ the set of all $\mathbb{F}$-linear combinations of the elements of $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{2m}\}$. Then $V$ can be regarded as a 2$m$-dimensional vector space over $\mathbb{F}$. We denote the projective spaces associated with $V$ and $V'$ by PG($V$) and PG($V'$), respectively. The projective space PG($V$) can be regarded in a natural way as a subgeometry of PG($V'$). Every subspace $\alpha$ of PG($V$) then generates a subspace $\alpha'$ of PG($V'$) of the same dimension as $\alpha$.

Now, let $\pi$ be an $(m - 1)$-dimensional subspace of PG($V'$) disjoint from PG($V$). Then there exist vectors $\tilde{e}_1, \tilde{f}_1, \tilde{e}_2, \tilde{f}_2, \ldots, \tilde{e}_m, \tilde{f}_m$ such that $\pi = \langle \tilde{e}_1 + \delta \tilde{f}_1, \tilde{e}_2 + \delta \tilde{f}_2, \ldots, \tilde{e}_m + \delta \tilde{f}_m \rangle$.

Lemma 3.1. $\{\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_m, \bar{f}_m\}$ is a basis of $V$.

Proof. If this were not the case, then there exist $a_1, b_1, \ldots, a_m, b_m \in \mathbb{F}$ with $(a_1, b_1, \ldots, a_m, b_m) \neq (0, 0, \ldots, 0, 0)$ such that $a_1 \bar{e}_1 + b_1 \bar{f}_1 + \cdots + a_m \bar{e}_m + b_m \bar{f}_m = \bar{o}$. Now, put $k_i := a_i + \frac{b_i}{\mu_2} \delta$ for every $i \in \{1, \ldots, m\}$. Then $(k_1, \ldots, k_m) \neq (0, \ldots, 0)$.
since \((a_1, b_1, \ldots, a_m, b_m) \neq (0, 0, \ldots, 0, 0)\). Since \(k_1(\bar{e}_1 + \delta f_1) + \cdots + k_m(\bar{e}_m + \delta f_m) = \delta(\bar{a}_1 f_1 + \frac{b_1}{\mu_2} e_1 + \frac{a_1}{\mu_2} f_1 + \cdots + a_m f_m + \frac{b_m}{\mu_2} e_m + \frac{a_m}{\mu_2} f_m)\), the subspace \(\pi\) is not disjoint from \(\text{PG}(V)\), a contradiction. So, \(\{\bar{e}_1, f_1, \ldots, \bar{e}_m, f_m\}\) is a basis of \(V\). \(\square\)

Now, let \(k \in \{1, \ldots, 2m\}\). Let \(W_1\) denote the subspace of \(\Lambda^k V\) generated by all vectors \(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k\) where \(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\) are \(k\) linearly independent vectors of \(V\) such that \((\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k)\)' meets \(\pi\). If there are no such vectors \(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\), then \(W_1 = 0\). We will prove by induction on \(m\) that \(\dim(W_1) \geq \binom{2m}{k} - 2 \cdot \binom{m}{k}\).

If \(k = 1\), then \(W_1 = 0\) since \(\pi \cap \text{PG}(V) = \emptyset\). Hence, \(\dim(W_1) = 0 = \binom{2m}{1} - 2 \cdot \binom{m}{1}\).

Suppose \(k = 2m\). Since \(\pi \subseteq (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{2m})\)' for every \(2m\) linearly independent vectors \(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{2m}\) of \(V\), we have \(W_1 = \Lambda^{2m} V\) and hence \(\dim(W_1) = 1 = \binom{2m}{1} - 2 \cdot \binom{m}{1}\).

In the sequel, we may suppose that \(m \geq 2\) and \(k \in \{2, \ldots, 2m - 1\}\). Put \(U = \langle \bar{e}_2, f_2, \ldots, \bar{e}_m, f_m \rangle\). Every vector \(\chi\) of \(\Lambda^k V\) can be written in a unique way as

\[
\bar{e}_1 \wedge f_1 \wedge \alpha(\chi) + \bar{e}_1 \wedge \beta(\chi) + \bar{f}_1 \wedge \gamma(\chi) + \delta(\chi),
\]

where \(\alpha(\chi) \in \Lambda^{k-2} U\), \(\beta(\chi) \in \Lambda^{k-1} U\), \(\gamma(\chi) \in \Lambda^{k-1} U\) and \(\delta(\chi) \in \Lambda^k U\). [Here, \(\Lambda^0 U = F\) and \(\Lambda^{2m-1} U = 0\).] Let \(\theta\) denote the linear map from \(W_1 \subseteq \Lambda^k V\) to \(\Lambda^{k-1} U\) mapping \(\chi\) to \(\gamma(\chi)\). Then by the rank-nullity theorem,

\[
\dim(W_1) = \dim(\ker(\theta)) + \dim(\text{Im}(\theta)).
\]

**Lemma 3.2.** We have \(\dim(\ker(\theta)) \geq \binom{2m-2}{k-2} + \binom{2m-1}{k-1} - 2 \cdot \binom{m}{k}\).

**Proof.** (a) If \(\bar{v}_3, \ldots, \bar{v}_k\) are \(k-2\) linearly independent vectors of \(U\), then \((\bar{e}_1, \bar{f}_1, \bar{v}_3, \ldots, \bar{v}_k)\)' meets \(\pi\) and hence \(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_k \in W_1\). It follows that \(\bar{e}_1 \wedge \bar{f}_1 \wedge \Lambda^{k-2} U = \ker(\theta)\).

(b) Let \(Z_1\) denote the subspace of \(\Lambda^{k-1} U\) generated by all vectors \(\bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_k\) where \(\bar{v}_2, \ldots, \bar{v}_k\) are \(k-1\) linearly independent vectors of \(U\) such that \((\bar{v}_2, \ldots, \bar{v}_k)\)' meets \((\bar{e}_2 + \delta f_2, \ldots, \bar{e}_m + \delta f_m)\). By the induction hypothesis, \(\dim(Z_1) \geq \binom{2m-2}{k-1} - 2 \cdot \binom{m-1}{k-1}\). Clearly, \(\bar{e}_1 \wedge Z_1 \subseteq \ker(\theta)\).

(c) Suppose \(k \leq 2m - 2\). Let \(Z_2\) denote the subspace of \(\Lambda^k U\) generated by all vectors \(\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k\) where \(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\) are \(k\) linearly independent vectors of \(U\) such that \((\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k)\)' meets \((\bar{e}_2 + \delta f_2, \ldots, \bar{e}_m + \delta f_m)\). By the induction hypothesis, \(\dim(Z_2) \geq \binom{2m-2}{k} - 2 \cdot \binom{m}{k}\). Clearly, \(Z_2 \subseteq \ker(\theta)\).

By (a), (b), (c) and the decomposition \(\Lambda^k V = (\bar{e}_1 \wedge \bar{f}_1 \wedge \Lambda^{k-2} U) \oplus (\bar{e}_1 \wedge \Lambda^{k-1} U) \oplus (\bar{f}_1 \wedge \Lambda^{k-1} U) \oplus (\Lambda^k U)\), we have \(\dim(\ker(\theta)) \geq \binom{2m-2}{k-2} + \binom{2m-2}{k-1} - 2 \cdot \binom{m-1}{k}\).
(m-1) + \binom{2m-2}{k-1} - 2 \cdot \binom{m-1}{k} = \binom{2m-2}{k} + \binom{2m-1}{k-2} - 2 \cdot \binom{m}{k}. Notice that this inequality remains valid if k = 2m - 1 since \binom{2m-2}{k} - 2 \cdot \binom{m}{k} = 0 in this case.

Lemma 3.3. We have \text{Im}(\theta) = \bigwedge^{k-1} U. Hence, \dim(\text{Im}(\theta)) = \binom{2m-2}{k-1}.

Proof. It suffices to prove that every vector of the form \bar{g}_2 \land g_3 \land \cdots \land g_k belongs to \text{Im}(\theta), where \bar{g}_2, g_3, \ldots, g_k are k-1 distinct elements of \{\bar{e}_2, f_2, \ldots, e_m, f_m\}. Without loss of generality, we may suppose that \bar{g}_2 \in \{\bar{e}_2, f_2\}. Since \langle (\bar{e}_1 + \bar{e}_2) + \delta(f_1 + f_2) \rangle belongs to \pi, \langle \bar{e}_1 + \bar{e}_2 \rangle \land (f_1 + f_2) \land g_3 \land \cdots \land g_k \in W_1 and hence \bar{e}_2 \land g_3 \land \cdots \land g_k \in \text{Im}(\theta). Since \langle (\bar{e}_1 + \delta f_1) + \delta(\bar{e}_2 + f_2) \rangle = \langle (\bar{e}_1 + \mu_2 f_2) + \delta(f_1 + e_2 + \mu_1 f_2) \rangle belongs to \pi, \langle e_1 + \mu_2 f_2 \rangle \land (f_1 + e_2 + \mu_1 f_2) \land g_3 \land \cdots \land g_k \in W_1 and hence f_2 \land g_3 \land \cdots \land g_k \in \text{Im}(\theta) (recall \mu_2 \neq 0).

Corollary 3.4. We have \dim(W_1) \geq \binom{2m}{k-1} - 2 \cdot \binom{m}{k}.

Proof. By equation (3.1) and Lemmas 3.2, 3.3, we have that \dim(W_1) \geq \binom{2m-2}{k-1} + \binom{2m-1}{k-2} - 2 \cdot \binom{m}{k} = \binom{2m-1}{k-1} + \binom{2m-1}{k-2} - 2 \cdot \binom{m}{k} = \binom{2m}{k} - 2 \cdot \binom{m}{k}.

3.2. Proof of Theorem 1.1. We continue with the notation introduced in Section 3.1. We suppose here that m \geq 2 and k \in \{1, \ldots, 2m - 1\}. Let S be the spread of PG(V) induced by the points of \pi (recall Proposition 2.2(a)) and let X_S denote the set of all (k - 1)-dimensional subspaces of PG(V) which contain at least one line of S.

Lemma 3.5. A (k - 1)-dimensional subspace \alpha of PG(V) contains a line of S if and only if \alpha' meets \pi.

Proof. Suppose \alpha contains a line L of S. Since \alpha' contains the line L' which meets \pi, \alpha' must also meet \pi.

Conversely, suppose that \alpha' meets \pi and let p be an arbitrary point in the intersection \alpha' \cap \pi. Then in the subspace \alpha' there exists a unique line L' through p which meets \alpha in a line L (recall Lemma 2.1). Since L is a line of PG(V), we must necessarily have L \in S. So, \alpha contains a line of S.

Corollary 3.6. If k \in \{m + 1, m + 2, \ldots, 2m - 1\}, then X_S consists of all (k - 1)-dimensional subspaces of PG(V).

Let W_2 denote the subspace of \bigwedge^k V consisting of all vectors \chi \in \bigwedge^k V satisfying \langle \bar{e}_1 + \delta f_1 \rangle \land \langle \bar{e}_2 + \delta f_2 \rangle \land \cdots \land \langle e_m + \delta f_m \rangle \land \chi = 0.

Lemma 3.7.

(1) The subspace PG(W_1) is generated by all points e_{gr}(\alpha) where \alpha is some element of X_S.
(2) A $(k - 1)$-dimensional subspace $\alpha$ of $\text{PG}(V)$ belongs to $X_3$ if and only if $e_{gr}(\alpha) \in \text{PG}(W_2)$.
(3) $\text{PG}(W_1) \subseteq \text{PG}(W_2)$.

Proof. Claim (1) is an immediate corollary of Lemma 3.5 and the definition of the subspace $W_1$. By Lemma 3.5, a $(k - 1)$-dimensional subspace $\alpha = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ of $\text{PG}(V)$ belongs to $X_3$ if and only if $\pi$ meets $\alpha' = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle'$, i.e. if and only if $(\bar{e}_1 + \delta f_1) \wedge (\bar{e}_2 + \delta f_2) \wedge \cdots \wedge (\bar{e}_m + \delta f_m) \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k = 0$, i.e. if and only if $e_{gr}(\alpha) \in \text{PG}(W_2)$. Claim (3) follows directly from Claims (1) and (2).

Lemma 3.8. We have $\dim(W_2) \leq \binom{2m}{k} - 2 \cdot \binom{m}{k}$.

Proof. If $k \in \{m + 1, \ldots, 2m - 1\}$, then $W_2 = \bigwedge^k V$ and hence $\dim(W_2) = \binom{2m}{k} = \binom{2m}{k} - 2 \cdot \binom{m}{k}$. We may therefore suppose that $k \in \{1, \ldots, m\}$.

Let $T$ denote the set of all $(m - k)$-tuples $(i_1, \ldots, i_{m-k})$, where $i_1, \ldots, i_{m-k} \in \{1, \ldots, m\}$ satisfies $i_1 < i_2 < \cdots < i_{m-k}$. We take the convention here that if $k = m$, then $|T| = 1$ and $T$ consists of the unique “0-tuple”. If $\tau \in T$, then $\chi \in W_2$ implies that

\[(\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_{m-k}}) \wedge (\bar{e}_1 + \delta f_1) \wedge \cdots \wedge (\bar{e}_m + \delta f_m) \wedge \chi = 0.\]

We can write (3.2) as

\[(\alpha_\tau + \delta \beta_\tau) \wedge \chi = 0,
\]

where

\[\alpha_\tau + \delta \beta_\tau = \frac{\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_{m-k}} \wedge (\bar{e}_1 + \delta f_1) \wedge \cdots \wedge (\bar{e}_m + \delta f_m)}{\delta^{m-k}},\]

\[\alpha_\tau, \beta_\tau \in \bigwedge^{2m-k} V.\]

Equation (3.3) is equivalent with

\[\begin{aligned}
\alpha_\tau \wedge \chi &= 0, \\
\beta_\tau \wedge \chi &= 0.
\end{aligned}\]

Consider now a basis $B$ of $\bigwedge^{2m-k} V$ which consists only of vectors of the form $\bar{g}_1 \wedge \bar{g}_2 \wedge \cdots \wedge \bar{g}_{2m-k}$, where $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{2m-k} \in \{\bar{e}_1, f_1, \ldots, \bar{e}_m, f_m\}$.

The $2 \cdot \binom{m}{k} = 2 \cdot \binom{m}{k}$ equations determined by (3.4) are linearly independent if and only if the $2 \cdot \binom{m}{k}$ vectors $\alpha_\tau, \beta_\tau$ $(\tau \in T)$ are linearly independent.

Suppose there exist $k_\tau, l_\tau \in \mathbb{F}$ $(\tau \in T)$ such that

\[\sum_{\tau \in T} (k_\tau \alpha_\tau + l_\tau \beta_\tau) = 0.\]
Take an arbitrary $\tau^* = (i_1, i_2, \ldots, i_{m-k})$ of $T$. If we write the left hand side of equation (3.5) as a linear combination of the elements of the basis $B$ of $\wedge^{2m-k} V$, then the sum of all terms which contain the factor $(\bar{e}_{i_1} \wedge \bar{f}_{i_1}) \wedge (\bar{e}_{i_2} \wedge \bar{f}_{i_2}) \wedge \cdots \wedge (\bar{e}_{i_{m-k}} \wedge \bar{f}_{i_{m-k}})$ must be 0. This implies that $k_{\tau^*} \alpha_{\tau^*} + l_{\tau^*} \beta_{\tau^*} = 0$. Now, the two vectors $\alpha_{\tau^*}$ and $\beta_{\tau^*}$ are linearly independent: $\alpha_{\tau^*}$ contains a term which is a multiple of $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_m \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_{m-k}$, while $\beta_{\tau^*}$ does not contain such a term; for every $j \in \{1, \ldots, m\} \setminus \{i_1, \ldots, i_{m-k}\}$, $\beta_{\tau^*}$ contains a term which is a multiple of $\bar{e}_1 \wedge \cdots \wedge \bar{e}_{j-1} \wedge \bar{e}_j \wedge \bar{e}_{j+1} \wedge \cdots \wedge \bar{e}_m \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_{m-k}$, while $\alpha_{\tau^*}$ does not contain such a term. We conclude that $k_{\tau^*} = l_{\tau^*} = 0$. Since $\tau^*$ was an arbitrary element of $T$, we can indeed conclude that the vectors $\alpha_{\tau^*}, \beta_{\tau^*}$ ($\tau \in T$) are linearly independent.

Since the vectors $\chi$ of $W_2$ satisfy a linear system of $2 \cdot \binom{m}{2}$ linearly independent equations (recall (3.4)), we can indeed conclude that $\dim(W_2) \leq \binom{2m}{k} - 2 \cdot \binom{m}{k}$. \[\Box\]

Theorem 1.1 is now an immediate consequence of Corollary 3.4 and Lemmas 3.7, 3.8.

4. On the classification of the regular spreads of $\text{PG}(3, F)$. Proposition 2.2(b) plays an essential role in this paper. The proof of Proposition 2.2(b) given in [1] consists of two parts. In [1, Section 3], the case $t = 2$ was treated and subsequently this classification was used in [1, Section 5] to obtain also a classification in the case $t \geq 3$. In the proof for the case $t = 2$, a gap seems to occur. Indeed, in [1, Section 3] the authors tacitly assume that the lines and reguli of a given regular spread determine a Möbius plane. This fact is trivial in the finite case, where one could use a simple counting argument to prove it, but not at all obvious in the infinite case.

The aim of this section is to fill this apparent gap. We give a proof for Proposition 2.2(b) in the case that $t$ is equal to 2. The methods used here will be different from the ones of [1]. Our treatment will be more geometric and based on the Klein correspondence. A discussion of regular spreads of finite 3-dimensional projective spaces can also be found in [3, Section 17.1]. Some of the tools we need here are already in [3], either explicitly or implicitly.

Let $V$ be a 4-dimensional vector space over a field $F$. For every line $L = \langle \bar{u}_1, \bar{u}_2 \rangle$ of $\text{PG}(V)$, let $\kappa(L)$ denote the point $\langle \bar{u}_1 \wedge \bar{u}_2 \rangle$ of $\text{PG}(\wedge^2 V)$. The image $Q$ of $\kappa$ is a nonsingular quadric of Witt index 3 of $\text{PG}(\wedge^2 V)$. If $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$ is a basis of $V$, then the equation of $Q$ with respect to the ordered basis $B^* := (\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \wedge \bar{e}_2 \wedge \bar{e}_3, \bar{e}_4)$ of $\wedge^2 V$ is equal to $X_1 X_6 - X_2 X_5 + X_3 X_4 = 0$. The bijective correspondence $\kappa$ between the set of lines of $\text{PG}(V)$ and the set of points of $Q$ is often referred to as the Klein correspondence. For every point $x$ of $\text{PG}(V)$, let $\mathcal{L}_x$ denote the set of lines of $\text{PG}(V)$ containing $x$ and for every plane $\pi$ of $\text{PG}(V)$, let
\( \mathcal{L}_\pi \) denote the set of lines of \( \text{PG}(V) \) contained in \( \pi \). The sets \( \kappa(\mathcal{L}_x) \) and \( \kappa(\mathcal{L}_\pi) \) are generators of \( Q \). Let \( \mathcal{M}^+ \) [respectively, \( \mathcal{M}^- \)] denote the set of generators of \( Q \) of the form \( \kappa(\mathcal{L}_x) \) [respectively, \( \kappa(\mathcal{L}_\pi) \)] for some point \( x \) [respectively, plane \( \pi \)] of \( \text{PG}(V) \). Then \( \mathcal{M}^+ \) and \( \mathcal{M}^- \) are the two families of generators of \( Q \), i.e. (i) \( \mathcal{M}^+ \cap \mathcal{M}^- = \emptyset \), (ii) \( \mathcal{M}^+ \cup \mathcal{M}^- \) consists of all generators of \( Q \), and (iii) two generators of \( Q \) belong to the same family \( \mathcal{M}^\epsilon \) for some \( \epsilon \in \{+, -\} \) if and only if they intersect in a subspace of even co-dimension. Every line of \( Q \) is contained in precisely two generators, one generator of \( \mathcal{M}^+ \) and one generator of \( \mathcal{M}^- \).

The following three lemmas are known and their proofs are straightforward.

**Lemma 4.1.** Let \( \mathcal{R} \) be a regulus of \( \text{PG}(V) \). Then there exists a 2-dimensional subspace \( \alpha \) of \( \text{PG}(\wedge^2 V) \) such that \( \kappa(\mathcal{R}) = \alpha \cap Q \) is a nonsingular quadric of Witt index 1 of \( \alpha \).

**Lemma 4.2.** Suppose \( \alpha \) is a 3-dimensional subspace of \( \text{PG}(\wedge^2 V) \) which intersects \( Q \) in a nonsingular quadric of Witt index 1 of \( \alpha \). Then the set \( S \) of all lines \( L \) of \( \text{PG}(V) \) for which \( \kappa(L) \in \alpha \) is a regular spread of \( \text{PG}(V) \).

**Lemma 4.3.** Suppose \( \alpha \) is a 3-dimensional subspace of \( \text{PG}(\wedge^2 V) \) and that \( S \) is a spread of \( \text{PG}(V) \) such that \( \alpha \cap Q \subseteq \kappa(S) \). Then \( \alpha \) intersects \( Q \) in a nonsingular quadric of Witt index 1 of \( \alpha \). Moreover, \( \alpha \cap Q = \kappa(S) \).

**Lemma 4.4.** Suppose \( F = \mathbb{F}_2 \). Then \( \text{PG}(V) = \text{PG}(3, 2) \). The following hold:

1. Every spread of \( \text{PG}(V) \) is regular.
2. Every regulus of \( \text{PG}(V) \) can be extended to a unique spread of \( \text{PG}(V) \).
3. If \( S \) is a regular spread of \( \text{PG}(V) \), then there exists a unique subspace \( \alpha \) of dimension 3 of \( \text{PG}(\wedge^2 V) \) such that \( \kappa(S) = \alpha \cap Q \) is a nonsingular quadric of Witt index 1 of \( \alpha \).

**Proof.** Claims (1) and (2) are well known and easy to prove. So, we will only give a proof for Claim (3). Suppose \( S \) is a (regular) spread of \( \text{PG}(V) \) and \( \mathcal{R} \) a regulus contained in \( S \). Then by Lemma 4.1 there exists a 2-dimensional subspace \( \beta \) of \( \text{PG}(\wedge^2 V) \) such that \( \kappa(\mathcal{R}) = \beta \cap Q \) is a nonsingular conic of \( \beta \). Now, by an easy counting argument there are three 3-dimensional subspaces \( \gamma_1 \) through \( \beta \) which intersect \( Q \) in a singular quadric of \( \gamma_1 \) (namely the subspaces \( \langle \beta, \kappa(M) \rangle \) where \( M \) is one of the three lines of \( \text{PG}(V) \) meeting each line of \( \mathcal{R} \)), three 3-dimensional subspaces \( \gamma_2 \) through \( \beta \) which intersect \( Q \) in a nonsingular hyperbolic quadric of \( \gamma_2 \) and one 3-dimensional subspace \( \alpha \) through \( \beta \) which intersects \( Q \) in a nonsingular elliptic quadric of \( \alpha \). Since \( \kappa^{-1}(\alpha \cap Q) \) is a spread containing \( \mathcal{R} \), \( \kappa^{-1}(\alpha \cap Q) = S \) by Claim (2). Hence, \( \alpha \cap Q = \kappa(S) \). \( \square \)

**Lemma 4.5.** Suppose \( |F| \geq 3 \). If \( S \) is a regular spread of \( \text{PG}(V) \), then there
exists a unique subspace $\alpha$ of dimension 3 of $\PG(\wedge^2 V)$ such that $\kappa(S) = \alpha \cap Q$ is a nonsingular quadric of Witt index 1 of $\alpha$.

Proof. Let $L_1$, $L_2$, $L_3$ and $L_4$ be four distinct lines of $S$ such that $L_4 \notin \mathcal{R}(L_1, L_2, L_3)$. Put $\mathcal{R}_1 = \mathcal{R}(L_1, L_2, L_3)$ and $\mathcal{R}_2 = \mathcal{R}(L_1, L_2, L_4)$. By Lemma 4.1, there exists a 2-dimensional subspace $\alpha_i$, $i \in \{1, 2\}$, of $\PG(\wedge^2 V)$ such that $\kappa(\mathcal{R}_i) = \alpha_i \cap Q$. Since $\mathcal{R}_1 \neq \mathcal{R}_2$, we have $\alpha_1 \neq \alpha_2$. Since $\kappa(L_1)$ and $\kappa(L_2)$ are contained in $\alpha_1$ and $\alpha_2$, $\alpha_1 \cap \alpha_2$ is a line and $\alpha := \langle \alpha_1, \alpha_2 \rangle$ is a 3-dimensional subspace of $\PG(\wedge^2 V)$.

We prove that every point $x$ of $\alpha \cap Q$ belongs to $\kappa(S)$. Clearly, $\alpha_1 \cap Q = \kappa(\mathcal{R}_1) \subseteq \kappa(S)$ and $\alpha_2 \cap Q = \kappa(\mathcal{R}_2) \subseteq \kappa(S)$. So, we may assume that $x \in (\alpha \cap Q) \setminus (\alpha_1 \cup \alpha_2)$. Let $M$ denote a line through $x$ which meets $\alpha_1$ in a point $y_1$ of $(\alpha_1 \cap Q) \setminus \alpha_2$ and let $y_2$ be the intersection of $M$ with $\alpha_2$. Since $|\mathbb{F}| \geq 3$, we may suppose that we have chosen $M$ in such a way that $y_2$ is not the kernel of the quadric $\alpha_2 \cap Q$ of $\alpha_2$ in the case the characteristic of $\mathbb{F}$ is equal to 2. Then there exists a line $N \subseteq \alpha_2$ through $y_2$ which intersects $Q \cap \alpha_2$ in two points, say $u$ and $v$. The plane $\alpha_3 := (M, N)$ through $M$ is contained in $\alpha$ and contains the points $y_1$, $u$ and $v$ of $\kappa(\mathcal{R}_1 \cup \mathcal{R}_2)$. So, there exist three distinct lines $U$, $V$ and $W$ of $\mathcal{R}_1 \cup \mathcal{R}_2$ such that $\kappa(U)$, $\kappa(V)$ and $\kappa(W)$ belong to $\alpha_3$. If $\mathcal{R}_3$ denotes the unique regulus of $\PG(V)$ containing $U$, $V$ and $W$, then $\kappa(\mathcal{R}_3) = \alpha_3 \cap Q$ by Lemma 4.1. Now, $\mathcal{R}_3 \subseteq S$ since $S$ is regular and $x \in \alpha_3 \cap Q$. So, there exists a line $L \in S$ such that $x = \kappa(L)$. This is what we needed to prove.

By the above, we know that $\alpha \cap Q \subseteq \kappa(S)$. Lemma 4.3 then implies that $\alpha \cap Q = \kappa(S)$ is a nonsingular quadric of Witt index 1 of $\alpha$. \qed

Now, let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$ (which is unique, up to isomorphism) and let $\overline{V}$ denote a 4-dimensional vector space over $\overline{\mathbb{F}}$ which also has $\{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4\}$ as basis. We will regard $\PG(V)$ as a subgeometry of $\PG(\overline{V})$ and $\PG(\wedge^2 V)$ as a subgeometry of $\PG(\wedge^2 \overline{V})$.

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$ which is contained in $\overline{\mathbb{F}}$. Let $\PG(V)_k$ denote the set of all $\mathbb{K}$-linear combinations of the elements of $\{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4\}$. Then $\PG(V)_k$ can be regarded as a vector space over $\mathbb{K}$. We will regard $\PG(V)$ as a subgeometry of $\PG(V)_k$ and $\PG(\wedge^2 V)_k$ as a subgeometry of $\PG(\wedge^2 V)$. Similarly, we will regard $\PG(\wedge^2 V)$ as a subgeometry of $\PG(\wedge^2 V_k)$ and $\PG(\wedge^2 V)_k$ as a subgeometry of $\PG(\wedge^2 \overline{V})$. Every subspace $\alpha$ of $\PG(V)$ (respectively $\PG(\wedge^2 V)$) then generates a subspace $\alpha_k$ of $\PG(V)_k$ (respectively $\PG(\wedge^2 V)_k$) with the same dimension as $\alpha$. We define $\overline{\alpha} := \alpha_k$ and $\overline{\alpha}_k := \alpha$.\overline{\alpha}

We denote by $Q_k$ the quadric of $\PG(\wedge^2 V_k)$ whose equation with respect to $B^*$ is equal to $X_1X_6 - X_2X_5 + X_3X_4 = 0$, and put $\overline{Q} := \overline{Q}_k$. Then $Q \subseteq Q_k \subseteq \overline{Q}$. The Klein correspondence between the set of lines of $\PG(V_k)$ and the points of $Q_k$ will be denoted by $\kappa_k$. We define $\overline{\kappa} := \kappa_k$. Notice that two distinct lines $L_1$ and $L_2$ of
PG(\(\bar{V}\)) meet if and only if the points \(\overline{\pi}(L_1)\) and \(\overline{\pi}(L_2)\) are \(\overline{Q}\)-collinear.

Now, suppose \(S\) is a regular spread of PG(\(V\)). Then by Lemmas 4.4 and 4.5, there exists a unique subspace \(\alpha\) of dimension 3 of PG(\(\Lambda^2 V\)) such that \(\kappa(S) = \alpha \cap Q\) is a non-singular quadric of Witt index 1 of \(\alpha\). With respect to a suitable reference system of \(\alpha\), the quadric \(\alpha \cap Q\) of \(\alpha\) has equation \(f(X_0, X_1) + X_2X_3 = 0\), where \(f(X_0, X_1)\) is an irreducible quadratic polynomial of \(F[X_0, X_1]\). Now, there exists a unique quadratic extension \(K\) of \(F\) such that \(f(X_0, X_1)\) is reducible when regarded as a polynomial of \(K[X_0, X_1]\). This quadratic extension \(K\) is independent from the reference system of \(\alpha\) with respect to which the equation of \(\alpha \cap Q\) is of the form \(f(X_0, X_1) + X_2X_3 = 0\). Now, we can distinguish two cases.

(I) The quadratic extension \(K/F\) is a Galois extension. Let \(\psi\) denote the unique element in \(\text{Gal}(K/F)\). Then \(f(X_0, X_1) = a(X_0 + \delta X_1)(X_0 + \delta^\psi X_1)\) for a certain \(a \in F \setminus \{0\}\) and a certain \(\delta \in K \setminus F\). It follows that \(\alpha_K \cap Q_K\) is a nonsingular quadric of Witt index 2 of \(\alpha_K\). If \((X_1, \ldots, X_6)\) are the coordinates of a point \(p\) of PG(\(\wedge^2 V_K\)) with respect to the ordered basis \(B^*\), then \(p^\psi\) denotes the point of PG(\(\wedge^2 V\)) whose coordinates with respect to \(B^*\) are equal to \((X_1^\psi, \ldots, X_6^\psi)\). Clearly, \(Q_K^\psi = Q_K\).

(II) The quadratic extension \(K/F\) is not a Galois extension. Then \(\text{char}(K) = 2\) and \(f(X_0, X_1) = a(X_0 + \delta X_1)^2\) for some \(a \in F \setminus \{0\}\) and some \(\delta \in K \setminus F\) satisfying \(\delta^2 \in F\). It follows that \(\alpha_K \cap Q_K\) is a singular quadric of \(\alpha_K\) having a unique singular point\(^1\).

Now, let \(X\) denote the set of all points \(x\) of \(\overline{Q}\) which are \(\overline{Q}\)-collinear with every point of \(\alpha \cap Q\). Notice that \(x \in X\) if and only if \(\overline{\pi}^{-1}(x)\) meets every line \(\overline{L}\) where \(L \in S\). We prove the following lemma which implies Proposition 2.2(b) in the case \(t = 2\).

**Lemma 4.6.**

1. We have \(X \subseteq Q_K\).
2. If \(K/F\) is a Galois extension, then \(|X| = 2\). Moreover, if \(X = \{x_1, x_2\}\), then \(x_2 = x_1^\psi\).
3. If \(K/F\) is not a Galois extension, then \(|X| = 1\).
4. If \(x \in X\), then the points of \(Q\) which are \(Q_K\)-collinear with \(x\) are precisely the points of \(\alpha \cap Q\), or equivalently, the lines of \(S\) are precisely those lines \(L\) of PG(\(V\)) for which \(L_K\) meets \(\kappa_K^{-1}(x)\). The line \(\kappa_K^{-1}(x)\) of PG(\(V_K\)) is disjoint from PG(\(V\)).

\(^1\)With a singular point of a quadric, we mean a point of the quadric with the property that every line through it is a tangent line, i.e. a line which intersects the quadric in either a singleton or the whole line. The tangent hyperplane in a singular point is not defined.
Proof. (I) Suppose the quadratic extension $\mathbb{K}/F$ is a Galois extension. Let $L_1$ and $L_2$ be two disjoint lines of $\alpha \cap Q$ and let $\beta_1, \beta_2$ denote the two planes of $Q_\mathbb{K}$ through $L_1$. Then $\beta_1$ and $\beta_2$ are the two planes of $Q_\mathbb{K}$ through $L_1$. Let $x_i, i \in \{1, 2\}$, denote the unique point of $\beta_i$ $Q_\mathbb{K}$-collinear with every point of $L_2$. Then $x_i$ is also the unique point of $\beta_i$ $Q_\mathbb{K}$-collinear with every point of $L_2$.

Let $i \in \{1, 2\}$. We prove that $x_i \notin \text{PG}(\Lambda^2 V)$, or equivalently, that $x_i \notin Q$. Suppose this is not the case and consider the hyperplane $T$ of $\text{PG}(\Lambda^2 V)$ which is tangent to $Q$ at the point $x_i$. Then $T_\mathbb{K}$ is the hyperplane of $\text{PG}(\Lambda^2 V_\mathbb{K})$ which is tangent to $Q_\mathbb{K}$ at the point $x_i$. Since $L_1 \cup L_2 \subseteq T_\mathbb{K}$, $\alpha$ is a hyperplane of $T$ not containing $x_i$ and hence $\alpha \cap Q$ would be a nonsingular quadric of Witt index 2 of $\alpha$, clearly a contradiction.

We prove that $X = \{x_1, x_2\}$. Clearly, $\{x_1, x_2\} \subseteq X$. Conversely, suppose that $x$ is a point of $X$. Since no point of $T_{\mathbb{K}}$ is $Q_\mathbb{K}$-collinear with every point of $L_2$, we have $x \notin T_{\mathbb{K}}$. Since $x$ is collinear with every point of $T_{\mathbb{K}}$, we have $\langle x, T_{\mathbb{K}} \rangle = \beta_i$ for some $i \in \{1, 2\}$. Since $x$ is $Q_\mathbb{K}$-collinear with every point of $L_2$, we necessarily have $x = x_i$. Hence, $X = \{x_1, x_2\} \subseteq Q_\mathbb{K}$. Since $x_1$ is $Q_\mathbb{K}$-collinear with every point of $\alpha \cap Q$, $x_1 \in Q_\mathbb{K}$-collinear with every point of $(\alpha \cap Q)^{\vee} = \alpha \cap Q$. It follows that $x_1 \in Q_\mathbb{K}$.

(II) Suppose the quadratic extension $\mathbb{K}/F$ is not a Galois extension. Then $\alpha \cap Q_\mathbb{K}$ is a singular quadric of $\alpha \mathbb{K}$ with a unique singular point $x^*$. Clearly, $x^* \notin \text{PG}(\Lambda^2 V)$ and $x^* \notin Q$.

We prove that $X = \{x^*\}$. Clearly, $x^* \in X$. Suppose now that there exists a point $x \in X \setminus \{x^*\}$. Then $x$ is $Q_\mathbb{K}$-collinear with every point of $\overline{\pi} \cap Q_\mathbb{K}$ and hence cannot be contained in $\overline{\pi}$ since $x \neq x^*$. The points of $Q_\mathbb{K}$ which are $Q_\mathbb{K}$-collinear with $x$ and $x^*$ are contained in a 3-dimensional subspace of $\text{PG}(\Lambda^2 V)$, namely the intersection of the tangent hyperplanes to $Q_\mathbb{K}$ at the points $x$ and $x^*$. This 3-dimensional subspace necessarily coincides with $\overline{\pi}$ and contains the points $x$ and $x^*$, a contradiction, since $x \notin \overline{\pi}$. So, we have that $X = \{x^*\} \subseteq Q_\mathbb{K}$.

Now, let $x$ be an arbitrary point of $X$. Then $x \in \text{PG}(\Lambda^2 V_\mathbb{K}) \setminus \text{PG}(\Lambda^2 V)$. By Lemma 2.1, there exist two distinct points $x_1$ and $x_2$ of $\text{PG}(\Lambda^2 V)$ such that $x \in x_1 \cap x_2$. Let $\zeta$ denote the orthogonal or symplectic polarity of $\text{PG}(\Lambda^2 V_\mathbb{K})$ associated to the quadric $Q_\mathbb{K}$. We prove that the points of $Q$ which are $Q_\mathbb{K}$-collinear with $x$ are precisely the points of $\alpha \cap Q$. Since $x \in X$, every point of $\alpha \cap Q$ is $Q_\mathbb{K}$-collinear with $x$. Conversely, suppose that $y$ is a point of $Q$ which is $Q_\mathbb{K}$-collinear with $x$. Then $x \in y^{\zeta}$. By Lemma 2.1 applied to the subspace $y^{\zeta}$, we see that $x_1, x_2 \in y^{\zeta}$ and hence $y \in x_1^{\zeta} \cap x_2^{\zeta}$. Now, $x_1^{\zeta} \cap x_2^{\zeta}$ is a 3-dimensional subspace of $\text{PG}(\Lambda^2 V_\mathbb{K})$ which necessarily coincides with $\alpha \mathbb{K}$ since every point of $\alpha \cap Q$ is $Q_\mathbb{K}$-collinear with $x$. So, $y \in \alpha \mathbb{K}$ and hence $y \in Q \cap \alpha$. 

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If $p$ would be a point of $\text{PG}(V)$ contained in $\kappa^{-1}(x)$, then every line of $\text{PG}(V)$ through $p$ would be contained in the spread $S$, clearly a contradiction.

**Remark 4.7.** If we go back to Proposition 2.2(b) and regard $\text{PG}(2t-1,\mathbb{F})$ as a subgeometry of $\text{PG}(2t-1,\overline{\mathbb{F}})$, where $\overline{\mathbb{F}}$ is a fixed algebraic closure of $\mathbb{F}$, then Lemma 4.6 implies that there exists a unique quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$ for which the corresponding subgeometry $\text{PG}(2t-1,\mathbb{F}')$ of $\text{PG}(2t-1,\overline{\mathbb{F}})$ satisfies the properties (i) or (ii) of Proposition 2.2(b).

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