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AN IMPROVED CHARACTERISATION OF THE INTERIOR OF THE COMPLETELY POSITIVE CONE*

PETER J.C. DICKINSON†

Abstract. A symmetric matrix is defined to be completely positive if it allows a factorisation BB^T, where B is an entrywise nonnegative matrix. This set is useful in certain optimisation problems. The interior of the completely positive cone has previously been characterised by Dür and Still [M. Dür and G. Still, Interior points of the completely positive cone, *Electronic Journal of Linear Algebra*, 17:48–53, 2008]. In this paper, we introduce the concept of the set of zeros in the nonnegative orthant for a quadratic form, and use the properties of this set to give a more relaxed characterisation of the interior of the completely positive cone.

Key words. Completely positive matrices, Copositive matrices, Cones of matrices.

AMS subject classifications. 15A23, 15B48.

1. Introduction. The copositive and completely positive cones have been found to be useful in mathematical programming, especially as they can be used to create a convex formulation of some NP-hard problems. For example, it was shown by Bomze et al. [3] that the maximum clique problem can be reformulated as a linear optimisation problem over either the copositive or completely positive cone, and Burer [5] showed that every quadratic problem with linear and binary constraints can be rewritten in such a form. Surveys on copositive and completely positive matrices and their cones are provided in [2, 7, 9].

A symmetric matrix is defined to be *completely positive* if it allows a factorisation BB^T, where B is an entrywise nonnegative matrix. Another way of saying this is that a symmetric matrix A is a completely positive matrix if it allows a factorisation $A = \sum_{i=1}^{m} a_i a_i^T$, where $a_i \in \mathbb{R}^n_+$ for all i. This is called a *rank-one decomposition of* A, and the decomposition is not unique.

A symmetric matrix A is defined to be *copositive* if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n_+$. The completely positive and copositive cones are both proper cones (as defined in [4, p. 43]) and have been shown to be the duals of each other, a proof of which can be found in [2, p. 71]. We will go into a bit more detail about this dual relationship in Section 3.

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In this paper, we will use the following notation:

Inner product for symmetric matrices, $\langle A, B\rangle := \mathrm{trace}(AB)$,
Nonnegative Orthant, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$,
Strictly Positive Orthant, $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n \mid x > 0\}$,
Cone of symmetric matrices, $\mathcal{S} := \{A \in \mathbb{R}^{n\times n} \mid A = A^T\}$,
Nonnegative cone, $\mathcal{N} := \{A \in \mathcal{S} \mid A \geq 0\}$,
Positive semidefinite cone, $\mathcal{S}_+ := \{A \in \mathcal{S} \mid A \succeq 0\}$,
Copositive cone, $\mathcal{C} := \{A \in \mathcal{S} \mid x^T Ax \geq 0\ \forall x \geq 0\}$,
Completely positive cone, $\mathcal{C}^* := \{B = \sum_{i=1}^m a_i a_i^T \mid a_i \in \mathbb{R}^n_+\ \forall i\}$.

It can be immediately seen from the definitions that

$$\mathcal{C}^* \subseteq \mathcal{S}_+ \cap \mathcal{N}.$$

This inclusion has been shown in [10] to hold with equality for $n \leq 4$ and be strict for $n \geq 5$.

We will now have a brief look at the interior of the completely positive cone. A characterisation of the interior of the completely positive cone could come in useful for creating an interior point algorithm for solving an optimisation problem over this cone. The use of this type of algorithm is motivated by its proven efficiency for semidefinite problems. From a paper by Dür and Still [8], we have that

$$\mathrm{int}(\mathcal{C}^*) = \left\{AA^T \mid A = [A_1|A_2] \text{ with } A_1 > 0 \text{ nonsingular}, A_2 \geq 0\right\}. \tag{1.1}$$

This characterisation of the interior is useful in that we can construct any matrix in the interior from it, and any matrix in the interior can be decomposed into this form. However, for a general completely positive matrix in the interior we can decompose it in a way that is not of this form, as can be seen in the following example:

$$\begin{pmatrix} 25 & 10 \\ 10 & 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}^T + \begin{pmatrix} 4 \\ 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 5 \\ 2 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}^T + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T.$$

How to tell if an arbitrary completely positive matrix is in the interior or not from a general rank-one decomposition of it is still an open question. In this paper we shall show that the characterisation of the interior previously given is more restricted than it need be, and we present a more relaxed version of it. However, to do this we first need to consider what we have called the *set of zeros in the nonnegative orthant* for a quadratic form.

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ELA
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2. The set of zeros in the nonnegative orthant. We define the set of zeros in the nonnegative orthant for a quadratic form as follows.

DEFINITION 2.1. The *set of zeros for x^TAx in the nonnegative orthant* is defined as

$$\mathcal{V}^A := \{v \in \mathbb{R}^n_+ \mid v^TAv = 0\}.$$

To our knowledge, this set has not been previously investigated for copositive matrices. We shall now present two useful properties of this set.

THEOREM 2.2. *If $A \in \mathcal{C}$ and $\mathcal{V}^A \cap \mathbb{R}^n_{++} \neq \emptyset$, then $A \in \mathcal{S}_+$.*

Proof. This comes directly from [6, Lemma 1]. Due to the importance of this theorem for our results we present our own proof here. For $A \in \mathcal{C}$ and $\mathcal{V}^A \cap \mathbb{R}^n_{++} \neq \emptyset$, let $x \in \mathcal{V}^A \cap \mathbb{R}_{++}$. Now let $u \in \mathbb{R}^n$ be arbitrary. Then there exists an $\bar{\varepsilon} > 0$ such that $(x + \varepsilon u) \in \mathbb{R}^n_+$ for all $\varepsilon \in (0, \bar{\varepsilon}]$. This implies that for all $\varepsilon \in (0, \bar{\varepsilon}]$,

$$0 \leq (x + \varepsilon u)^T A(x + \varepsilon u) = 2\varepsilon u^TAx + \varepsilon^2 u^TAu.$$

Letting $\varepsilon \to 0$ we get $0 \leq u^TAx$. Since u was arbitrary, this implies that $Ax = 0$. Consequently, for $\varepsilon > 0$, we have $0 \leq u^TAu$. Since u was arbitrary, this implies that $A \in \mathcal{S}_+$. ◻

THEOREM 2.3. *Let $U = \sum_{i=1}^m u_iu_i^T$ such that $u_i \in \mathbb{R}^n_+$ for all $i = 1, \ldots, m$, and let $A \in \mathcal{C}$. Then*

$$\langle U, A\rangle = 0 \Leftrightarrow \{u_1, \ldots, u_m\} \subseteq \mathcal{V}^A.$$

Proof. We have

$$\langle U, A\rangle = \left\langle \textstyle\sum_{i=1}^m u_iu_i^T, A\right\rangle = \textstyle\sum_{i=1}^m u_i^TAu_i.$$

From the definition of a copositive matrix, $0 \leq u_i^TAu_i$ for all i. Therefore,

$$0 = \langle U, A\rangle \Leftrightarrow 0 = u_i^TAu_i \text{ for all } i \Leftrightarrow \{u_1, \ldots, u_m\} \subseteq \mathcal{V}^A. \quad ◻$$

3. Interior of the completely positive cone. For a general cone $\mathcal{K} \subseteq \mathcal{S}$, the dual cone $\mathcal{K}^*$ is defined as

$$\mathcal{K}^* := \{A \in \mathcal{S} \mid \langle A, B\rangle \geq 0 \text{ for all } B \in \mathcal{K}\}.$$

It has been shown in [1] that

$$\operatorname{int}(\mathcal{K}^*) = \{A \in \mathcal{S} \mid \langle A, B\rangle > 0 \text{ for all } B \in \mathcal{K} \setminus \{0\}\}.$$

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ELA
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As stated in the introduction, the copositive and completely positive cones are duals of each other, from which we immediately get the following result.

LEMMA 3.1. *For an arbitrary* $U \in \mathcal{C}^*$*,*

$$U \in \operatorname{bd}(\mathcal{C}^*) \Leftrightarrow \exists A \in \mathcal{C} \setminus \{0\} \ \textit{s.t.} \ \langle U, A\rangle = 0,$$
$$U \in \operatorname{int}(\mathcal{C}^*) \Leftrightarrow \nexists A \in \mathcal{C} \setminus \{0\} \ \textit{s.t.} \ \langle U, A\rangle = 0.$$

We can now combine this with one of the properties of the set of zeros in the nonnegative orthant (Theorem 2.3) to get the following.

THEOREM 3.2. *Let* $U = \sum_{i=1}^m u_i u_i^T$ *such that* $u_i \in \mathbb{R}^n_+$ *for all* $i = 1, \dots, m$*. Then*

$$U \in \operatorname{bd}(\mathcal{C}^*) \Leftrightarrow \exists A \in \mathcal{C} \setminus \{0\} \ \textit{s.t.} \ \{u_1, \dots, u_m\} \subseteq \mathcal{V}^A,$$
$$U \in \operatorname{int}(\mathcal{C}^*) \Leftrightarrow \nexists A \in \mathcal{C} \setminus \{0\} \ \textit{s.t.} \ \{u_1, \dots, u_m\} \subseteq \mathcal{V}^A.$$

We will now prove a sufficient condition for a matrix being in the interior of the completely positive cone from its rank-one decomposition.

THEOREM 3.3. *Let* $U = \sum_{i=1}^m u_i u_i^T$ *such that* $u_i \in \mathbb{R}^n_+$ *for all* $i = 1, \dots, m$*, let* $\operatorname{span}\{u_1, \dots, u_m\} = \mathbb{R}^n$*, and assume there exists* $u \in \{u_1, \dots, u_m\}$ *such that* $u \in \mathbb{R}_{++}$*. Then* $U \in \operatorname{int}(\mathcal{C}^*)$*.*

Proof. For the sake of contradiction, assume that $U \in \operatorname{bd}(\mathcal{C}^*)$. From Theorem 3.2, this implies that there exists an $A \in \mathcal{C} \setminus \{0\}$ such that $\{u_1, \dots, u_m\} \subseteq \mathcal{V}^A$. We therefore have that $u \in \mathcal{V}^A \cap \mathbb{R}_{++}$, so from Theorem 2.2 we get $A \in \mathcal{S}_+$. It follows that

$$\begin{aligned} A \in \mathcal{S}_+, \ u_i^T A u_i = 0 \text{ for all } i &\Rightarrow A u_i = 0 \text{ for all } i \\ &\Rightarrow Av = 0 \text{ for all } v \in \operatorname{span}\{u_1, \dots, u_m\} \\ &\Rightarrow Av = 0 \text{ for all } v \in \mathbb{R}^n \\ &\Rightarrow A = 0. \quad \square \end{aligned}$$

The conditions in our previous theorem appear at first glance to be fairly restrictive, however we will see next that the condition $\operatorname{span}\{u_1, \dots, u_m\} = \mathbb{R}^n$ was a necessary condition for $U \in \operatorname{int}(\mathcal{C}^*)$.

THEOREM 3.4. *Let* $U = \sum_{i=1}^m u_i u_i^T$ *such that* $u_i \in \mathbb{R}^n_+$ *for all* i*, and let* $\operatorname{span}\{u_1, \dots, u_m\} \neq \mathbb{R}^n$*. Then* $U \in \operatorname{bd}(\mathcal{C}^*)$*.*

Proof. As stated previously, it can be easily seen that $\mathcal{C}^* \subseteq \mathcal{S}_+$. The condition $\operatorname{span}\{u_1, \dots, u_m\} \neq \mathbb{R}^n$ implies that $\operatorname{rank} U < n$. From this, we must have that $U \in \operatorname{bd}(\mathcal{S}_+)$, which in turn implies that $U \in \operatorname{bd}(\mathcal{C}^*)$. $\square$

We will now create a more relaxed characterisation of the interior of the completely positive cone than has been done previously.

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ELA
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THEOREM 3.5. *We have*

$$\begin{aligned}\operatorname{int}(\mathcal{C}^*) &= \left\{ \sum_{i=1}^m a_i a_i^T \mid \begin{array}{c} a_1 \in \mathbb{R}_{++},\ a_i \in \mathbb{R}^n_+ \ \forall i, \\ \operatorname{span}\{a_1, \dots, a_m\} = \mathbb{R}^n \end{array} \right\} \\ &= \left\{ AA^T \mid \operatorname{rank} A = n,\ A = [a|B],\ a \in \mathbb{R}_{++},\ B \geq 0 \right\}.\end{aligned}$$

Proof. Let

$$\begin{aligned}\mathcal{M} &:= \left\{ \sum_{i=1}^m a_i a_i^T \mid \begin{array}{c} a_1 \in \mathbb{R}_{++},\ a_i \in \mathbb{R}^n_+ \ \forall i, \\ \operatorname{span}\{a_1, \dots, a_m\} = \mathbb{R}^n \end{array} \right\} \\ &= \left\{ AA^T \mid \operatorname{rank} A = n,\ A = [a|B],\ a \in \mathbb{R}_{++},\ B \geq 0 \right\}.\end{aligned}$$

It can be immediately seen from Theorem 3.3 that $\mathcal{M} \subseteq \operatorname{int}(\mathcal{C}^*)$. From characterisation (1.1) of the interior, by considering the columns of A_1 and A_2 it is also immediately apparent that

$$\mathcal{M} \supseteq \left\{ AA^T \mid A = [A_1|A_2] \text{ with } A_1 > 0 \text{ nonsingular, } A_2 \geq 0 \right\} = \operatorname{int}(\mathcal{C}^*). \qquad \square$$

Another way of putting this is that rather than the initial characterisation requiring that all entries of A_1 are strictly positive, we actually need only require that the entries of one column of A_1 are strictly positive.

It is interesting to note that there is also a more restricted characterisation of the interior of the completely positive cone. For this, we will need the Krein-Milman Theorem, as stated in [2, p. 45].

THEOREM 3.6. (Krein-Milman Theorem) *If T is a set of extreme vectors of a closed convex cone K which generate all the extreme rays of K, then $K = \operatorname{cl}(\operatorname{cone} T)$.*

From this we get the following lemma, which we will use to construct the more restricted characterisation.

LEMMA 3.7. *If $\mathcal{D}$ is a convex cone contained in a proper cone $\mathcal{K}$ such that all the extreme rays of $\mathcal{K}$ are contained in $\operatorname{cl}(\mathcal{D})$, then $\operatorname{int}(\mathcal{K}) \subseteq \mathcal{D}$.*

Proof. It can be seen from the Krein-Milman Theorem that $\mathcal{K} = \operatorname{cl}(\mathcal{D})$. Now for the sake of contradiction, assume that there exists an $x \in \operatorname{int}(\mathcal{K}) \setminus \mathcal{D}$. It is a standard result that as $\mathcal{D}$ is convex there must exist a hyperplane through x giving a closed halfspace H such that $\mathcal{D} \subseteq H$. This implies that $\operatorname{cl}(\mathcal{D}) \subseteq H$. However, we also have that $x \in \mathcal{K} = \operatorname{cl}(\mathcal{D}) \subseteq H$. The fact that the hyperplane goes through x implies that $x \in \operatorname{bd}(H)$, so we must have $x \in \operatorname{bd}(\mathcal{K})$, which is a contradiction. $\square$

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Volume 20, pp. 723-729, November 2010

Theorem 3.8. *We have*

$$\begin{aligned}\operatorname{int}(\mathcal{C}^*) &= \left\{ \sum_{i=1}^m a_i a_i^T \;\middle|\; \begin{array}{l} a_i \in \mathbb{R}_{++} \ \forall i, \\ \operatorname{span}\{a_1, \dots, a_m\} = \mathbb{R}^n \end{array} \right\} \\ &= \{AA^T \mid A > 0, \operatorname{rank} A = n\}.\end{aligned}$$

Proof. Let $\mathcal{D} = \{AA^T \mid A > 0\} \cup \{0\}$. It is not difficult to see that this is a convex cone which is contained in the completely positive cone. From [2, p. 71], we have that the completely positive cone is a proper cone, with the extreme rays being the matrices bb^T where $b \in \mathbb{R}^n_+ \setminus \{0\}$. These are obviously members of $\operatorname{cl}(\mathcal{D})$. Therefore, from Lemma 3.7, we must have that $\operatorname{int}(\mathcal{C}^*) \subseteq \mathcal{D}$. For arbitrary $A > 0$, using Theorems 3.3 and 3.4, we have that

$$AA^T \in \operatorname{int}(\mathcal{C}^*) \Leftrightarrow \operatorname{rank} A = n. \qquad \square$$

This characterisation is interesting due to the fact that although it is very restricted, any matrix in the interior of the completely positive cone can still be written in this form.

These characterisations may be both more relaxed for one and more restricted for the other, but they are still not sufficient in telling if a completely positive matrix is in the interior from a general rank-one decomposition of it, as can be seen in the following example:

$$\begin{aligned}\begin{pmatrix} 18 & 9 & 9 \\ 9 & 18 & 9 \\ 9 & 9 & 18 \end{pmatrix} &= \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}^T + \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}^T + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}^T + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}^T \\ &= \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}^T + \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}^T + \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}^T \\ &= \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}^T + \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}^T + \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}^T .\end{aligned}$$

Although the relaxed characterisation does not solve the problem of using a general rank-one decomposition to tell if a matrix is in the interior of the copositive cone, it is nonetheless a fairly relaxed characterisation giving the interior of the completely positive cone.

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