Generalized Pascal $k$-eliminated functional matrix with $2n$ variables

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Abstract. In this paper, we introduce the Pascal $k$-eliminated functional matrix and the Pascal symmetric functional matrix with $2n$ variables. Some algebraic properties of these matrices are presented and proved. In addition, we demonstrate a direct application of these properties for LU decompositions of some well-known matrices (such as symmetric Pascal matrices).

Key words. Pascal matrix, Pascal $k$-eliminated functional matrix, Pascal symmetric functional matrix, LU decompositions, Cholesky factorization.

AMS subject classifications. 15A06, 34A30.

1. Introduction. In [1], the Pascal matrix $P_n[x]$ is defined by

$$ (P_n[x])_{ij} = \begin{cases} \binom{i}{j} x^{i-j}, & i \geq j \\ 0, & i < j \end{cases}, \quad (i, j = 0, 1, \ldots, n). $$

Call and Velleman [1] discussed the inverse and exponential property of $P_n[x]$, i.e., $P_n[x + y] = P_n[x]P_n[y]$ for all $x, y$, and a few basic properties of this matrix. In [2], the symmetric Pascal matrix $Q_n$ is defined by

$$ (Q_n)_{ij} = \binom{i+j}{j}, \quad (i, j = 0, 1, \ldots, n), $$

and it has been shown that $Q_n$ can be expressed as the product of a lower triangular Pascal matrix, $P_n[1]$, and an upper triangular Pascal matrix, $P_n^T[1]$.

In [3], the extended generalized lower triangular Pascal matrix for two variables $\Phi_n[x, y]$ is defined by

$$ (\Phi_n[x, y])_{ij} = \begin{cases} \binom{i}{j} x^{i-j} y^{i+j}, & i \geq j \\ 0, & i < j \end{cases}, \quad (i, j = 0, 1, \ldots, n), $$

and the extended generalized rectangular Pascal matrix $\Psi_n[x, y]$ is defined by

$$ (\Psi_n[x, y])_{ij} = x^{i-j} y^{i+j} \binom{i+j}{j}, \quad (i, j = 0, 1, \ldots, n). $$

*Received by the editors on July 13, 2010. Accepted for publication on April 2, 2011. Handling Editor: Miroslav Fiedler.

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In [3], the authors also demonstrated that $\Psi_{n}[x, y]$ has the LU decomposition $\Psi_{n}[x, y] = \Phi_{n}[x, y]P_{n}^{T}[y]$. The Pascal functional matrix was introduced in [4], by

$$\text{(1.5)} \quad (P_{n, \lambda}[x])_{i,j} = \begin{cases} \binom{i}{j}x^{i-j}y^{\lambda}, & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \ldots, n),$$

where $x^{n|\lambda}$ is the generalized upper factorial, which is defined as follows:

$$x^{n|\lambda} = \begin{cases} x(x + \lambda)(x + 2\lambda)\cdots(x + (n-1)\lambda), & n \geq 1 \\ 0, & n = 0 \end{cases}.$$

In [5], the notion of Pascal functional matrix was extended to a more general Pascal functional matrix, $G_{n}[x]$, defined by

$$\text{(1.6)} \quad (G_{n}[x])_{i,j} = \begin{cases} \binom{i}{j}g_{i-j}(x), & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \ldots, n),$$

where $\{g_{n}(x)\}$ is an arbitrary sequence of binomial-type polynomials, i.e., for all $n$,

$$g_{n}(x + y) = \sum_{k=0}^{n} \binom{n}{k}g_{k}(x)g_{n-k}(y) \quad \text{for any } x \text{ and } y.$$

In [6], the Pascal $k$-eliminated functional matrix with two variables, is defined by

$$\text{(1.7)} \quad (P_{n, k}[x, y])_{i,j} = \begin{cases} \binom{i+k}{j+k}x^{i-j}y^{k}, & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \ldots, n, k \in \mathbb{N} \cup \{0\}).$$

Another variant of Pascal functional matrix was introduced in [8], defined by

$$\text{(1.8)} \quad (P_{n}[f(t, x)])_{i,j} = \begin{cases} \binom{i}{j}f^{(i-j)}(t, x), & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \ldots, n),$$

where $f^{(k)}(t, x)$ is $k$th order derivative of $f$ with respect to $t$. In [8], it was shown that all well-known variants of Pascal matrices in [4,5] are special cases of this generalization of Pascal functional matrix. In [3–8], the authors proved some algebraic properties of such matrices and derived combinatorial identities from these properties.

2. The Pascal $k$-eliminated functional matrix with $2n$ variables.

**Definition 2.1.** Using the sequence $t_{0} = 1, t_{1}, t_{2}, \ldots$, we define the sequence $t^{[0]} = 1, t^{[1]}, t^{[2]}, \ldots$ by the relation

$$t^{[n]} = t_{n}t^{[n-1]}.$$
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**Notation.** For convenience, we accept the following notation:

$$
\ell[i]+[j] := \ell[i] \cdot \ell[j] \quad \text{and} \quad \ell[i]−[j] := \ell[i]/\ell[j].
$$

**Definition 2.2.** Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be any $2n$ variables. The Pascal $k$-eliminated functional matrix with $2n$ variables $\Phi_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n]$ for $k \in \mathbb{N} \cup \{0\}$ is defined by

$$
(2.1) \quad (\Phi_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n])_{ij} = \binom{i+k}{j+k} x^{i−[j]} y^{[i]+[j]} \quad (i, j = 0, 1, \ldots, n),
$$

where it is assumed that

$$
\binom{i}{j} = 0 \quad \text{if} \quad j > i.
$$

It can be shown that all well-known variants of Pascal matrices in $[1,3,4,6,7]$ are special cases of this new generalization of Pascal functional matrix with $2n$ variables. At first we study a special case of this matrix where $x_1 = \cdots = x_n = x, y_1 = \cdots = y_n = y$ and $k \in \mathbb{N} \cup \{0\}$:

$$
\Phi_{n,k}[x; y] = \Phi_{n,k}[x, \ldots, x; y, \ldots, y],
$$
called the Pascal $k$-eliminated functional matrix with two variables $[4]$. It is easily seen that for $k = 0$, $\Phi_{n,0}[x, y]$ is the Pascal matrix $\Phi_n[x, y]$ in $[3]$, and for $k = 0$ and $y = 1$, $\Phi_{n,0}[x, 1]$ is the Pascal matrix $P_n[x]$ in $[1]$. Also taking $x_1 = x, x_2 = x + \lambda, \ldots, x_n = x + (n−1)\lambda$, $y_1 = y, y_2 = y + \lambda, \ldots, y_n = y + (n−1)\lambda$ and $k = 0$ in (2.1), it yields (1.5).

**Theorem 2.3.** For any four real numbers $x_1, x_2, x_3, x_4$ and $k \in \mathbb{N} \cup \{0\}$, we have

(i) $\Phi_{n,k}[x_1; y_1].\Phi_{n,k}[x_2; y_2] = \Phi_{n,k}\left[\frac{x_1}{y_1} + x_2y_1; y_1y_2\right].$

Also for any two real numbers $x$ and $y$, we have

(ii) $\Phi_{n,k}[-x; y] = \Phi_{n,k}[x; −y],$

(iii) $\Phi_{n,k}^{−1}[x; y] = \Phi_{n,k}\left[-x; \frac{1}{y}\right] = \Phi_{n,k}\left[x, -\frac{1}{y}\right] \quad (y \neq 0),$

(iv) $\prod_{i=1}^{n} \Phi_{n,k}[x_i; y_i] = \Phi_{n,k}\left[(x_1 + \sum_{i=2}^{k} x_i y_{2i} \cdots y_{2i-1})^2 y_{2i} \prod_{j=2}^{k} \frac{1}{y_{2j-1}} \prod_{i=1}^{m} y_{2i}\right].$

**Proof.** Let $(\Phi_{n,k}[x_1; y_1].\Phi_{n,k}[x_2; y_2])_{ij} = a_{ij}$. Then

$$
a_{ij} = \sum_{t=0}^{n} \binom{i+k}{t+k} x_1^{i−t} y_1^{t+k} \binom{t+k}{j+k} x_2^{t−j} y_2^{t+j}.
$$
Now, using (i) and \( \Phi_{n,k} \), we have completed the proof.

It holds that
\[
\Phi_{n,k}^m[x; y] = \Phi_{n,k} \left[ xy^{-(m-1)} (1 + y^2 + y^4 + \cdots + y^{2(m-1)}); y^m \right] = \Phi_{n,k} \left[ \frac{xy^{-(m-1)}}{1 - y^2}; y^m \right].
\]

**Example 2.5.**
\[
[\Phi_{3,2}[x; y], \Phi_{3,2} \left[ -x; \frac{1}{y} \right]] = \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
3xy & y^2 & 0 & 0 \\
6x^2y^2 & 4xy^3 & y^4 & 0 \\
10x^3y^3 & 10x^2y^4 & 5xy^5 & y^6
\end{array} \right]\left[ \begin{array}{cccc}
-\frac{3x}{y} & 1 & 0 & 0 \\
\frac{6x^2}{y^2} & \frac{4x}{y^3} & \frac{1}{y^4} & 0 \\
-10x^3 & \frac{10x^2}{y} & \frac{10y^3}{y^2} & \frac{5y^2}{y^3} \\
-\frac{5y}{y^4} & \frac{1}{y^5} & -\frac{5}{y^6} & 0
\end{array} \right] = I_4.
\]

The matrices \( \Phi_{n,0}[x; y] \) and \( \Phi_{n,0}[x; 1] \) are the same as \( \Phi_n[x; y] \) and \( P_n[x] \), respectively, which are defined in [1,3]. We also consider the \((n+1) \times (n+1)\) matrices
\[
(W_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n])_{ij} = \begin{cases}
\frac{(k+1-i)}{k} x_i^{(i-j)} y_i^{(i+j)}, & i \geq j \\
0, & i < j
\end{cases}
\]
and
\[
(U_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n])_{ij} = \begin{cases}
(-1)^m \frac{(k+1-i)}{m} y_i^{(i+j)}, & i = j + m, m = 0, 1, \ldots, k + 1 \\
0, & \text{otherwise}
\end{cases}
\]
where \( i, j = 0, 1, \ldots, n \).
Theorem 2.6. For \( k \in \mathbb{N} \cup \{0\} \), we have

\( (i) \Phi_{n,k} \left[ -x_1, \ldots, -x_n; y_1, \ldots, y_n \right] = \Phi_{n,k} \left[ x_1, \ldots, x_n; -y_1, \ldots, -y_n \right] \),

\( (ii) \Phi_{n,k}^{-1} \left[ x_1, \ldots, x_n; y_1, \ldots, y_n \right] = \Phi_{n,k} \left[ -x_1, \ldots, -x_n; \frac{1}{y_1}, \ldots, \frac{1}{y_n} \right] \),

\( (iii) W_{n,k}^{-1} \left[ x_1, \ldots, x_n; y_1, \ldots, y_n \right] = U_{n,k} \left[ x_1, \ldots, x_n; y_1, \ldots, y_n \right] \).

Proof. Here we prove (iii). Let

\[
(W_{n,k} [x_1, \ldots, x_n; y_1, \ldots, y_n] U_{n,k} [x_1, \ldots, x_n; y_1, \ldots, y_n])_{ij} = a_{ij}.
\]

Obviously, \( a_{ii} = 1 \) (i = 0, 1, …, n) and \( a_{ij} = 0 \) (i < j). When i > j we have

\[
a_{ij} = \sum_{t=0}^{i} \binom{k + i - t}{t} x^{i-[i]} y^{i+[i]} (-1)^{t-j} \binom{k + 1}{t-j} \binom{k}{t-j} = x^{i-[j]} y^{i-[j]} \sum_{t=0}^{i-j} (-1)^{s} \binom{k + i - j - s}{s} \binom{k}{s} = 0.
\]

Example 2.7.

\[
W_{3,3} [x_1, x_2, x_3; y_1, y_2, y_3] U_{3,3} [x_1, x_2, x_3; y_1, y_2, y_3]
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
2x_1 y_1 & y_1^2 & 0 & 0 \\
3x_1 x_2 y_1 y_2 & 2x_2 y_1 y_2^2 & y_1^2 y_2^2 & 0 \\
4x_1 x_2 x_3 y_1 y_2 y_3 & 3x_2 x_3 y_1 y_2 y_3 & 2x_3 y_1 y_2^2 y_3 & y_1^2 y_2^2 y_3
\end{bmatrix}
\]

\[
\times \begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{y_1} & 0 & 0 & 0 \\
\frac{1}{y_1 y_2} & \frac{1}{y_1^2 y_2} & 0 & 0 \\
0 & \frac{1}{y_1 y_2 y_3} & \frac{1}{y_1^2 y_2 y_3} & \frac{1}{y_1^2 y_2 y_3}
\end{bmatrix} = I_4.
\]

Again, we need the matrices \( P_n [x_1, \ldots, x_n] \), \( S_n [x_1, \ldots, x_n] \), \( T_m [x_1, \ldots, x_m] \) and \( G_n [x_1, \ldots, x_n] \):

\( P_n [x_1, \ldots, x_n] := \Phi_{n,0} [x_1, \ldots, x_n; 1, \ldots, 1] \),

\( S_n [x_1, \ldots, x_n] := W_{n,0} [x_1, \ldots, x_n; 1, \ldots, 1] \),

\( T_m [x_1, \ldots, x_m] = \begin{bmatrix} 1 \ O^T \end{bmatrix} \in \mathbb{R}^{(m+2) \times (m+2)} \quad (m \geq 0) \),
Obviously, \( G_m[x_1, \ldots, x_m] = \begin{bmatrix} I_n & 0 \\ O & S_m \end{bmatrix}^{O^T} \in \mathbb{R}^{(n+1) \times (n+1)} \) \((m = 0, 1, \ldots, n - 1),\)

\( G_n[x_1, \ldots, x_n] := S_n[x_1, \ldots, x_n]. \)

**Lemma 2.8.** For \( m \in \mathbb{N} \) and \( k \in \mathbb{N} \cup \{0\}, \) we have

\[
W_{m,k}[x_1, \ldots, x_m; y_1, \ldots, y_m] P_m^{-1} \begin{bmatrix} x_2 \\ y_2 \\ \vdots \\ x_m \\ y_m \end{bmatrix} = P_{m,k}[x_1, \ldots, x_m; y_1, \ldots, y_m].
\]

**Proof.** Let

\[
\left( W_{m,k}[x_1, \ldots, x_m; y_1, \ldots, y_m] P_m^{-1} \begin{bmatrix} x_2 \\ y_2 \\ \vdots \\ x_m \\ y_m \end{bmatrix} \right)_{ij} = a_{ij}.
\]

Obviously, \( a_{ii} = (y^{[i]}_i)^2 \) \((i = 0, 1, \ldots, m)\) and \( a_{ij} = 0 \) \((i < j)\). When \( i > j \) we have

\[
a_{ij} = \sum_{s=j}^{i} (k + i - s) \frac{x^{[s]}_i y^{[s]}_j}{y^{[s]}_j} \frac{(t-1)}{(t-1)} = \frac{x^{[i]}_i y^{[j]}_j}{y^{[j]}_j} \frac{(t-1)}{(t-1)} = \frac{x^{[i]}_i y^{[j]}_j}{y^{[j]}_j} \frac{(i+k)}{(j+k)}. \]

**Example 2.9.**

\[
W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] P_2 \begin{bmatrix} x_2 \\ y_2 \\ y_3 \end{bmatrix} =
\]

\[
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2x_1 y_1 & y_1^2 & 0 & 0 & 0 \\ 3x_1 x_2 y_1 y_2 & x_2 y_1^2 y_2 & y_1 y_2^2 & 0 \\ 4x_1 x_2 x_3 y_1 y_2 y_3 & 3x_2 x_3 y_1^2 y_2 y_3 & 2x_3 y_1^2 y_2^2 y_3 & y_1^2 y_2^2 y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_2 y_2 & 1 & 0 \\ 0 & x_2 x_3 y_2 y_3 & x_3 y_2 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1 y_1 & y_1^2 & 0 & 0 \\ 3x_1 x_2 y_1 y_2 & x_2 y_1^2 y_2 & y_1 y_2^2 & 0 \\ 4x_1 x_2 x_3 y_1 y_2 y_3 & 3x_2 x_3 y_1^2 y_2 y_3 & 2x_3 y_1^2 y_2^2 y_3 & y_1^2 y_2^2 y_3 \end{bmatrix} = P_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3].
\]

By Lemma 2.8 and the definition of \( G_m[x_1, \ldots, x_m] \), we get the following result:
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**Theorem 2.10.** The extended generalized Pascal matrix $\Phi_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n]$ can be factorized by $W_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n]$ and $G_m[x_1, \ldots, x_m]$ ($m = 0, 1, \ldots, n - 1$):

$$
\Phi_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n] = W_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n] G_{n-1} \begin{bmatrix} x_2/y_2 & \cdots & x_n/y_n \end{bmatrix} G_{n-2} \begin{bmatrix} x_3/y_3 & \cdots & x_n/y_n \end{bmatrix} \cdots G_1 \begin{bmatrix} x_n/y_n \end{bmatrix}.
$$

**Example 2.11.**

$$
\Phi_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\
2x_1y_1 & y_1^2 & 0 & 0 \\
3x_1x_2y_1y_2 & 3x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\
4x_1x_2x_3y_1y_2y_3 & 6x_2x_3y_1^2y_2^2y_3 & 4x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \frac{x_2}{y_3} & 1 \end{bmatrix} = W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] G_2 \begin{bmatrix} x_2/y_2 & x_3/y_3 \end{bmatrix} G_1 \begin{bmatrix} x_3/y_3 \end{bmatrix}.
$$

If we take $x_1 = \cdots = x_n = x$, $y_1 = \cdots = y_n = y$ and $k = 0$ (or $x_1 = \cdots = x_n = x$, $y_1 = \cdots = y_n = y = 1$) in Theorem 2.10, then we get the following results [3,4]:

**Corollary 2.12.** It holds that

$$
\Phi_{n}[x; y] = W_{n,k}[x; y] G_{n-1} \begin{bmatrix} x/y \end{bmatrix} G_{n-2} \begin{bmatrix} x/y \end{bmatrix} \cdots G_1 \begin{bmatrix} x/y \end{bmatrix},
$$

where

$$
\Phi_{n}[x; y] = P_{n,0}[x, \ldots, x; y, \ldots, y] \quad \text{and} \quad W_{n}[x; y] = W_{n,0}[x, \ldots, x; y, \ldots, y].
$$

**Corollary 2.13.** It holds that

$$
P_{n,k}[x] = (G_n[x])^k G_{n-1}[x] G_{n-2}[x] \cdots G_1[x],
$$

where

$$
P_{n,k}[x] = P_{n,k}[x, \ldots, x] \quad \text{and} \quad (G_n[x])^k = W_{n,k}[x, \ldots, x; x, \ldots, x].$$
For the inverse of the extended generalized Pascal matrix $\Phi_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]$, applying Theorems 2.6 and 2.10 yields the following.

Theorem 2.14. It holds that

$$\Phi^{-1}_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n] = \Phi_{n,k} \left[ -x_1,\ldots,-x_n; y_1,\ldots,\frac{1}{y_n} \right] = \Phi_{n,k} \left[ x_1,\ldots,x_n; -\frac{1}{y_1},\ldots,\frac{1}{y_n} \right]$$

$$= G^{-1}_{n,k} \left[ \frac{x_n}{y_n} \right] G^{-1}_{n,k} \left[ \frac{x_{n-1}}{y_{n-1}} \right] \cdots G^{-1}_{n,k} \left[ \frac{x_1}{y_1} \right] W^{-1}_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]$$

$$= F_1 \left[ \frac{x_n}{y_n} \right] F_2 \left[ \frac{x_{n-1}}{y_{n-1}} \right] \cdots F_{n-1} \left[ \frac{x_1}{y_1} \right] U_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n],$$

where

$$F_m \left[ \frac{x_{n-m+1}}{y_{n-m+1}},\ldots,\frac{x_n}{y_n} \right] = G^{-1}_{m-k} \left[ \frac{x_{n-m+1}}{y_{n-m+1}},\ldots,\frac{x_n}{y_n} \right]$$

$$= \begin{bmatrix} I_{n-k} & O \\ O & D_m \left[ \frac{x_{n-m+1}}{y_{n-m+1}},\ldots,\frac{x_n}{y_n} \right] \end{bmatrix},$$

for $m = 1,\ldots,n-1$, and $D_n[x_1,\ldots,x_n] = U_{n,0}[x_1,\ldots,x_n;1,\ldots,1]$.

In particular,

$$P^{-1}_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n] = J_n[y_1,\ldots,y_n] P_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n] J_n[y_1,\ldots,y_n],$$

where

$$J_n[y_1,\ldots,y_n] = \text{diag}\left(1,\frac{1}{y_1^2},\ldots,(-1)^n\frac{1}{y_1^2\cdots y_n^2}\right).$$

3. The symmetric Pascal matrix with $2n$ variables. In this section, we define the extended generalized symmetric Pascal matrix $\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n]$ by

$$(\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n])_{ij} = \binom{i+j}{i} x^{i-[j]} y^{[i]+[j]},$$

where $i,j = 0,1,\ldots,n$.

Theorem 3.1. It holds that

$$P_n^T \left[ \frac{y_1}{x_1},\ldots,\frac{y_n}{x_n} \right] = F_1 \left[ \frac{x_n}{y_n} \right] F_2 \left[ \frac{x_{n-1}}{y_{n-1}} \right] \cdots F_{n-1} \left[ \frac{x_1}{y_1} \right] \times U_{n,0}[x_1,\ldots,x_n;y_1,\ldots,y_n] \Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n].$$
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$$P^T_{n,0}[x_1, \ldots, x_n; y_1, \ldots, y_n] = F_1 \left[ \frac{x_n}{y_n} \right] F_2 \left[ \frac{x_{n-1}}{y_{n-1}}; \frac{x_n}{y_n} \right] \cdots F_{n-1} \left[ \frac{x_2}{y_2}; \frac{x_{n-1}}{y_{n-1}}; \frac{x_n}{y_n} \right]$$

and the Cholesky factorization of $\Psi_n[x_1, \ldots, x_n; y_1, \ldots, y_n]$ is given by

(3.1) $\Psi_n[x_1, \ldots, x_n; y_1, \ldots, y_n] = P^T_{n,0}[1, \ldots, 1; x_1y_1, \ldots, x_ny_n]$ 

\[ \times P_{n,0} \left[ y_1, \ldots, y_n; \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right] , \]

(3.2) $\Psi_n[x_1, \ldots, x_n; y_1, \ldots, y_n] = P^T_{n,0}[x_1, \ldots, x_n; y_1, \ldots, y_n] P^T_{n} \left[ y_1, \ldots, y_n \right] \left[ \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right] . \]

Proof. Let $\left( P^T_{n,0}[1, \ldots, 1; x_1y_1, \ldots, x_ny_n] P^T_{n,0} \left[ y_1, \ldots, y_n; \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right] \right)_{ij} = a_{ij}$.

Then

$$a_{ij} = \left\{ \begin{array}{ll}
\sum_{k=0}^{i} \binom{i}{k} \binom{j}{k} x^{i-j} y^{j-i}, & i \geq j \\
\sum_{k=0}^{i} \binom{i}{k} \binom{j}{k} x^{i-j} y^{j-i}, & i < j
\end{array} \right\} ,$$

which by using the Vandermonde identities

$$\sum_{k=0}^{i} \binom{i}{k} \binom{j}{k} = \sum_{k=0}^{i} \binom{i}{k} \binom{j}{j-k} = \binom{i+j}{j}$$

and

$$\sum_{k=0}^{j} \binom{i}{k} \binom{j}{k} = \sum_{k=0}^{j} \binom{i}{i-k} \binom{j}{j-k} = \binom{i+j}{j} ,$$

implies (3.1). Similarly we get (3.2). \( \square \)

Example 3.2.

$$W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] P^2 \left[ \frac{x_2}{y_2}, \frac{x_3}{y_3} \right] =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
2x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 \\
3x_1x_2y_1y_2 & 2x_2y_1^3y_2 & y_1^2y_2^2 & 0 & 0 & 0 \\
4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2^2y_3 & 2x_3y_1^3y_2^2y_3 & y_1^2y_2^2y_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2x_1y_1 & y_1^2 & 0 & 0 & 0 \\
3x_1x_2y_1y_2 & 2x_2y_1^3y_2 & y_1^2y_2^2 & 0 & 0 & 0 \\
4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2^2y_3 & 2x_3y_1^3y_2^2y_3 & y_1^2y_2^2y_3 & 0 & 0 \\
0 & 2x_1y_1 & y_1^2 & 0 & 0 & 0 \\
0 & 2x_2y_1^3y_2 & y_1^2y_2^2 & 0 & 0 & 0 \\
0 & 3x_1x_2y_1y_2 & 2x_2y_1^3y_2 & y_1^2y_2^2 & 0 & 0 \\
4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2^2y_3 & 2x_3y_1^3y_2^2y_3 & y_1^2y_2^2y_3 & 0 & 0 \\
0 & 3x_1x_2y_1y_2 & 2x_2y_1^3y_2 & y_1^2y_2^2 & 0 & 0 \\
0 & 4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2^2y_3 & 2x_3y_1^3y_2^2y_3 & y_1^2y_2^2y_3 & 0 \end{bmatrix} = P_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] .$$
By using Theorems 2.6 and 3.1, we have:

**Corollary 3.3.** It holds that

\[
\Psi_n^{-1}[x_1, \ldots, x_n; y_1, \ldots, y_n] = P_n^T \left[ -\frac{y_1}{x_1}, \ldots, -\frac{y_n}{x_n} \right] P_{n,0} \left[ x_1, \ldots, x_n; -\frac{1}{y_1}, \ldots, -\frac{1}{y_n} \right] = P_{n,0}^T [y_1, \ldots, y_n; -x_1, \ldots, -x_n] P_{n,0} \left[ 1, \ldots, 1; -\frac{1}{x_1 y_1}, \ldots, -\frac{1}{x_n y_n} \right].
\]

Applying Theorems 2.14 and 3.1, we get the following.

**Corollary 3.4.** It holds that

\[
\Psi_n^{-1}[x_1, \ldots, x_n; y_1, \ldots, y_n] = J_n \left[ 1, \ldots, 1 \right] P_n^T \left[ \frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n} \right] J_n \left[ 1, \ldots, 1 \right] \times J_n \left[ y_1, \ldots, y_n \right] P_{n,0} \left[ 1, \ldots, 1; y_1, \ldots, y_n \right] J_n \left[ 1, \ldots, 1 \right] \\
\times J_n \left[ y_1, \ldots, y_n \right] P_{n,0}^T \left[ 1, \ldots, 1; y_1, \ldots, y_n \right] J_n \left[ 1, \ldots, 1 \right] J_n \left[ y_1, \ldots, y_n \right] P_{n,0} \left[ 1, \ldots, 1; y_1, \ldots, y_n \right] J_n \left[ 1, \ldots, 1 \right].
\]

For the previous two kinds of extended generalized Pascal matrices, we can get:

**Corollary 3.5.** It holds that

\[
\det \Phi_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n] = y_1^{2n} y_2^{2(n-1)} \cdots y_n^{2}, \\
\det \Phi_{n,k}^{-1}[x_1, \ldots, x_n; y_1, \ldots, y_n] = y_1^{-2n} y_2^{-2(n-1)} \cdots y_n^{-2}, \\
\det \Psi_n[x_1, \ldots, x_n; y_1, \ldots, y_n] = y_1^{2n} y_2^{2(n-1)} \cdots y_n^{2}, \\
\det \Psi_n^{-1}[x_1, \ldots, x_n; y_1, \ldots, y_n] = y_1^{-2n} y_2^{-2(n-1)} \cdots y_n^{-2}.
\]

**REFERENCES**


