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## A DIAZ–METCALF TYPE INEQUALITY FOR POSITIVE LINEAR MAPS AND ITS APPLICATIONS\*

MOHAMMAD SAL MOSLEHIAN<sup>†</sup>, RITSUO NAKAMOTO<sup>‡</sup>, AND YUKI SEO<sup>§</sup>

**Abstract.** We present a Diaz–Metcalf type operator inequality as a reverse Cauchy–Schwarz inequality and then apply it to get some operator versions of Pólya–Szegő’s, Greub–Rheinboldt’s, Kantorovich’s, Shisha–Mond’s, Schweitzer’s, Cassels’ and Klamkin–McLenaghan’s inequalities via a unified approach. We also give some operator Grüss type inequalities and an operator Ozeki–Izumino–Mori–Seo type inequality. Several applications are included as well.

**Key words.** Diaz–Metcalf type inequality, Reverse Cauchy–Schwarz inequality, Positive map, Ozeki–Izumino–Mori–Seo inequality, Operator inequality.

**AMS subject classifications.** 46L08, 26D15, 46L05, 47A30, 47A63.

**1. Introduction.** The Cauchy–Schwarz inequality plays an essential role in mathematical analysis and its applications. In a semi-inner product space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  the Cauchy–Schwarz inequality reads as follows

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \quad (x, y \in \mathcal{H}).$$

There are interesting generalizations of the Cauchy–Schwarz inequality in various frameworks, e.g., finite sums, integrals, isotone functionals, inner product spaces,  $C^*$ -algebras and Hilbert  $C^*$ -modules; see [5, 6, 7, 9, 11, 13, 17, 20] and references therein. There are several reverses of the Cauchy–Schwarz inequality in the literature: Diaz–Metcalf’s, Pólya–Szegő’s, Greub–Rheinboldt’s, Kantorovich’s, Shisha–Mond’s, Ozeki–Izumino–Mori–Seo’s, Schweitzer’s, Cassels’ and Klamkin–McLenaghan’s inequalities.

Inspired by the work of J.B. Diaz and F.T. Metcalf [4], we present several reverse Cauchy–Schwarz type inequalities for positive linear maps. We give a unified treatment of some reverse inequalities of the classical Cauchy–Schwarz type for positive

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linear maps.

Throughout the paper  $\mathbb{B}(\mathcal{H})$  stands for the algebra of all bounded linear operators acting on a Hilbert space  $\mathcal{H}$ . We simply denote by  $\alpha I$  of the identity operator  $I \in \mathbb{B}(\mathcal{H})$ . For self-adjoint operators  $A, B$  the partially ordered relation  $B \leq A$  means that  $\langle B\xi, \xi \rangle \leq \langle A\xi, \xi \rangle$  for all  $\xi \in \mathcal{H}$ . In particular, if  $0 \leq A$ , then  $A$  is called positive. If  $A$  is a positive invertible operator, then we write  $0 < A$ . A linear map  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  between  $C^*$ -algebras is said to be positive if  $\Phi(A)$  is positive whenever  $A$  is. We say that  $\Phi$  is unital if  $\Phi$  preserves the identity. The reader is referred to [9, 19] for undefined notations and terminologies.

**2. Operator Diaz–Metcalfe type inequality.** We start this section with our main result. Recall that the geometric operator mean  $A \sharp B$  for positive operators  $A, B \in \mathbb{B}(\mathcal{H})$  is defined by

$$A \sharp B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}}$$

if  $0 < A$ .

**THEOREM 2.1.** *Let  $A, B \in \mathbb{B}(\mathcal{H})$  be positive invertible operators and  $\Phi : \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$  be a positive linear map.*

(i) *If  $m^2 A \leq B \leq M^2 A$  for some positive real numbers  $m < M$ , then the following inequalities hold:*

- *Operator Diaz–Metcalfe inequality of first type*

$$Mm\Phi(A) + \Phi(B) \leq (M + m)\Phi(A \sharp B);$$

- *Operator Cassels inequality*

$$\Phi(A) \sharp \Phi(B) \leq \frac{M + m}{2\sqrt{Mm}} \Phi(A \sharp B);$$

- *Operator Klamkin–McLenaghan inequality*

$$\Phi(A \sharp B)^{-\frac{1}{2}} \Phi(B) \Phi(A \sharp B)^{-\frac{1}{2}} - \Phi(A \sharp B)^{\frac{1}{2}} \Phi(A)^{-1} \Phi(A \sharp B)^{\frac{1}{2}} \leq (\sqrt{M} - \sqrt{m})^2;$$

- *Operator Kantorovich inequality*

$$\Phi(A) \sharp \Phi(A^{-1}) \leq \frac{M^2 + m^2}{2Mm}.$$

(ii) *If  $m_1^2 \leq A \leq M_1^2$  and  $m_2^2 \leq B \leq M_2^2$  for some positive real numbers  $m_1 < M_1$  and  $m_2 < M_2$ , then the following inequalities hold:*

- *Operator Diaz–Metcalfe inequality of second type*

$$\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \Phi(A \sharp B);$$

- *Operator Pólya–Szegő inequality*

$$\Phi(A) \sharp \Phi(B) \leq \frac{1}{2} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right) \Phi(A \sharp B);$$

- *Operator Shisha–Mond inequality*

$$\begin{aligned} \Phi(A \sharp B)^{-\frac{1}{2}} \Phi(B) \Phi(A \sharp B)^{-\frac{1}{2}} - \Phi(A \sharp B)^{\frac{1}{2}} \Phi(A)^{-1} \Phi(A \sharp B)^{\frac{1}{2}} \\ \leq \left( \sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}} \right)^2; \end{aligned}$$

- *Operator Grüss type inequality*

$$\Phi(A) \sharp \Phi(B) - \Phi(A \sharp B) \leq \frac{\sqrt{M_1 M_2} (\sqrt{M_1 M_2} - \sqrt{m_1 m_2})^2}{2\sqrt{m_1 m_2}} \min \left\{ \frac{M_1}{m_1}, \frac{M_2}{m_2} \right\}.$$

*Proof.* (i) If  $m^2 A \leq B \leq M^2 A$  for some positive real numbers  $m < M$ , then  $m^2 \leq A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq M^2$ .

(ii) If  $m_1^2 \leq A \leq M_1^2$  and  $m_2^2 \leq B \leq M_2^2$  for some positive real numbers  $m_1 < M_1$  and  $m_2 < M_2$ , then

$$m^2 = \frac{m_2^2}{M_1^2} \leq A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq \frac{M_2^2}{m_1^2} = M^2. \quad (2.1)$$

In any case we then have

$$\left( M - \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{1/2} \right) \left( \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{1/2} - m \right) \geq 0,$$

whence

$$Mm + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq (M + m) \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}}.$$

Hence

$$Mm A + B \leq (M + m) A^{1/2} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{1/2} = (M + m) A \sharp B. \quad (2.2)$$

Since  $\Phi$  is a positive linear map, (2.2) yields the *operator Diaz–Metcalfe inequality of first type* as follows:

$$Mm \Phi(A) + \Phi(B) \leq (M + m) \Phi(A \sharp B). \quad (2.3)$$

In the case when (ii) holds we get the following, which is called the *operator Diaz–Metcalf inequality of second type*:

$$\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \Phi(A \sharp B).$$

Following the strategy of [21], we apply the operator geometric–arithmetic inequality to  $Mm\Phi(A)$  and  $\Phi(B)$  to get:

$$\sqrt{Mm}(\Phi(A) \sharp \Phi(B)) = (Mm\Phi(A)) \sharp \Phi(B) \leq \frac{1}{2} (Mm\Phi(A) + \Phi(B)). \quad (2.4)$$

It follows from (2.3) and (2.4) that

$$\Phi(A) \sharp \Phi(B) \leq \frac{M + m}{2\sqrt{Mm}} \Phi(A \sharp B),$$

which is said to be the *operator Cassels inequality* under the assumption (i); see also [16]. Under the case (ii) we can represent it as the following inequality being called the *operator Pólya–Szegő inequality* or the *operator Greub–Rheinboldt inequality*:

$$\Phi(A) \sharp \Phi(B) \leq \frac{1}{2} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right) \Phi(A \sharp B). \quad (2.5)$$

It follows from (2.5) that

$$\begin{aligned} \Phi(A) \sharp \Phi(B) - \Phi(A \sharp B) &\leq \left( \frac{1}{2} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right) - 1 \right) \Phi(A \sharp B) \\ &= \frac{(\sqrt{M_1 M_2} - \sqrt{m_1 m_2})^2}{2\sqrt{m_1 m_2} \sqrt{M_1 M_2}} \Phi(A \sharp B). \end{aligned} \quad (2.6)$$

It follows from (2.1) that

$$\frac{m_2}{M_1} A \leq A^{1/2} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{1/2} A^{1/2} \leq \frac{M_2}{m_1} A,$$

so

$$\frac{m_1^2 m_2}{M_1} \leq A \sharp B \leq \frac{M_1^2 M_2}{m_1}. \quad (2.7)$$

Now, (2.6) and (2.7) yield that

$$\Phi(A) \sharp \Phi(B) - \Phi(A \sharp B) \leq \frac{(\sqrt{M_1 M_2} - \sqrt{m_1 m_2})^2}{2\sqrt{m_1 m_2} \sqrt{M_1 M_2}} \frac{M_1^2 M_2}{m_1}.$$

An easy symmetric argument then follows that

$$\Phi(A) \sharp \Phi(B) - \Phi(A \sharp B) \leq \frac{\sqrt{M_1 M_2} (\sqrt{M_1 M_2} - \sqrt{m_1 m_2})^2}{2\sqrt{m_1 m_2}} \min \left\{ \frac{M_1}{m_1}, \frac{M_2}{m_2} \right\},$$

presenting a Grüss type inequality.

If  $A$  is invertible and  $\Phi$  is unital and  $m_1^2 = m^2 \leq A \leq M^2 = M_1^2$ , then by putting  $m_2^2 = 1/M^2 \leq B = A^{-1} \leq 1/m^2 = M_2^2$  in (2.5) we get the following *operator Kantorovich inequality*:

$$\Phi(A)\sharp\Phi(A^{-1}) \leq \frac{M^2 + m^2}{2Mm}.$$

It follows from (2.3) that

$$\begin{aligned} & \Phi(A\sharp B)^{-\frac{1}{2}}\Phi(B)\Phi(A\sharp B)^{-\frac{1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \\ & \leq M + m - Mm\Phi(A\sharp B)^{-\frac{1}{2}}\Phi(A)\Phi(A\sharp B)^{-\frac{1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \\ & \leq M + m - 2\sqrt{Mm} - \left(\sqrt{Mm}\left(\Phi(A\sharp B)^{-\frac{1}{2}}\Phi(A)\Phi(A\sharp B)^{-\frac{1}{2}}\right)^{1/2}\right. \\ & \quad \left. - \left(\Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}}\right)^{1/2}\right)^2 \\ & \leq (\sqrt{M} - \sqrt{m})^2, \end{aligned} \tag{2.8}$$

that is, an *operator Klakmin–Mclenaghan inequality* when (i) holds. Under (ii), we get the following *operator Shisha–Szegő inequality* from (2.8):

$$\Phi(A\sharp B)^{-\frac{1}{2}}\Phi(B)\Phi(A\sharp B)^{-\frac{1}{2}} - \Phi(A\sharp B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(A\sharp B)^{\frac{1}{2}} \leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2. \quad \square$$

**3. Applications.** If  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  are  $n$ -tuples of real numbers with  $0 < m_1 \leq a_i \leq M_1$  ( $1 \leq i \leq n$ ),  $0 < m_2 \leq b_i \leq M_2$  ( $1 \leq i \leq n$ ), we can consider the positive linear map  $\Phi(T) = \langle Tx, x \rangle$  on  $\mathbb{B}(\mathbb{C}^n) = M_n(\mathbb{C})$  and let  $A = \text{diag}(a_1^2, \dots, a_n^2)$ ,  $B = \text{diag}(b_1^2, \dots, b_n^2)$  and  $x = (1, \dots, 1)^t$  in the operator inequalities above to get the following classical inequalities:

- Diaz–Metcalf inequality [4]

$$\sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^n a_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right) \sum_{k=1}^n a_k b_k.$$

- Pólya–Szegő inequality [23]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2;$$

- Shisha–Mond inequality [24]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leq \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}}\right)^2;$$

- A Grüss type inequality

$$\begin{aligned} \left(\sum_{k=1}^n a_k^2\right)^{1/2} \left(\sum_{k=1}^n b_k^2\right)^{1/2} - \sum_{k=1}^n a_k b_k \\ \leq \frac{\sqrt{M_1 M_2} (\sqrt{M_1 M_2} - \sqrt{m_1 m_2})^2}{2\sqrt{m_1 m_2}} \min \left\{ \frac{M_1}{m_1}, \frac{M_2}{m_2} \right\}. \end{aligned}$$

Using the same argument with a positive  $n$ -tuple  $(a_1, \dots, a_n)$  of real numbers with  $0 < m \leq a_i \leq M$  ( $1 \leq i \leq n$ ),  $x = \frac{1}{\sqrt{n}}(1, \dots, 1)^t$ , we get from Kantorovich inequality that

- Schweitzer inequality [2]

$$\left(\frac{1}{n} \sum_{i=1}^n a_i^2\right) \left(\frac{1}{n} \sum_{i=1}^n a_i^{-2}\right) \leq \frac{(M^2 + m^2)^2}{4M^2 m^2}.$$

If  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  are  $n$ -tuples of real numbers with  $0 < m \leq a_i/b_i \leq M$  ( $1 \leq i \leq n$ ), we can consider the positive linear map  $\Phi(T) = \langle Tx, x \rangle$  on  $\mathbb{B}(\mathbb{C}^n) = M_n(\mathbb{C})$  and let  $A = \text{diag}(a_1^2, \dots, a_n^2)$ ,  $B = \text{diag}(b_1^2, \dots, b_n^2)$  and  $x = (\sqrt{w_1}, \dots, \sqrt{w_n})^t$  based on the weight  $\bar{w} = (w_1, \dots, w_n)$ , in the operator inequalities above to get the following classical inequalities:

- Cassels inequality [25]

$$\frac{\sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \leq \frac{(M + m)^2}{4mM};$$

- Klamkin–McLenaghan inequality [14]

$$\sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2 - \left(\sum_{k=1}^n w_k a_k b_k\right)^2 \leq (\sqrt{M} - \sqrt{m})^2 \sum_{k=1}^n w_k a_k b_k \sum_{k=1}^n w_k a_k^2.$$

Using the same argument, we obtain a weighted form of the Pólya–Szegő inequality as follows:

- Grueb–Rheinboldt inequality [10]

$$\frac{\sum_{k=1}^n w_k a_k^2 \sum_{k=1}^n w_k b_k^2}{\left(\sum_{k=1}^n w_k a_k b_k\right)^2} \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4m_1 m_2 M_1 M_2}.$$

One can assert the integral versions of discrete results above by considering  $L^2(X, \mu)$ , where  $(X, \mu)$  is a probability space, as a Hilbert space via  $\langle h_1, h_2 \rangle = \int_X h_1 \overline{h_2} d\mu$ , multiplication operators  $A, B \in \mathbb{B}(L^2(X, \mu))$  defined by  $A(h) = f^2 h$

and  $B(h) = g^2h$  for bounded  $f, g \in L^2(X, \mu)$  and a positive linear map  $\Phi$  by  $\Phi(T) = \int_X T(1)d\mu$  on  $\mathbb{B}(L^2(X, \mu))$ . For instance, let us state integral versions of the Cassels and Klamkin–McLenaghan inequalities. These two inequalities are obtained, first for bounded positive functions  $f, g \in L^2(X, \mu)$  and next for general positive functions  $f, g \in L^2(X, \mu)$  as the limits of sequences of bounded positive functions.

**COROLLARY 3.1.** *Let  $(X, \mu)$  be a probability space and  $f, g \in L^2(X, \mu)$  with  $0 \leq mg \leq f \leq Mg$  for some scalars  $0 < m < M$ . Then*

$$\int_X f^2 d\mu \int_X g^2 d\mu \leq \frac{(M+m)^2}{4Mm} \left( \int_X fg d\mu \right)^2$$

and

$$\int_X f^2 d\mu \int_X g^2 d\mu - \left( \int_X fg d\mu \right)^2 \leq (\sqrt{M} - \sqrt{m})^2 \int_X fg d\mu \int_X f^2 d\mu.$$

Considering the positive linear functional  $\Phi(R) = \sum_{i=1}^n \langle R\xi_i, \xi_i \rangle$  on  $\mathbb{B}(\mathcal{H})$ , where  $\xi_1, \dots, \xi_n \in \mathcal{H}$ , we get the following versions of the Diaz–Metcalf and Pólya–Szegő inequalities in a Hilbert space.

**COROLLARY 3.2.** *Let  $\mathcal{H}$  be a Hilbert space, let  $\xi_1, \dots, \xi_n \in \mathcal{H}$  and let  $T, S \in \mathbb{B}(\mathcal{H})$  be positive operators satisfying  $0 < m_1 \leq T \leq M_1$  and  $0 < m_2 \leq S \leq M_2$ . Then*

$$\frac{M_2 m_2}{M_1 m_1} \sum_{i=1}^n \|T\xi_i\|^2 + \sum_{i=1}^n \|S\xi_i\|^2 \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{i=1}^n \|(T^2 \sharp S^2)^{1/2} \xi_i\|^2$$

and

$$\begin{aligned} & \left( \sum_{i=1}^n \|T\xi_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n \|S\xi_i\|^2 \right)^{1/2} \\ & \leq \frac{1}{2} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right) \sum_{i=1}^n \|(T^2 \sharp S^2)^{1/2} \xi_i\|^2. \end{aligned}$$

**4. A Grüss type inequality.** In this section we obtain another Grüss type inequality, see also [18]. Let  $\mathcal{A}$  be a  $C^*$ -algebra and let  $\mathcal{B}$  be a  $C^*$ -subalgebra of  $\mathcal{A}$ . Following [1], a positive linear map  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  is called a left multiplier if  $\Phi(XY) = \Phi(X)Y$  for every  $X \in \mathcal{A}, Y \in \mathcal{B}$ .

The following lemma is interesting on its own right.



LEMMA 4.1. *Let  $\Phi$  be a unital positive linear map on  $\mathcal{A}$ ,  $A \in \mathcal{A}$  and  $M, m$  be complex numbers such that*

$$\operatorname{Re}((M - A)^*(A - m)) \geq 0. \quad (4.1)$$

Then

$$\Phi(|A|^2) - |\Phi(A)|^2 \leq \frac{1}{4}|M - m|^2.$$

*Proof.* For any complex number  $c \in \mathbb{C}$ , we have

$$\Phi(|A|^2) - |\Phi(A)|^2 = \Phi(|A - c|^2) - |\Phi(A - c)|^2. \quad (4.2)$$

Since for any  $T \in \mathcal{A}$  the operator equality

$$\frac{1}{4}|M - m|^2 - \left| T - \frac{M + m}{2} \right|^2 = \operatorname{Re}((M - T)(T - m)^*)$$

holds, the condition (4.1) implies that

$$\Phi \left( \left| A - \frac{M + m}{2} \right|^2 \right) \leq \frac{1}{4}|M - m|^2. \quad (4.3)$$

Therefore, it follows from (4.2) and (4.3) that

$$\begin{aligned} \Phi(|A|^2) - |\Phi(A)|^2 &\leq \Phi \left( \left| A - \frac{M + m}{2} \right|^2 \right) \\ &\leq \frac{1}{4}|M - m|^2. \quad \square \end{aligned}$$

REMARK 4.2. If (i)  $\Phi$  is a unital positive linear map and  $A$  is a normal operator or (ii)  $\Phi$  is a 2-positive linear map and  $A$  is an arbitrary operator, then it follows from [3] that

$$0 \leq \Phi(|A|^2) - |\Phi(A)|^2. \quad (4.4)$$

Condition (4.4) is stronger than positivity and weaker than 2-positivity; see [8]. Another class of positive linear maps satisfying (4.4) are left multipliers, cf. [1, Corollary 2.4].

LEMMA 4.3. *Let a positive linear map  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  be a unital left multiplier. Then*

$$|\Phi(A^*B) - \Phi(A)^*\Phi(B)|^2 \leq \|\Phi(|A|^2) - |\Phi(A)|^2\| (\Phi(|B|^2) - |\Phi(B)|^2) \quad (4.5)$$

*Proof.* If we put  $[X, Y] := \Phi(X^*Y) - \Phi(X)^*\Phi(Y)$ , then  $\mathcal{A}$  is a right pre-inner product  $C^*$ -module over  $\mathcal{B}$ , since  $\Phi(X^*Y)$  is a right pre-inner product  $\mathcal{B}$ -module, see [1, Corollary 2.4]. It follows from the Cauchy–Schwarz inequality in pre-inner product  $C^*$ -modules (see [15, Proposition 1.1]) that

$$\begin{aligned} |\Phi(A^*B) - \Phi(A)^*\Phi(B)|^2 &= [B, A][A, B] \\ &\leq \|[A, A]\|[B, B] \\ &= \|\Phi(A^*A) - \Phi(A)^*\Phi(A)\|(\Phi(B^*B) - \Phi(B)^*\Phi(B)) \end{aligned}$$

and hence (4.5) holds.  $\square$

**THEOREM 4.4.** *Let a positive linear map  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  be a unital left multiplier. If  $M_1, m_1, M_2, m_2 \in \mathbb{C}$  and  $A, B \in \mathcal{A}$  satisfy the following conditions:*

$$\operatorname{Re}(M_1 - A)^*(A - m_1) \geq 0 \quad \text{and} \quad \operatorname{Re}(M_2 - B)^*(B - m_2) \geq 0,$$

then

$$|\Phi(A^*B) - \Phi(A)^*\Phi(B)| \leq \frac{1}{4}|M_1 - m_1||M_2 - m_2|.$$

*Proof.* By Löwner–Heinz theorem, we have

$$\begin{aligned} &|\Phi(A^*B) - \Phi(A)^*\Phi(B)| \\ &\leq \|\Phi(|A|^2) - |\Phi(A)|^2\|^{\frac{1}{2}} \|\Phi(|B|^2) - |\Phi(B)|^2\|^{\frac{1}{2}} \quad (\text{by Lemma 4.3}) \\ &\leq \frac{1}{4}|M_1 - m_1||M_2 - m_2| \quad (\text{by Lemma 4.1}). \quad \square \end{aligned}$$

**5. Ozeki–Izumino–Mori–Seo type inequality.** Let  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  be  $n$ -tuples of real numbers satisfying

$$0 \leq m_1 \leq a_i \leq M_1 \quad \text{and} \quad 0 \leq m_2 \leq b_i \leq M_2 \quad (i = 1, \dots, n).$$

Then Ozeki–Izumino–Mori–Seo inequality [12, 22] asserts that

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left( \sum_{i=1}^n a_i b_i \right)^2 \leq \frac{n^2}{3} (M_1 M_2 - m_1 m_2)^2. \quad (5.1)$$

In [12] they also showed the following operator version of (5.1): If  $A$  and  $B$  are positive operators in  $\mathbb{B}(\mathcal{H})$  such that  $0 < m_1 \leq A \leq M_1$  and  $0 < m_2 \leq B \leq M_2$  for some scalars  $m_1 \leq M_1$  and  $m_2 \leq M_2$ , then

$$(A^2 x, x)(B^2 x, x) - (A^2 \sharp B^2 x, x)^2 \leq \frac{1}{4\gamma^2} (M_1 M_2 - m_1 m_2)^2 \quad (5.2)$$

for every unit vector  $x \in H$ , where  $\gamma = \max\{\frac{m_1}{M_1}, \frac{m_2}{M_2}\}$ .

Based on the Kantorovich inequality for the difference, we present an extension of Ozeki–Izumino–Mori–Seo inequality (5.2) as follows.

**THEOREM 5.1.** *Suppose that  $\Phi : \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$  is a positive linear map such that  $\Phi(I)$  is invertible and  $\Phi(I) \leq I$ . Assume that  $A, B \in \mathbb{B}(\mathcal{H})$  are positive invertible operators such that  $0 < m_1 \leq A \leq M_1$  and  $0 < m_2 \leq B \leq M_2$  for some scalars  $m_1 \leq M_1$  and  $m_2 \leq M_2$ . Then*

$$|\Phi(B^2)^{\frac{1}{2}}\Phi(A^2)\Phi(B^2)^{\frac{1}{2}} - |\Phi(B^2)^{-\frac{1}{2}}\Phi(A^2\sharp B^2)\Phi(B^2)^{\frac{1}{2}}|^2 \leq \frac{(M_1M_2 - m_1m_2)^2}{4} \times \frac{M_2^2}{m_2^2} \tag{5.3}$$

and

$$|\Phi(A^2)^{\frac{1}{2}}\Phi(B^2)\Phi(A^2)^{\frac{1}{2}} - |\Phi(A^2)^{-\frac{1}{2}}\Phi(A^2\sharp B^2)\Phi(A^2)^{\frac{1}{2}}|^2 \leq \frac{(M_1M_2 - m_1m_2)^2}{4} \times \frac{M_1^2}{m_1^2}. \tag{5.4}$$

*Proof.* Define a normalized positive linear map  $\Psi$  by

$$\Psi(X) := \Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}.$$

By using the Kantorovich inequality for the difference, it follows that

$$\Psi(X^2) - \Psi(X)^2 \leq \frac{(M - m)^2}{4} \tag{5.5}$$

for all  $0 < m \leq X \leq M$  with some scalars  $m \leq M$ . As a matter of fact, we have

$$\begin{aligned} \Psi(X^2) - \Psi(X)^2 &\leq \Psi((M + m)X - Mm) - \Psi(X)^2 \\ &= -\left(\Psi(X) - \frac{M + m}{2}\right)^2 + \frac{(M - m)^2}{4} \\ &\leq \frac{(M - m)^2}{4}. \end{aligned}$$

If we put  $X = (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}$ , then due to

$$0 < (m =) \sqrt{\frac{m_2}{M_1}} \leq X \leq \sqrt{\frac{M_2}{m_1}} (= M)$$

we deduce from (5.5) that

$$|\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}} - \left(\Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}\sharp B)\Phi(A)^{-\frac{1}{2}}\right)^2 \leq \frac{(\sqrt{M_1M_2} - \sqrt{m_1m_2})^2}{4M_1m_1}.$$

Pre- and post-multiplying both sides by  $\Phi(A)$ , we obtain

$$\begin{aligned} |\Phi(A)^{\frac{1}{2}}\Phi(B)\Phi(A)^{\frac{1}{2}} - |\Phi(A)^{-\frac{1}{2}}\Phi(A\sharp B)\Phi(A)^{\frac{1}{2}}|^2 &\leq \frac{(\sqrt{M_1M_2} - \sqrt{m_1m_2})^2}{4M_1m_1}\Phi(A)^2 \\ &\leq \frac{(\sqrt{M_1M_2} - \sqrt{m_1m_2})^2}{4} \times \frac{M_1}{m_1}, \end{aligned}$$

since  $0 \leq \Phi(A)^2 \leq M_1^2$ . Replacing  $A$  and  $B$  by  $A^2$  and  $B^2$  respectively, we have the desired inequality (5.4). Similarly, one can obtain (5.3).  $\square$

REMARK 5.2. If  $\Phi$  is a vector state in (5.3) and (5.4), then we get Ozeki–Izumino–Mori–Seo inequality (5.2).

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