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THE EIGENVALUE DISTRIBUTION OF BLOCK DIAGONALLY DOMINANT MATRICES AND BLOCK $H-$MATRICES*

CHENG-YI ZHANG†, SHUANGHUA LUO‡, AIQUN HUANG§, AND JUNXIANG LU¶

Abstract. The paper studies the eigenvalue distribution of some special matrices, including block diagonally dominant matrices and block $H-$matrices. A well-known theorem of Taussky on the eigenvalue distribution is extended to such matrices. Conditions on a block matrix are also given so that it has certain numbers of eigenvalues with positive and negative real parts.

Key words. Eigenvalues, Block diagonally dominant, Block $H-$matrix, Non-Hermitian positive (negative) definite.

AMS subject classifications. 15A15, 15F10.

1. Introduction. The eigenvalue distribution of a matrix has important consequences and applications (see e.g., [4], [6], [9], [12]). For example, consider the ordinary differential equation (cf. Section 2.0.1 of [4])

$$\frac{dx}{dt} = A[x(t) - \hat{x}], \tag{1.1}$$

where $A \in C^{n\times n}$ and $x(t), \hat{x} \in C^n$. The vector $\hat{x}$ is an equilibrium of this system. It is not difficult to see that $\hat{x}$ of system is globally stable if and only if each eigenvalue of $-A$ has positive real part, which concerns the eigenvalue distribution of the matrix A. The analysis of stability of such a system appears in mathematical biology, neural networks, as well as many problems in control theory. Therefore, there is considerable interest in the eigenvalue distribution of some special matrices A and some results are classical. For example, Taussky in 1949 [14] stated the following theorem.

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THEOREM 1.1. ([14]) *Let $A = (a_{ij}) \in C^{n\times n}$ be strictly or irreducibly diagonally dominant with positive real diagonal entries a_{ii} for all $i \in N = \{1, 2, \cdots, n\}$. Then for arbitrary eigenvalue λ of A, we have $Re(\lambda) > 0$.*

Tong [15] improved Taussky's result in [14] and proposed the following theorem on the eigenvalue distribution of strictly or irreducibly diagonally dominant matrices.

THEOREM 1.2. ([15]) *Let $A = (a_{ij}) \in C^{n\times n}$ be strictly or irreducibly diagonally dominant with real diagonal entries a_{ii} for all $i \in N$. Then A has $|J_+(A)|$ eigenvalues with positive real part and $|J_-(A)|$ eigenvalues with negative real part, where $J_+(A) = \{i \mid a_{ii} > 0,\ i \in N\},\ J_-(A) = \{i \mid a_{ii} < 0,\ i \in N\}$.*

Later, Jiaju Zhang [23], Zhaoyong You et al. [18] and Jianzhou Liu et al. [6] extended Tong's results in [15] to conjugate diagonally dominant matrices, generalized conjugate diagonally dominant matrices and $H-$matrices, respectively. Liu's result is as follows.

THEOREM 1.3. ([6]) *Let $A = (a_{ij}) \in H_n$ with real diagonal entries a_{ii} for all $i \in N$. Then A has $|J_+(A)|$ eigenvalues with positive real part and $|J_-(A)|$ eigenvalues with negative real part.*

Recently, Cheng-yi Zhang et al. ([21], [22]) generalized Tong's results in [15] to nonsingular diagonally dominant matrices with complex diagonal entries and established the following conclusion.

THEOREM 1.4. ([21], [22]) Given a matrix $A \in C^{n\times n}$, if $\widehat{A}$ is nonsingular diagonally dominant, where $\widehat{A} = (\widehat{a}_{ij}) \in C^{n\times n}$ is defined by

$$\widehat{a}_{ij} = \begin{cases} Re(a_{ii}), & \text{if} \quad i = j, \\ a_{ij}, & \text{if} \quad i \neq j, \end{cases}$$

then A has $|J_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)|$ eigenvalues with negative real part, where $J_{R_+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in N\},\ J_{R_-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in N\}$.

However, there exists a dilemma in practical application. That is, for a large-scale matrix or a matrix which is neither a diagonally dominant matrix nor an $H-$matrix, it is very difficult to obtain the property of this class of matrices.

On the other hand, David G. Feingold and Richard S. Varga [2], Zhao-yong You and Zong-qian Jiang [18] and Shu-huang Xiang [17], respectively, generalized the concept of diagonally dominant matrices and proposed two classes of block diagonally dominant matrices, i.e., $I-$block diagonally dominant matrices [2] and II$-$block diagonally dominant matrices [14], [15]. Later, Ben Polman [10], F. Robert [11], Yong-zhong Song [13], L.Yu. Kolotilina [5] and Cheng-yi Zhang and Yao-tang Li [20]

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also presented two classes of block $H-$matrices such as $I-$block $H-$matrices[11] and II$-$block $H-$matrices[10] on the basis of the previous work.

It is known that a block diagonally dominant matrix is not always a diagonally dominant matrix (an example is seen in [2, (2.6)]). So suppose a matrix A is not strictly (or irreducibly) diagonally dominant nor an $H-$matrix. Using and appropriate partitioning of A, can we obtain its eigenvalue distribution when it is block diagonally dominant or a block $H-$matrix?

David G. Feingold and Richard S. Varga (1962) showed that an $I-$block strictly or irreducibly diagonally dominant diagonally dominant matrix has the same property as the one in Theorem 1.1. The result reads as follows.

THEOREM 1.5. ([2]) *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ be $I-$block strictly or irreducibly diagonally dominant with all the diagonal blocks being $M-$matrices. Then for arbitrary eigenvalue λ of A, we have $Re(\lambda) > 0$.*

The purpose of this paper is to establish some theorems on the eigenvalue distribution of block diagonally dominant matrices and block H-matrices. Following the result of David G. Feingold and Richard S. Varga, the well-known theorem of Taussky on the eigenvalue distribution is extended to block diagonally dominant matrices and block $H-$matrices with each diagonal block being non-Hermitian positive definite. Then, the eigenvalue distribution of some special matrices, including block diagonally dominant matrices and block H-matrices, is studied further to give conditions on the block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ such that the matrix A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part; here $J_P^+(A)$ ($J_P^-(A)$) denotes the set of all indices of non-Hermitian positive (negative) definite diagonal blocks of A and n_k is the order of the diagonal block A_{kk} for $k \in J_P^+(A) \cup J_P^-(A)$.

The paper is organized as follows. Some notation and preliminary results about special matrices including block diagonally dominant matrices and block $H-$matrices are given in Section 2. The theorem of Taussky on the eigenvalue distribution is extended to block diagonally dominant matrices and block H-matrices in Section 3. Some results on the eigenvalue distribution of block diagonally dominant matrices and block H-matrices are then presented in Section 4. Conclusions are given in Section 5.

2. Preliminaries. In this section we present some notions and preliminary results about special matrices that are used in this paper. Throughout the paper, we denote the conjugate transpose of the vector x, the conjugate transpose of the matrix A, the spectral norm of the matrix A and the cardinality of the set α by x^H, A^H, $\|A\|$ and $|\alpha|$, respectively. $C^{m\times n}$ ($R^{m\times n}$) will be used to denote the set of all $m \times n$ complex (real) matrices. Let $A = (a_{ij}) \in R^{m\times n}$ and $B = (b_{ij}) \in R^{m\times n}$, we write $A \geq B$,

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if $a_{ij} \geq b_{ij}$ holds for all $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$. A matrix $A = (a_{ij}) \in R^{n\times n}$ is called a $Z-$matrix if $a_{ij} \leq 0$ for all $i \neq j$. We will use Z_n to denote the set of all $n \times n$ $Z-$matrices. A matrix $A = (a_{ij}) \in R^{n\times n}$ is called an $M-$matrix if $A \in Z_n$ and $A^{-1} \geq 0$. M_n will be used to denote the set of all $n \times n$ $M-$matrices.

The comparison matrix of a given matrix $A = (a_{ij}) \in C^{n\times n}$, denoted by $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

It is clear that $\mu(A) \in Z_n$ for a matrix $A \in C^{n\times n}$. A matrix $A \in C^{n\times n}$ is called $H-$matrix if $\mu(A) \in M_n$. H_n will denote the set of all $n \times n$ $H-$matrices.

A matrix $A \in C^{n\times n}$ is called Hermitian if $A^H = A$ and skew-Hermitian if $A^H = -A$. A Hermitian matrix $A \in C^{n\times n}$ is called Hermitian positive definite if $x^H Ax > 0$ for all $0 \neq x \in C^n$ and Hermitian negative definite if $x^H Ax < 0$ for all $0 \neq x \in C^n$. A matrix $A \in C^{n\times n}$ is called non-Hermitian positive definite if $Re(x^H Ax) > 0$ for all $0 \neq x \in C^n$ and non-Hermitian negative definite if $Re(x^H Ax) < 0$ for all $0 \neq x \in C^n$. Let $A \in C^{n\times n}$, then $H = (A + A^H)/2$ and $S = (A - A^H)/2$ are called the Hermitian part and the skew-Hermitian

part of the matrix A, respectively. Furthermore, A is non-Hermitian positive (negative) definite if and only if H is Hermitian positive (negative) definite (see [3,7,8]).

Let $x = (x_1, x_2, \cdots, x_n)^T \in C^n$. The Euclidean norm of the vector x is defined by $\|x\| = \sqrt{(x^H x)} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ and the spectral norm of the matrix $A \in C^{n\times n}$ is defined by

$$\|A\| = \sup_{0\neq x\in C^n} \left(\frac{\|Ax\|}{\|x\|} \right) = \sup_{\|y\|=1} (\|Ay\|) . \tag{2.1}$$

If A is nonsingular, it is useful to point out that

$$\|A^{-1}\|^{-1} = \inf_{0\neq x\in C^n} \left(\frac{\|Ax\|}{\|x\|} \right). \tag{2.2}$$

With (2.2), we can then define $\|A^{-1}\|^{-1}$ by continuity to be zero whenever A is singular. Therefore, for $B \in C^{n\times n}$ and $0 \neq C \in C^{n\times m}$,

$$\|B^{-1}C\|^{-1} = \inf_{0\neq x\in C^n} \left(\frac{\|B^H x\|}{\|C^H x\|} \right) \tag{2.3}$$

if B is nonsingular, and

$$\|B^{-1}C\|^{-1} \to 0 \Rightarrow \|B^{-1}C\| \to \infty \tag{2.4}$$

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by continuity if B is singular.

A matrix $A \in C^{n\times n}(n \geq 2)$ is called reducible if there exists an $n \times n$ permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where $A_{11} \in C^{r\times r}$, $A_{22} \in C^{(n-r)\times(n-r)}$, $1 \leq r < n$. If no such permutation matrix exists, then A is called irreducible. $A = (a_{11}) \in C^{1\times 1}$ is irreducible if $a_{11} \neq 0$, and reducible, otherwise.

A matrix $A = (a_{ij}) \in C^{n\times n}$ is diagonally dominant by row if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}| \tag{2.5}$$

holds for all $i \in N = \{1, 2, \cdots, n\}$. If inequality in (2.5) holds strictly for all $i \in N$, A is called strictly diagonally dominant by row; if A is irreducible and diagonally dominant with inequality (2.5) holding strictly for at least one $i \in N$, A is called irreducibly diagonally dominant by row.

By D_n, SD_n and ID_n denote the sets of matrices which are $n \times n$ diagonally dominant, $n \times n$ strictly diagonally dominant and $n \times n$ irreducibly diagonally dominant, respectively.

Let $A = (a_{ij}) \in C^{n\times n}$ be partitioned as the following form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{bmatrix}, \tag{2.6}$$

where A_{ll} is an $n_l \times n_l$ nonsingular principal submatrix of A for all $l \in S = \{1, 2, \cdots, s\}$ and $\sum_{l=1}^{s} n_l = n$. By $C_s^{n\times n}$ denote the set of all $s \times s$ block matrices in $C^{n\times n}$ partitioned as (2.6). Note: $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ implies that each diagonal block A_{ll} of the block matrix A is nonsingular for all $l \in S$.

Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ and $S = \{1, 2, \cdots, s\}$, we define index sets

$$J_P^+(A) = \{i \mid A_{ii}\ is\ non-Hermitian\ positive\ definite,\ i \in S\},$$
$$J_P^-(A) = \{i \mid A_{ii}\ is\ non-Hermitian\ negative\ definite,\ i \in S\}.$$

From the index sets above, we know that $J_P^+(A) = S$ shows that each diagonal block A_{ll} of A is non-Hermitian positive definite for all $l \in S$ and $J_P^+(A) \cup J_P^-(A) = S$

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shows that each diagonal block A_{ll} of A is either non-Hermitian positive definite or non-Hermitian negative definite for all $l \in S$.

Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$. Then A is called block irreducible if either its $I-$block comparison matrix $\mu_I(A) = (w_{lm}) \in R^{s\times s}$ or its $\text{II}-$block comparison matrix $\mu_{\text{II}}(A) = (m_{lm}) \in R^{s\times s}$ is irreducible, where

$$(2.7)\quad w_{lm} = \begin{cases} \|A_{ll}^{-1}\|^{-1}, & l = m \\ -\|A_{lm}\|, & l \neq m \end{cases}, \quad m_{lm} = \begin{cases} 1, & l = m \\ -\parallel A_{ll}^{-1}A_{lm} \parallel, & l \neq m \end{cases}.$$

A block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ is called $I-$block diagonally dominant if its $I-$block comparison matrix comparison matrix, $\mu_I(A) \in D_s$. If $\mu_I(A) \in SD_s$, A is $I-$block strictly diagonally dominant; and if $\mu_I(A) \in ID_s$, A is called $I-$block irreducibly diagonally dominant.

Similarly, a block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ is called $\text{II}-$block diagonally dominant if its $\text{II}-$block comparison matrix, $\mu_{\text{II}}(A) \in D_s$. If $\mu_{\text{II}}(A) \in SD_s$, A is $\text{II}-$block strictly diagonally dominant; and if $\mu_{\text{II}}(A) \in ID_s$, A is called $\text{II}-$block irreducibly diagonally dominant.

A block matrix $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ is called an $I-$block $H-$matrix (resp., a $\text{II}-$block $H-$matrix) if its $I-$block comparison matrix $\mu_I(A) = (w_{lm}) \in R^{s\times s}$ (resp., its $\text{II}-$block comparison matrix $\mu_{\text{II}}(A) = (m_{lm}) \in R^{s\times s}$) is an $s \times s$ $M-$matrix.

In the rest of this paper, we denote the set of all $s\times s$ $I-$block (strictly, irreducibly) diagonally dominant matrices, all $s \times s$ $\text{II}-$block (strictly, irreducibly) diagonally dominant matrices, all $s\times s$ $I-$block $H-$matrices and all $s\times s$ $\text{II}-$block $H-$matrices by $IBD_s(IBSD_s,\ IBID_s)$, $\text{II}BD_s(\text{II}BSD_s,\ \text{II}BID_s)$, IBH_s and $\text{II}BH_s$, respectively.

It follows that we will give some lemmas to be used in the following sections.

Lemma 2.1. (see [2,5]) *If A block matrix $A = (A_{lm})_{s\times s} \in IBSD_s \cup IBID_s$, then A is nonsingular.*

Lemma 2.2. $IBSD_s \cup IBID_s \subset IBH_s$ and $\text{II}BSD_s \cup \text{II}BID_s \subset \text{II}BH_s$

Proof. According to the definition of $I-$block strictly or irreducibly diagonally dominant matrices, $\mu_I(A) \in SD_s \cup ID_s$ for any block matrix $A \in IBSD_s \cup IBID_s$. Since $SD_s \cup ID_s \subset H_s$ (see Lemma 2.3 in [21]), $\mu_I(A) \in H_s$. As a result, $A \in IBH_s$ coming from the definition of $I-$block $H-$matrices. Therefore, $IBSD_s \cup IBID_s \subset IBH_s$. Similarly, we can prove $\text{II}BSD_s \cup \text{II}BID_s \subset \text{II}BH_s$. ∎

Lemma 2.3. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$. Then $A \in \text{II}BH_s$ (IBH_s) if and only if there exists a block diagonal matrix $D = diag(d_1I_1, \cdots, d_sI_s)$, where $d_l > 0$, I_l is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $AD \in \text{II}BSD_s$ $(IBSD_s)$.*

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Proof. Using Theorem 6.2.3 (M_{35}) in [1, pp.136-137], the conclusion of this lemma is obtained immediately. ◻

LEMMA 2.4. (see [7) *If a matrix $A \in C^{n\times n}$ is non-Hermitian positive definite, then for arbitrary eigenvalue λ of A, we have $Re(\lambda) > 0$.*

LEMMA 2.5. (see [16]) *Let $A \in C^{n\times n}$. Then*

$$\|A\| = \rho(A^H A), \tag{2.8}$$

the spectral radius of $A^H A$. In particular, if A is Hermitian, then $\|A\| = \rho(A)$.

3. Some generalizations of Taussky's theorem. In this section, the famous Taussky's theorem on the eigenvalue distribution is extended to block diagonally dominant matrices and block H-matrices. The following lemmas will be used in this section.

LEMMA 3.1. (see [8]) *Let $A = (a_{ij}) \in C^{n\times n}$ be non-Hermitian positive definite with Hermitian part $H = (A + A^H)/2$. Then*

$$\|A^{-1}\| \le \|H^{-1}\|. \tag{3.1}$$

LEMMA 3.2. *Let $A \in C^{n\times n}$ be non-Hermitian positive definite with Hermitian part $H = (A+A^H)/2$. Then for arbitrary complex number $\alpha \neq 0$ with $Re(\alpha) \ge 0$, we have*

$$\|(\alpha I + A)^{-1}\| \le \|H^{-1}\|, \tag{3.2}$$

where I is the identity matrix and $\|A\|$ is the spectral norm of the matrix A.

Proof. Since A is non-Hermitian positive definite, for arbitrary complex number $\alpha \neq 0$ with $Re(\alpha) \ge 0$, we have $\alpha I + A$ is non-Hermitian positive definite. It then follows from Lemma 3.1 that

$$\|(\alpha I + A)^{-1}\| \le \|[Re(\alpha)I + H]^{-1}\|. \tag{3.3}$$

Since H is Hermitian positive definite, so is $Re(\alpha)I+H$. Thus, the smallest eigenvalue of $Re(\alpha)I + H$ is $\tau(Re(\alpha)I + H) = Re(\alpha) + \tau(H)$. Following Lemma 2.5, we have

$$\begin{aligned} \|[Re(\alpha)I + H]^{-1}\| &= \rho([Re(\alpha)I + H]^{-1}) = \frac{1}{\tau(Re(\alpha)I + H)} \\ &= \frac{1}{Re(\alpha) + \tau(H)} \le \frac{1}{\tau(H)} = \rho(H^{-1}) \\ &= \|H^{-1}\|. \end{aligned} \tag{3.4}$$

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Then it follows from (3.3) and (3.4) that $\|(\alpha I + A)^{-1}\| \le \|H^{-1}\|$, which completes the proof. □

LEMMA 3.3. (see [2], The generalization of the Gersgorin Circle Theorem) *Let $A \in C^{n\times n}$ be partitioned as* (2.6)*. Then each eigenvalue λ of A satisfies that*

$$\|(\lambda I - A_{ll})^{-1}\|^{-1} \le \sum_{m=1, m\neq l}^{s} \|A_{lm}\| \tag{3.5}$$

holds for at least one $l \in S$.

THEOREM 3.4. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in IBSD_s$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined by*

$$\widehat{A}_{lm} = \begin{cases} H_{ll} = (A_{ll} + A_{ll}^H)/2, & l = m \\ A_{lm}, & otherwise, \end{cases} \tag{3.6}$$

then for any eigenvalue λ of A, we have $Re(\lambda) > 0$.

Proof. The conclusion can be proved by contradiction. Assume that λ be any eigenvalue of A with $Re(\lambda) \le 0$. Following Lemma 3.3, (3.5) holds for at least one $l \in S$. Since $J_P^+(A) = S$ shows that each diagonal block A_{ll} of A is non-Hermitian positive definite for all $l \in S$, it follows from Lemma 3.2 that

$$\|(\lambda I - A_{ll})^{-1}\| \le \|H_{ll}^{-1}\|$$

and hence

$$\|(\lambda I - A_{ll})^{-1}\|^{-1} \ge \|H_{ll}^{-1}\|^{-1} \tag{3.7}$$

for all $l \in S$. According to (3.5) and (3.7) we have that

$$\|H_{ll}^{-1}\|^{-1} \le \sum_{m=1, m\neq l}^{s} \|A_{lm}\|$$

holds for at least one $l \in S$. This shows $\mu_I(\widehat{A}) = (w_{lm}) \notin SD_s$. As a result, $\widehat{A} \notin IBSD_s$, which is in contradiction with the assumption $\widehat{A} \in IBSD_s$. Therefore, the conclusion of this theorem holds. □

THEOREM 3.5. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in IBH_s$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in* (3.6)*, then for any eigenvalue λ of A, we have $Re(\lambda) > 0$.*

Proof. Since $\widehat{A} \in IBH_s$, it follows from Lemma 2.3 that there exists a block diagonal matrix $D = diag(d_1 I_1, \cdots, d_s I_s)$, where $d_l > 0$, I_l is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $\widehat{A}D \in IBSD_s$, i.e.,

$$\|H_{ll}^{-1}\|^{-1} d_l > \sum_{m=1, m\neq l}^{s} \|A_{lm}\| d_m \tag{3.8}$$

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holds for all $l \in S$. Multiply the inequality (3.8) by d_l^{-1}, then

$$\|H_{ll}^{-1}\|^{-1} > \sum_{m=1, m\neq l}^{s} d_l^{-1}\|A_{lm}\|d_m = \sum_{m=1, m\neq l}^{s} \|d_l^{-1}A_{lm}d_m\| \tag{3.9}$$

holds for all $l \in S$. Since $J_P^+(D^{-1}AD) = J_P^+(A) = S$ and (3.9) shows $D^{-1}\widehat{A}D \in IBSD_s$, Theorem 3.4 yields that for any eigenvalue λ of $D^{-1}AD$, we have $Re(\lambda) > 0$. Again, since A has the same eigenvalues as $D^{-1}AD$, for any eigenvalue λ of A, we have $Re(\lambda) > 0$. This completes the proof. ◻

Using Lemma 2.2 and Theorem 3.5, we can obtain the following corollary.

Corollary 3.6. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in IBID_s$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in* (3.6)*, then for any eigenvalue λ of A, we have $Re(\lambda) > 0$.*

The following will extend the result of Theorem 1.1 to II$-$block diagonally dominant matrices and II$-$block $H-$matrices. First, we will introduce some relevant lemmas.

Lemma 3.7. (see [3]) *Let $A \in C^{n\times n}$. Then the following conclusions are equivalent.*

1. *A is Hermitian positive definite;*
2. *A is Hermitian and each eigenvalue of A is positive;*
3. *A^{-1} is also Hermitian positive definite.*

Lemma 3.8. *Let $A \in C^{n\times n}$ be nonsingular. Then*

$$\|A^{-1}\|^{-1} = \tau(A^H A), \tag{3.10}$$

where $\tau(A^H A)$ denote the minimal eigenvalue of the matrix $A^H A$.

Proof. It follows from equality (2.8) in Lemma 2.5 that

$$\|A^{-1}\| = \rho[(AA^H)^{-1}] = \rho[(A^H A)^{-1}] = \frac{1}{\tau(A^H A)}, \tag{3.11}$$

which yields equality (3.10). ◻

Lemma 3.9. *Let $A \in C^{n\times n}$ be Hermitian positive definite and let $B \in C^{n\times m}$. Then for arbitrary complex number $\alpha \neq 0$ with $Re(\alpha) \geq 0$, we have*

$$\|(\alpha I + A)^{-1}B\| \leq \|A^{-1}B\|, \tag{3.12}$$

where I is identity matrix and $\|A\|$ is the spectral norm of the matrix A.

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Proof. The theorem is obvious if $B = 0$. Since A is Hermitian positive definite and $\alpha \neq 0$ with $Re(\alpha) \geq 0$, $\alpha I + A$ is non-Hermitian positive definite. Hence, $\alpha I + A$ is nonsingular. As a result, $(\alpha I + A)^{-1}B \neq 0$ and consequently $\|(\alpha I + A)^{-1}B\| \neq 0$ for $B \neq 0$. Thus, if $B \neq 0$, it follows from (2.1) and (2.2) that for arbitrary vector $x,\ y \in C^m,\ \|x\| = \|y\| = 1$, we have

$$\begin{aligned}
\frac{\|A^{-1}B\|}{\|(\alpha I + A)^{-1}B\|} &= \frac{\sup\limits_{\|x\|=1} (\|A^{-1}Bx\|)}{\sup\limits_{\|y\|=1} (\|(\alpha I + A)^{-1}By\|)} \\
&\geq \frac{\|A^{-1}By\|}{\sup\limits_{\|y\|=1} (\|(\alpha I + A)^{-1}By\|)} \quad (set\ x = y) \\
&= \inf_{\|y\|=1} \left(\frac{\|A^{-1}By\|}{\|(\alpha I + A)^{-1}By\|} \right) \\
&= \inf_{0 \neq z \in C^n} \left(\frac{\|A^{-1}z\|}{\|(\alpha I + A)^{-1}z\|} \right) \quad (set\ z = By \in C^n) \\
&= \inf_{0 \neq u \in C^n} \left(\frac{\|A^{-1}(\alpha I + A)u\|}{\|u\|} \right) \quad (u = (\alpha I + A)^{-1}z \in C^n) \\
&= \|[A^{-1}(\alpha I + A)]^{-1}\|^{-1} \quad (from (2.2)) \\
&= \|(\alpha A^{-1} + I)^{-1}\|^{-1}.
\end{aligned} \tag{3.13}$$

According to Lemma 3.8,

$$\begin{aligned}
\|(\alpha A^{-1} + I)^{-1}\|^{-1} &= \tau[(\alpha A^{-1} + I)^H(\alpha A^{-1} + I)] \\
&= \tau(I + 2Re(\alpha)A^{-1} + |\alpha|^2 A^{-2}).
\end{aligned} \tag{3.14}$$

Since A is Hermitian positive definite, it follows from Lemma 3.7 that A^{-1} and A^{-2} are also. Therefore, $\alpha \neq 0$, together with $Re(\alpha) \geq 0$, implies that $2Re(\alpha)A^{-1} + |\alpha|^2A^{-2}$ is also Hermitian positive definite. As a result, $\tau(2Re(\alpha)A^{-1} + |\alpha|^2A^{-2}) > 0$. Thus, following (3.13) and (3.14), we get

$$\begin{aligned}
\frac{\|A^{-1}B\|}{\|(\alpha I + A)^{-1}B\|} &\geq \|(\alpha A^{-1} + I)^{-1}\|^{-1} \\
&= \tau(I + 2Re(\alpha)A^{-1} + |\alpha|^2A^{-2}) \\
&= 1 + \tau(2Re(\alpha)A^{-1} + |\alpha|^2A^{-2}) \\
&> 1,
\end{aligned} \tag{3.15}$$

from which it is easy to obtain (3.12). This completes the proof. ◻

Theorem 3.10. *Let $A = (A_{lm})_{s\times s} \in \text{II}BSD_s \cup \text{II}BID_s$ with $J_P^+(A) = S$. If $\widehat{A} = A$, where $\widehat{A}$ is defined in* (3.6)*, then for any eigenvalue λ of A, we have* $Re(\lambda) > 0$.

Proof. We prove by contradiction. Assume that λ is any eigenvalue of A with $Re(\lambda) \leq 0$. Then the matrix $\lambda I - A$ is singular. Since $\widehat{A} = A$ and (3.6) imply that the

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diagonal block A_{ll} of A is Hermitian for all $l \in S$, $J_P^+(A) = S$ yields that the diagonal block A_{ll} of A is Hermitian positive definite for all $l \in S$. Thus $\lambda I - A_{ll}$ is non-Hermitian negative definite for all $l \in S$. As a result, $D(\lambda) = diag(\lambda I - A_{11}, \cdots, \lambda I - A_{ss})$ is also non-Hermitian negative definite. Therefore, the matrix

$$\mathscr{A}(\lambda) := [D(\lambda)]^{-1}(\lambda I - A) = (\mathscr{A}_{lm})_{s\times s} \in C_s^{n\times n}$$

is singular, where $\mathscr{A}_{ll} = I_l$, the $n_l \times n_l$ identity matrix and $\mathscr{A}_{lm} = (\lambda I - A_{ll})^{-1}A_{lm}$ for $l \neq m$ and $l, m \in S$. It follows from Lemma 3.9 that

$$\|(\lambda I - A_{ll})^{-1}A_{lm}\| \leq \|A_{ll}^{-1}A_{lm}\| \tag{3.16}$$

for $l \neq m$ and $l, m \in S$. Since $A \in \text{II}BSD_s \cup \text{II}BID_s$,

$$1 \geq \sum_{m=1, m\neq l}^{s} \|A_{ll}^{-1}A_{lm}\| \tag{3.17}$$

holds for all $l \in S$ and the inequality in (3.17) holds strictly for at least one $i \in S$. Then from (3.16) and (3.17), we have

$$1 \geq \sum_{m=1, m\neq l}^{s} \|(\lambda I - A_{ll})^{-1}A_{lm}\| \tag{3.18}$$

holds for all $l \in S$ and the inequality in (3.18) holds strictly for at least one $i \in S$. If $A \in \text{II}BSD_s$, then the inequality in (3.17) and hence the one in (3.18) both hold strictly for all $i \in S$. That is, $\mathscr{A}(\lambda) \in IBSD_s$. If A is block irreducible, then so is $\mathscr{A}(\lambda)$. As a result, $A \in \text{II}BID_s$ yields $\mathscr{A}(\lambda) \in IBID_s$. Therefore, $A \in \text{II}BSD_s \cup \text{II}BID_s$ which implies $\mathscr{A}(\lambda) \in IBSD_s \cup IBID_s$. Using Lemma 2.1, $\mathscr{A}(\lambda)$ is nonsingular and consequently $\lambda I - A$ is nonsingular, which contradicts the singularity of $\lambda I - A$. This shows that the assumption is incorrect. Thus, for any eigenvalue λ of A, we have $Re(\lambda) > 0$. □

THEOREM 3.11. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in \text{II}BSD_s \cup \text{II}BID_s$, where $\widehat{A}$ is defined by* (3.6)*, then for any eigenvalue λ of A, we have $Re(\lambda) > 0$.*

Proof. Since $\widehat{A} \in \text{II}BSD_s \cup \text{II}BID_s$, it follows from Lemma 2.2 that $\widehat{A} \in \text{II}BH_s$ and $\widehat{A}^H \in \text{II}BH_s$. Following Lemma 2.3, there exists a block diagonal matrix $D = diag(d_1 I_1, \cdots, d_s I_s)$, where $d_l > 0$, I_l is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $\widehat{A}^H D = (D\widehat{A})^H \in \text{II}BSD_s$. Again, since $\widehat{A}$ is II−block strictly or irreducibly diagonally dominant by row, so is $D\widehat{A}$. Furthermore, (3.6) implies that diagonal blocks of $\widehat{A}$ are all Hermitian, and hence, so are the diagonal blocks of $D\widehat{A}$ and $(D\widehat{A})^H$. Therefore, from the definition of II−block strictly or irreducibly

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diagonally dominant matrix by row, we have

$$1 \geq \sum_{m=1,m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_l\widehat{A}_{lm})\| = \sum_{m=1,m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_lA_{lm})\| \tag{3.19}$$

and

$$1 > \sum_{m=1,m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_m\widehat{A}_{ml}^H)\| = \sum_{m=1,m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_mA_{ml}^H)\| \tag{3.20}$$

hold for all $l \in S$. Therefore, according to (3.19) and (3.20), we have

$$\begin{aligned}
&1 - \sum_{m=1,m\neq l}^{s} \|(d_lA_{ll} + d_lA_{ll}^H)^{-1}(d_lA_{lm} + d_mA_{ml}^H)\| \\
&= 1 - \sum_{m=1,m\neq l}^{s} \|(2d_l\widehat{A}_{ll})^{-1}(d_lA_{lm} + d_mA_{ml}^H)\| \\
&\geq 1 - \frac{1}{2}\sum_{m=1,m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_lA_{ml}) + (d_l\widehat{A}_{ll})^{-1}(d_mA_{ml}^H)\| \\
&\geq 1 - \frac{1}{2}\sum_{m=1,m\neq l}^{s} \left[\|(d_l\widehat{A}_{ll})^{-1}(d_lA_{ml}\| + \|d_l\widehat{A}_{ll})^{-1}(d_mA_{ml}^H)\|\right] \\
&\geq \frac{1}{2}\left[(1 - \sum_{m=1,m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_lA_{ml}\|) + (1 - \sum_{m=1,m\neq l}^{s} \|(d_l\widehat{A}_{ll})^{-1}(d_mA_{lm}^H\|)\right] \\
&> 0,
\end{aligned}$$

which indicates that $DA + (DA)^H = D\widehat{A} + (D\widehat{A})^H \in \text{II}BSD_s$. Again, since $J_P^+(A) = S$, the diagonal block of $A + A^H$, $A_{ll} + A_{ll}^H = 2\widehat{A}_{ll}$ is Hermitian positive definite for all $l \in S$. As a result, the diagonal block of $DA + (DA)^H$, $d_lA_{ll} + d_lA_{ll}^H = 2d_l\widehat{A}_{ll}$ is also Hermitian positive definite for all $l \in S$. Thus, it follows from Theorem 3.10 that for any eigenvalue μ of $DA + (DA)^H$, $Re(\mu) > 0$. Since $DA + (DA)^H$ is Hermitian, each eigenvalue μ of $DA + (DA)^H$ is positive. Then Lemma 3.7 yields that $DA + (DA)^H$ is Hermitian positive definite. From the proof above, we conclude that there exists a Hermitian positive definite matrix $D = diag(d_1I_1, \cdots, d_sI_s)$, where $d_l > 0$, I_l is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $DA + (DA)^H$ is Hermitian positive definite. It follows from Lyapunov's theorem (see [4], pp.96) that A is positive stable, i.e., for any eigenvalue λ of A, we have $Re(\lambda) > 0$, which completes the proof. ▯

THEOREM 3.12. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) = S$. If $\widehat{A} \in \text{II}BH_s$, where $\widehat{A}$ is defined in* (3.6)*, then for any eigenvalue λ of A, we have $Re(\lambda) > 0$.*

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Proof. Similar to the proof of Theorem 3.5, we can obtain the proof of this theorem by Lemma 2.3 and Theorem 3.11. □

Using Theorem 3.5 and Theorem 3.12, we obtain a sufficient condition for the system (1.1) to be globally stable.

COROLLARY 3.13. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(-A) = S$. If $\widehat{A} \in \mathrm{II}BH_s$ (IBH_s), where $\widehat{A}$ is defined in* (3.6)*, then the equilibrium $\hat{x}$ of system* (1.1) *is globally stable.*

4. The eigenvalue distribution of block diagonally dominant matrices and block $H-$matrices. In this section, some theorems on the eigenvalue distribution of block diagonally dominant matrices and block $H-$matrices are presented, generalizing Theorem 1.2, Theorem 1.3 and Theorem 1.4.

THEOREM 4.1. *Let $A = (A_{lm})_{s\times s} \in C_s^{n\times n}$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} \in IBSD_s$, where $\widehat{A}$ is defined in* (3.6)*, then A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.*

Proof. Suppose every block Gersgorin disk of the matrix A given in (3.5)

$$\Gamma_l : \|(A_{ll} - \lambda I)^{-1}\|^{-1} \le \sum_{m=1, m\ne l}^{s} \|A_{lm}\|, \quad l \in S.$$

Let

$$R_1 = \bigcup_{k\in J_P^+(A)} \Gamma_k, \quad R_2 = \bigcup_{k\in J_P^-(A)} \Gamma_k.$$

Since $J_P^+(A) \cup J_P^-(A) = S$, then $R_1 \cup R_2 = \bigcup_{l\in S} \Gamma_l$. Therefore, it follows from Lemma 3.3 that each eigenvalue λ of the matrix A lies in $R_1 \cup R_2$. Furthermore, R_1 lies on the right of imaginary axis, R_2 lies on the left of the imaginary axis in the imaginary coordinate plane. Then A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part in R_1, and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part in R_2. Otherwise, A has an eigenvalue $\lambda_{k_0} \in R_1$ with $Re(\lambda_{k_0}) \le 0$ such that for at least one $l \in J_P^+(A)$,

$$\|(A_{ll} - \lambda_0 I)^{-1}\|^{-1} \le \sum_{m=1, m\ne l}^{s} \|A_{lm}\|. \tag{4.1}$$

Then it follows from Lemma 3.2 that

$$\|H_{ll}^{-1}\|^{-1} \le \|(A_{ll} - \lambda_{k_0} I)^{-1}\|^{-1} \tag{4.2}$$

for at least one $l \in J_P^+(A)$. It then follows from (4.1) and (4.2) that

$$\|H_{ll}^{-1}\|^{-1} \le \sum_{m=1, m\ne l}^{s} \|A_{lm}\| \tag{4.3}$$

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for at least one $l \in J_P^+(A)$. Inequality (4.3) contradicts $\widehat{A} \in IBSD_s$. By the same method, we can obtain the same result if there exists an eigenvalue with nonnegative real part in R_2. Hence, if $\widehat{A} \in IBSD_s$ and $J_P^+(A) \cup J_P^-(A) = S$, then for an arbitrary eigenvalue λ_i of A, we have $Re(\lambda_i) \neq 0$ for $i = 1, 2, \cdots, n$. Again, $J_P^+(A) \cap J_P^-(A) = \emptyset$ yields $R_1 \cap R_2 = \emptyset$. Since R_1 and R_2 are both closed sets and $R_1 \cap R_2 = \emptyset$, λ_i in R_1 can not jump into R_2 and $\lambda_i \in R_2$ can not jump into R_1. Thus, A has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part in R_1 and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part in R_2. This completes the proof. ∎

THEOREM 4.2. *Let $A = (A_{lm})_{s \times s} \in C_s^{n \times n}$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} \in IBH_s$, where $\widehat{A}$ is defined in* (3.6)*, then A has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part.*

Proof. Since $\widehat{A} \in IBH_s$, it follows from Lemma 2.3 and the proof of Theorem 3.5 that there exists a block diagonal matrix $D = diag(d_1 I_1, \cdots, d_s I_s)$, where $d_l > 0$, I_l is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $D^{-1}\widehat{A}D \in IBSD_s$, i.e.,

$$\|H_{ll}^{-1}\|^{-1} > \sum_{m=1, m\neq l}^{s} d_l^{-1} \|A_{lm}\| d_m = \sum_{m=1, m\neq l}^{s} \|d_l^{-1} A_{lm} d_m\| \tag{4.4}$$

holds for all $l \in S$. Since $J_P^+(D^{-1}AD) = J_P^+(A)$ and $J_P^-(D^{-1}AD) = J_P^-(A)$, Theorem 3.4 yields that $D^{-1}AD$ has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part. Again, since A has the same eigenvalues as $D^{-1}AD$, A has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part. This completes the proof. ∎

Following Lemma 2.2 and Theorem 4.2, we have the following corollary.

COROLLARY 4.3. *Let $A = (A_{lm})_{s \times s} \in C_s^{n \times n}$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} \in IBID_s$, where $\widehat{A}$ is defined in* (3.6)*, then A has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part.*

Now, we consider the eigenvalue distribution of II−block diagonally dominant matrices and II−block $H-$matrices. In the following lemma, a further extension of the *Gersgorin Circle Theorem* is given.

LEMMA 4.4. *If for each block row of the block matrix $A = (A_{lm})_{s \times s} \in C_s^{n \times n}$, there exists at least one off-diagonal block not equal to zero, then for each eigenvalue λ of A,*

$$\sum_{m=1, m\neq l}^{s} \|(A_{ll} - \lambda I)^{-1} A_{lm}\| \geq 1 \tag{4.5}$$

holds for at least one $l \in S$, where $\|(A_{ll} - \lambda I)^{-1} A_{lm}\| \to \infty$ (defined in (2.4)*) if $A_{ll} - \lambda I$*

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is singular and $A_{lm} \neq 0$ *for* $l \neq m$ *and* $l, m \in S$.

Proof. The proof is by contradiction. Assume that λ is an arbitrary eigenvalue of A such that

$$\sum_{m=1, m\neq l}^{s} \|(A_{ll} - \lambda I)^{-1} A_{lm}\| < 1 \tag{4.6}$$

holds for all $l \in S$. It follows from (4.6) that $A_{ll} - \lambda I$ is nonsingular for all $l \in S$. Otherwise, there exists at leas one $l_0 \in S$ such that $A_{l_0 l_0} - \lambda I$ is singular. Since there exists at least one off-diagonal block $A_{l_0 m} \neq 0$ for $m \in S$ in the l_0th block row, (2.4) yields $\|(A_{l_0 l_0} - \lambda I)^{-1} A_{l_0 m}\| \to \infty$ and consequently, $\sum_{m=1, m\neq l}^{s} \|(A_{l_0 l_0} - \lambda I)^{-1} A_{l_0 m}\| \to \infty$, which contradicts (4.6). Therefore, $A_{ll} - \lambda I$ is nonsingular for all $l \in S$, which leads to the nonsingularity of the block diagonal matrix $D(\lambda) = diag(A_{11} - \lambda I, \cdots, A_{ss} - \lambda I)$. Further, (4.6) also shows $A - \lambda I \in \mathrm{II}BSD_s$ Thus, $\mathscr{A}(\lambda) := [D(\lambda)]^{-1}(A - \lambda I) = (\mathscr{A}_{lm})_{s\times s} \in IBSD_s$. Then, we have from Lemma 2.1 that $\mathscr{A}(\lambda)$ is nonsingular. As a result, $A - \lambda I$ is nonsingular, which contradicts the assumption that λ is an arbitrary eigenvalue of A. Hence, if λ is an arbitrary eigenvalue of A, then $A - \lambda I$ cannot be II$-$block diagonally dominant, which gives the conclusion of this lemma. ◻

Theorem 4.5. *Let* $A = (A_{lm})_{s\times s} \in \mathrm{II}BSD_s$ *with* $J_P^+(A) \cup J_P^-(A) = S$. *If* $\widehat{A} = A$, *where* $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ *is defined in* (3.6), *then* A *has* $\sum_{k\in J_P^+(A)} n_k$ *eigenvalues with positive real part and* $\sum_{k\in J_P^-(A)} n_k$ *eigenvalues with negative real part.*

Proof. The proof proceeds with the following two cases.

(i) If for each block row of the block matrix A, there exists at least one off-diagonal block not equal to zero, one may suppose that every block Gersgorin disk of the matrix A given in (4.5) is

$$G_l : 1 \leq \sum_{m=1, m\neq l}^{s} \|(A_{ll} - \lambda I)^{-1} A_{lm}\|, \quad l \in S.$$

Let

$$\widetilde{R}_1 = \bigcup_{k\in J_P^+(A)} G_k, \quad \widetilde{R}_2 = \bigcup_{k\in J_P^-(A)} G_k.$$

Since $J_P^+(A) \cup J_P^-(A) = S$, then $\widetilde{R}_1 \cup \widetilde{R}_2 = \bigcup_{l\in S} G_l$. Therefore, it follows from Lemma 4.4 that each eigenvalue λ of the matrix A lies in $\widetilde{R}_1 \cup \widetilde{R}_2$. Furthermore, $\widetilde{R}_1$ lies on the right of imaginary axis, $\widetilde{R}_2$ lies on the left of the imaginary axis in the imaginary coordinate plane. Then A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part in $\widetilde{R}_1$, and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part in $\widetilde{R}_2$. Otherwise, assume

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that A has an eigenvalue $\lambda_{k_0} \in \widetilde{R}_1$ such that $Re(\lambda_{k_0}) \leq 0$. Therefore, we have from Lemma 4.4 that

$$(4.7) \qquad 1 \leq \sum_{m=1, m\neq l}^{s} \|(A_{ll} - \lambda_0 I)^{-1} A_{lm}\|$$

holds for at least $l \in J_P^+(A)$. Since (3.6) and $\widehat{A} = A$ imply that the diagonal blocks of A are all Hermitian, the diagonal block A_{ll} of the block matrix A is Hermitian positive definite for all $l \in J_P^+(A)$. Then it follows from Lemma 3.9 that

$$(4.8) \qquad \|(\lambda I - A_{ll})^{-1} A_{lm}\| \leq \|A_{ll}^{-1} A_{lm}\|$$

for all $l \in J_P^+(A)$ and $m \neq l$, $m \in S$. Inequalities (4.7) and (4.8) yield that

$$(4.9) \qquad 1 \leq \sum_{m=1, m\neq l}^{s} \|A_{ll}^{-1} A_{lm}\|$$

holds for at least $l \in J_P^+(A)$. Inequality (4.9) contradicts $A \in \text{II}BSD_s$. In the same method, we can obtain the same result if there exists an eigenvalue with nonnegative real part in $\widetilde{R}_2$. Hence, if $A \in \text{II}BSD_s$ and $J_P^+(A) \cup J_P^-(A) = S$, then for arbitrary eigenvalue λ_i of A, we have $Re(\lambda_i) \neq 0$ for $i = 1, 2, \cdots, n$. Again, $J_P^+(A) \cap J_P^-(A) = \emptyset$ yields $\widetilde{R}_1 \cap \widetilde{R}_2 = \emptyset$. Since $\widetilde{R}_1$ and $\widetilde{R}_2$ are all closed set and $\widetilde{R}_1 \cap \widetilde{R}_2 = \emptyset$, λ_i in $\widetilde{R}_1$ can not jump into $\widetilde{R}_2$ and $\lambda_i \in \widetilde{R}_2$ can not jump into $\widetilde{R}_1$. Thus, A has $\sum_{k \in J_P^+(A)} n_k$ eigenvalues with positive real part in $\widetilde{R}_1$ and $\sum_{k \in J_P^-(A)} n_k$ eigenvalues with negative real part in $\widetilde{R}_2$. This completes the proof of (i).

(ii) The following will prove the case when there exist some block rows of the block matrix A with all their off-diagonal blocks equal to zero. Let $\omega \subseteq S$ denote the set containing block row indices of such block rows. Then there exists an $n \times n$ permutation matrix P such that

$$(4.10) \qquad PAP^T = \begin{bmatrix} A(\omega') & A(\omega', \omega) \\ 0 & A(\omega) \end{bmatrix},$$

where $\omega' = S - \omega$, $A(\omega') = (A_{lm})_{l,m \in \omega'}$ has no block rows with all their off-diagonal blocks equal to zero, $A(\omega) = (A_{lm})_{l,m \in \omega}$ is a block diagonal matrix and $A(\omega', \omega) = (A_{lm})_{l \in \omega', m \in \omega}$. It is easy to see that the partition of (4.10) does not destroy the partition of (2.6). Further, (4.10) shows that

$$(4.11) \qquad \sigma(A) = \sigma(A(\omega')) \cup \sigma(A(\omega)),$$

where $\sigma(A)$ denotes the spectrum of the matrix A. Since and $A(\omega')$ is a block principal submatrix of A and $J_P^+(A) \cup J_P^-(A) = S$, $J_P^+[A(\omega')] \cup J_P^-[A(\omega')] = \omega'$. Further,

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$A = \widehat{A} \in \text{II}BSD_s$ gives $A(\omega') = \widehat{A}(\omega') \in \text{II}BSD_{|\omega'|}$. Again, since $A(\omega') = (A_{lm})_{l,m\in\omega'}$ has no block rows with all their off-diagonal blocks equal to zero, i.e., for each block row of the block matrix $A(\omega')$, there exists at least one off-diagonal block not equal to zero, it follows from the proof of (i) that $A(\omega')$ has $\sum_{k\in J_P^+[A(\omega')]} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-[A(\omega')]} n_k$ eigenvalues with negative real part.
Let's consider the matrix $A(\omega)$. $A(\omega)$ being a block principal submatrix of the block matrix $A = (A_{lm})_{s\times s} \in \text{II}BSD_s$ with $J_P^+(A) \cup J_P^-(A) = S$ gives $J_P^+[A(\omega)] \cup J_P^-[A(\omega)] = \omega$. Since $A(\omega) = (A_{lm})_{l,m\in\omega}$ is a block diagonal matrix, the diagonal block A_{ll} of $A(\omega)$ is either non-Hermitian positive definite or non-Hermitian negative definite for each $l \in \omega$, and consequently

$$(4.12)\quad \sigma(A(\omega)) = \bigcup_{l\in\omega} \sigma(A_{ll}) = \left(\bigcup_{l\in J_P^+[A(\omega)]} \sigma(A_{ll})\right) \bigcup \left(\bigcup_{l\in J_P^-[A(\omega)]} \sigma(A_{ll})\right).$$

The equality (4.12) and Lemma 2.4 shows that $A(\omega)$ has $\sum_{k\in J_P^+[A(\omega)]} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-[A(\omega)]} n_k$ eigenvalues with negative real part.
Since

$$\begin{aligned} J_P^+(A) \cup J_P^-(A) &= S = \omega \cup \omega' \\ &= \left(J_P^+[A(\omega)] \cup J_P^-[A(\omega)]\right) \cup \left(J_P^+[A(\omega')] \cup J_P^-[A(\omega')]\right), \\ &= \left(J_P^+[A(\omega)] \cup J_P^+[A(\omega')]\right) \cup \left(J_P^-[A(\omega) \cup J_P^-[A(\omega')]\right) \end{aligned}$$

we have

$$(4.13)\qquad J_P^+(A) = J_P^+[A(\omega)] \cup J_P^+[A(\omega')],\quad J_P^-(A) = J_P^-[A(\omega)] \cup J_P^-[A(\omega')].$$

Again, $\omega' = S - \omega$ implies $\omega \cap \omega' = \emptyset$, which yields

$$(4.14)\qquad J_P^+[A(\omega)] \cap J_P^+[A(\omega')] = \emptyset,\quad J_P^-[A(\omega)] \cap J_P^-[A(\omega')] = \emptyset.$$

According to (4.13), (4.14) and the partition (2.6) of A, it is not difficult to see that

$$(4.15)\qquad \begin{aligned} \textstyle\sum_{k\in J_P^+(A)} n_k &= \textstyle\sum_{k\in J_P^+[A(\omega)]} n_k + \sum_{k\in J_P^+[A(\omega')]} n_k, \\ \textstyle\sum_{k\in J_P^-(A)} n_k &= \textstyle\sum_{k\in J_P^-[A(\omega)]} n_k + \sum_{k\in J_P^-[A(\omega')]} n_k. \end{aligned}$$

From (4.15) and the eigenvalue distribution of $A(\omega')$ and $A(\omega)$ given above, it is not difficult to see that A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part. We conclude from the proof of (i) and (ii) that the proof of this theorem is completed. □

THEOREM 4.6. *Let $A = (A_{lm})_{s\times s} \in \text{II}BH_s$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} = A$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in* (3.6), *then A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.*

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Proof. Since $\widehat{A} = A \in \mathrm{II}BH_s$, it follows from Lemma 2.3 that there exists a block diagonal matrix $D = diag(d_1 I_{n_1}, d_2 I_{n_2}, \cdots, d_s I_{n_s})$, where $d_l > 0$, I_{n_l} is the $n_l \times n_l$ identity matrix for all $l \in S$ and $\sum_{l=1}^{s} n_l = n$, such that $\widehat{A}D = AD \in \mathrm{II}BSD_s$, i.e.,

$$(4.16)\ 1 > \sum_{m=1, m\neq l}^{s} \|(A_{ll}d_l)^{-1}(A_{lm}d_m)\| = \sum_{m=1, m\neq l}^{s} \|(d_l^{-1}A_{ll}d_l)^{-1}(d_l^{-1}A_{lm}d_m)\|$$

holds for all $l \in S$. Inequality (4.16) shows $B = D^{-1}AD = D^{-1}\widehat{A}D = \widehat{B} \in \mathrm{II}BSD_s$. Since $B = D^{-1}AD$ has the same diagonal blocks as the matrix A, it follows from Theorem 4.5 that B has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part, so does A. ∎

Corollary 4.7. *Let $A = (A_{lm})_{s\times s} \in \mathrm{II}BID_s$ with $J_P^+(A) \cup J_P^-(A) = S$. If $\widehat{A} = A$, where $\widehat{A} = (\widehat{A}_{lm})_{s\times s}$ is defined in* (3.6)*, then A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.*

Proof. The proof is obtain directly by Lemma 2.2 and Theorem 4.6. ∎

5. Conclusions. This paper concerns the eigenvalue distribution of block diagonally dominant matrices and block block $H-$matrices. Following the result of Feingold and Varga, a well-known theorem of Taussky on the eigenvalue distribution is extended to block diagonally dominant matrices and block $H-$matrices with each diagonal block being non-Hermitian positive definite. Then, the eigenvalue distribution of some special matrices including block diagonally dominant matrices and block H-matrices is studied further to give the conditions on the block matrix A such that the matrix A has $\sum_{k\in J_P^+(A)} n_k$ eigenvalues with positive real part and $\sum_{k\in J_P^-(A)} n_k$ eigenvalues with negative real part.

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