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SOME NEW LOWER BOUNDS FOR THE MINIMUM EIGENVALUE OF THE HADAMARD PRODUCT OF AN M-MATRIX AND ITS INVERSE

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1. Introduction. For a positive integer $n$, $N$ denotes the set $\{1, 2, \ldots, n\}$. For $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, we write $A \geq 0$ ($A > 0$) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all $i, j \in N$. If $A \geq 0$, we say $A$ is a nonnegative matrix, and if $A > 0$, we say $A$ is a positive matrix. The Perron eigenvalue of an $n \times n$ nonnegative matrix $P$ is denoted by $\rho(P)$.

A matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called an $M$-matrix if there exists a nonnegative matrix $B$ and a nonnegative real number $\lambda$, such that $A = \lambda I - B$ with $\lambda \geq \rho(B)$, where $I$ is the identity matrix. If $\lambda > \rho(B)$ (resp., $\lambda = \rho(B)$), then the $M$-matrix $A$ is nonsingular (resp., singular); see [1].

For $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, define $\tau(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of $A$.

The Hadamard product of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathbb{R}^{n \times n}$ is the matrix $A \circ B = [a_{ij}b_{ij}] \in \mathbb{R}^{n \times n}$. If $A$ and $B$ are $M$-matrices, then it was proved in [5] that $A \circ B^{-1}$ is also an $M$-matrix. For an $M$-matrix $A$, Fiedler et al. showed in [4] that $0 < \tau(A \circ A^{-1}) \leq 1$. In [5], Fiedler and Markham gave a lower bound on

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\[ \tau(A \circ A^{-1}) \]  
(1.1)  
and proposed the following conjecture:

\[ \tau(A \circ A^{-1}) \geq \frac{2}{n} \]  
(1.2)

This conjecture has been proved by Yong ([13, 14]), Song ([10]) and Chen ([3]) independently.

In [12], Xiang used the spectral radius of the Jacobi iterative matrix of an \( n \times n \) M-matrix \( A \), and proved that

\[ \tau(A \circ A^{-1}) \geq 1 - \rho(J_A)^2 \]  
(1.3)  
and

\[ \tau(A \circ A^{-1}) \geq \frac{1 + \rho(J_A)}{1 + (n-1)\rho(J_A)^2} \]  
(1.4)

where \( \rho(J_A) \) denotes the spectral radius of the Jacobi iterative matrix of \( A \).

Obviously, the lower bounds (1.1) and (1.2) are simple, but they are not accurate enough. For the lower bounds (1.3) and (1.4), it is difficult to calculate the lower bound of \( \tau(A \circ A^{-1}) \) by using these formulas, since it is difficult to calculate \( \rho(J_A) \) when the order of \( A \) is large.

In [7], Li obtained the following result:

\[ \tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}} \right\}, \]  
(1.5)

which only depends on the entries of \( A = [a_{ij}] \), where \( R_i = \sum_{k \neq i} |a_{ik}|, d_i = \frac{R_i}{|a_{ii}|}, i \in N; \)
\( s_{ji} = \frac{|a_{ij}| + \sum_{k \neq i} |a_{jk}| d_k}{|a_{jj}|}, j \neq i, j \in N; s_i = \max \{ s_{ij} \}, i \in N. \) In [8], Li improved the bound (1.5) in some cases, and obtained the following result:

\[ \tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j \neq i} m_{ji}} \right\}, \]  
(1.6)

where \( r_{li} = \frac{|a_{li}|}{|a_{li}| - \sum_{k \neq l, i} |a_{lk}|}, l \neq i; r_i = \max_{l \neq i} \{ r_{li} \}, i \in N; m_{ji} = \frac{|a_{ji}| + \sum_{k \neq i} |a_{jk}| r_i}{|a_{jj}|}, j \neq i; m_i = \max_{j \neq i} \{ m_{ij} \}, i \in N. \)
Recently, in [9], Li has proved the following bound:

$$
\tau(B \circ A^{-1}) \geq \min_i \left\{ \frac{b_{ii} - n_i \sum_{j \neq i} |b_{ji}|}{a_{ii}} \right\},
$$

where

$$
r_{li} = \frac{|a_{li}|}{|a_{ll}|} - \sum_{k \neq l,i} |a_{lk}|, \quad l \neq i; \quad r_i = \max_{l \neq i} \{r_{li}\}, \quad i \in N; \quad n_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|r_k}{|a_{jj}|}, \quad j \neq i; \quad n_i = \max_{j \neq i} \{n_{ij}\}, \quad i \in N.
$$

When $B = A$, the bound gives a lower bound of $\tau(A \circ A^{-1})$:

$$
(1.7) \quad \tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - n_i R_i}{1 + \sum_{j \neq i} n_{ji}} \right\}.
$$

In this paper, we present some new lower bounds on $\tau(A \circ A^{-1})$. The bounds improve the results in [7, 8].

2. Preliminaries and notation. In this section, we give some lemmas which give bounds on the entries of the inverse matrix $A^{-1}$ of a nonsingular matrix $A$. The following is the list of notations that we use throughout: For $i, j, k, l \in N$,

$$
R_i = \sum_{k \neq i} |a_{ik}|, \quad C_i = \sum_{k \neq i} |a_{ki}|, \quad d_i = \frac{R_i}{|a_{ii}|}, \quad \hat{c}_i = \frac{C_i}{|a_{ii}|};
$$

$$
r_{li} = \frac{|a_{li}|}{|a_{ll}|} - \sum_{k \neq l,i} |a_{lk}|, \quad l \neq i; \quad r_i = \max_{l \neq i} \{r_{li}\}, \quad i \in N;
$$

$$
c_{il} = \frac{|a_{il}|}{|a_{ll}|} - \sum_{k \neq i,l} |a_{lk}|, \quad l \neq i; \quad c_i = \max_{l \neq i} \{c_{il}\}, \quad i \in N;
$$

$$
m_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|r_i}{|a_{jj}|}, \quad j \neq i; \quad m_i = \max_{j \neq i} \{m_{ij}\}, \quad i \in N;
$$

$$
n_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|r_k}{|a_{jj}|}, \quad j \neq i; \quad n_i = \max_{j \neq i} \{n_{ij}\}, \quad i \in N;
$$

$$
s_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|d_k}{|a_{jj}|}, \quad j \neq i, \quad j \in N; \quad s_i = \max_{j \neq i} \{s_{ij}\}, \quad i \in N;
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\[ T_{ji} = \min\{m_{ji}, n_{ji}\}, \ j \neq i; \ T_i = \max\{T_{ij}\}, \ i \in N. \]

**Lemma 2.1.** [8, Lemma 2.2] Let \( A \) be an \( n \times n \) real matrix.

(a) If \( A = [a_{ij}] \) is a strictly row diagonally dominant \( M \)-matrix, then \( A^{-1} = [b_{ij}] \) satisfies

\[ b_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{kj}|r_k}{a_{jj}} b_{ii}, \ i, j \in N, \ i \neq j. \]

(b) If \( A = [a_{ij}] \) is a strictly column diagonally dominant \( M \)-matrix, then \( A^{-1} = [b_{ij}] \) satisfies

\[ b_{ij} \leq \frac{|a_{ij}| + \sum_{k \neq j,i} |a_{ik}|c_k}{a_{jj}} b_{ii}, \ i, j \in N, \ i \neq j. \]

**Lemma 2.2.** [9, Lemma 2.2] Let \( A \) be an \( n \times n \) real matrix.

(a) If \( A = [a_{ij}] \) is a strictly row diagonally dominant \( M \)-matrix, then \( A^{-1} = [b_{ij}] \) satisfies

\[ b_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|r_k}{a_{jj}} b_{ii}, \ i, j \in N, \ i \neq j. \]

(b) If \( A = [a_{ij}] \) is a strictly column diagonally dominant \( M \)-matrix, then \( A^{-1} = [b_{ij}] \) satisfies

\[ b_{ij} \leq \frac{|a_{ij}| + \sum_{k \neq j,i} |a_{ik}|c_k}{a_{jj}} b_{ii}, \ i, j \in N, \ i \neq j. \]

**Lemma 2.3.** If \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is a strictly row diagonally dominant \( M \)-matrix, then \( A^{-1} = [b_{ij}] \) satisfies

\[ b_{ji} \leq T_{ji}b_{ii}, \ i, j \in N, \ i \neq j. \]

**Proof.** By Lemma 2.1 (a) and Lemma 2.2 (a), we have

\[ b_{ji} \leq n_{ji}b_{ii}, \ b_{ji} \leq m_{ji}b_{ii}, \ i, j \in N, \ i \neq j. \]

From \( T_{ji} = \min\{m_{ji}, n_{ji}\} \), we get

\[ b_{ji} \leq T_{ji}b_{ii}, \ i, j \in N, \ i \neq j. \]
Lemma 2.4. [8, Theorem 3.1] If $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is an M-matrix and $A^{-1} = [b_{ij}]$ is a doubly stochastic matrix, then

$$b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} m_{ji}}, \; i \in N.$$ 

Lemma 2.5. [7, Theorem 2.1] Let $A$ be an $n \times n$ real matrix.

(a) If $A = [a_{ij}]$ is a strictly row diagonally dominant matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$|b_{ji}| \leq |a_{ji}| + \sum_{k \neq j, i} k_{jk} \frac{|a_{jk}|}{|a_{jj}|} |b_{ii}|, \; i, j \in N, \; i \neq j.$$ 

(b) If $A = [a_{ij}]$ is a strictly column diagonally dominant matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$|b_{ij}| \leq |a_{ij}| + \sum_{k \neq j, i} k_{kj} \frac{|a_{kj}|}{|a_{jj}|} |b_{ii}|, \; i, j \in N, \; i \neq j.$$ 

Lemma 2.6. [7, Theorem 2.3] If $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = [b_{ij}]$ satisfies

$$b_{ii} \geq \frac{1}{a_{ii}}, \; i \in N.$$ 

Lemma 2.7. [14, Lemma 2.3] If $A^{-1}$ is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = [1, 1, \ldots, 1]^T$. 

Lemma 2.8. [11, P. 719] Let $A = [a_{ij}]$ be an $n \times n$ complex matrix and $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of $A$ lie in the region

$$\bigcup_i \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq x_i \sum_{j \neq i} \frac{1}{x_j} |a_{ji}|, \; i \in N \right\}.$$ 

Lemma 2.9. [14, Lemma 2.1] If $P$ is an irreducible M-matrix, and $Pz \succeq kz$ for a nonnegative nonzero vector $z$, then $\tau(P) \geq k$.

The following result can be found in [2].

Lemma 2.10. If $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is an M-matrix, then there exists a diagonal matrix $D$ with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix.
Lemma 2.11. [6, Lemma 5.1.2] Let $A, B \in \mathbb{R}^{n \times n}$, and suppose that $D \in \mathbb{R}^{n \times n}$ and $E \in \mathbb{R}^{n \times n}$ are diagonal matrices. Then

$$D(A \odot B)E = (DAE) \odot B = (DA) \odot (BE) = (AE) \odot (DB) = A \odot (DBE).$$

3. Main results. In this section, we present some new lower bounds for $\tau(A \odot A^{-1})$.

Theorem 3.1. If $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is an M-matrix, and $A^{-1} = [b_{ij}]$ is a doubly stochastic matrix, then

$$b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} n_{ji}}, \quad i \in N; \quad \text{and} \quad b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} T_{ji}}, \quad i \in N.$$

Proof. We first prove $b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} n_{ji}}, \quad i \in N$. Since $A^{-1}$ is doubly stochastic, by Lemma 2.7, we know that $Ae = e$, so $A$ is a strictly diagonally dominant matrix by row. By Lemma 2.2 (a), for $i \in N$,

$$1 = b_{ii} + \sum_{j \neq i} |b_{ij}|$$

$$\leq b_{ii} + \sum_{j \neq i} \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}r_k|}{|a_{jj}|} b_{ii}$$

$$= \left(1 + \sum_{j \neq i} \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}r_k|}{|a_{jj}|}\right) b_{ii}$$

$$= (1 + \sum_{j \neq i} n_{ji}) b_{ii},$$

i.e.,

$$b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} n_{ji}}, \quad i \in N.$$

Similarly, we can prove $b_{ii} \geq \frac{1}{1 + \sum_{j \neq i} T_{ji}}, \quad i \in N$. \Box

Theorem 3.2. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an irreducible M-matrix, and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then

$$\tau(A \odot A^{-1}) \geq \min \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum_{j \neq i} T_{ji}} \right\}.$$
Proof. Since $A$ is irreducible, from Lemma 2.7, we know that $A\mathbf{e} = \mathbf{e}$, so $A$ is a strictly diagonally dominant matrix by row. Therefore, $0 < T_i < 1$, $i = 1, 2, \ldots, n$.

Let $\tau(A \circ A^{-1}) = \lambda$. By Lemma 2.8, there exists $i_0 \in \mathbb{N}$, such that

$$|\lambda - a_{i_0i_0} b_{i_0i_0}| \leq T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j,i_0} b_{j,i_0}|.$$ 

Hence,

$$|\lambda| \geq a_{i_0i_0} b_{i_0i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j,i_0} b_{j,i_0}|$$

$$\geq a_{i_0i_0} b_{i_0i_0} - T_{i_0} \sum_{j \neq i_0} \frac{1}{T_j} |a_{j,i_0}| T_{j,i_0} b_{i_0i_0} \quad \text{(by Lemma 2.3)}$$

$$\geq a_{i_0i_0} b_{i_0i_0} - T_{i_0} R_{i_0} b_{i_0i_0}$$

$$\geq \frac{a_{i_0i_0} - T_{i_0} R_{i_0}}{1 + \sum_{j \neq i_0} T_{j,i_0}} \quad \text{(by Theorem 3.1)}$$

$$\geq \min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum_{j \neq i} T_{ji}} \right\}.$$ 

Remark 3.3. If $A$ is reducible, without loss of generality, we can assume that $A$ is a block upper triangular matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & \cdots & \cdots & A_{kk} \end{bmatrix}$$

with irreducible diagonal blocks $A_{ii}$, $i \in K = \{1, 2, \ldots, k\}$. Then $\tau(A \circ A^{-1}) = \min_{i \in K} \tau(A_{ii} \circ A_{ii}^{-1})$. Thus, the problem of the reducible matrix $A$ is reduced to those of irreducible diagonal blocks $A_{ii}$, $i \in K$. The result of Theorem 3.2 also holds.

Theorem 3.4. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an irreducible strictly row diagonally dominant $M$-matrix. Then

$$\tau(A \circ A^{-1}) \geq \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| T_{ji} \right\}.$$
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Proof. Since $A$ is irreducible, then $A^{-1} = [b_{ij}] > 0$, and $A \circ A^{-1}$ is again irreducible. Note that

$$\tau(A \circ A^{-1}) = \tau((A \circ A^{-1})^T) = \tau(A^T \circ (A^T)^{-1}).$$

Let

$$(A^T \circ (A^T)^{-1})c = [g_1, g_2, \ldots, g_n]^T,$$ where $c = [1, 1, \ldots, 1]^T$. Without loss of generality, we may assume that $g_1 = \min_i \{g_i\}$, by Lemma 2.3 , we have

$$g_1 = \sum_{j=1}^{n} |a_{j1}b_{j1}|$$

$$= a_{11}b_{11} - \sum_{j \neq 1} |a_{j1}b_{j1}|$$

$$\geq a_{11}b_{11} - \sum_{j \neq 1} |a_{j1}|T_{j1}b_{11} \quad \text{(by Lemma 2.3)}$$

$$= (a_{11} - \sum_{j \neq 1} |a_{j1}|T_{j1})b_{11}$$

$$= \frac{a_{11} - \sum_{j \neq 1} |a_{j1}|T_{j1}}{a_{11}} \quad \text{(by Lemma 2.6)}$$

$$\geq 1 - \frac{1}{a_{11}} \sum_{j \neq 1} |a_{j1}|T_{j1}.$$ Therefore, $(A^T \circ (A^T)^{-1})c \geq (1 - \frac{1}{a_{11}} \sum_{j \neq 1} |a_{j1}|T_{j1})c$. From Lemma 2.9, we have

$$\tau(A \circ A^{-1}) = \tau(A^T \circ (A^T)^{-1}) \geq \min_i \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}|T_{ji} \right\}. \quad \square$$

Remark 3.5. If $A$ is an $M$-matrix, then by Lemma 2.10, we know that there exists a diagonal matrix $D$ with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant $M$-matrix. So the result of Theorem 3.4 also holds for a general $M$-matrix.

Theorem 3.6. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an $M$-matrix, and let $A^{-1} = [b_{ij}]$ be a doubly stochastic matrix. Then

$$\min_i \left\{ \frac{a_{ii} - T_iR_i}{1 + \sum_{j \neq i} T_{ji}} \right\} \geq \min_i \left\{ \frac{a_{ii} - m_iR_i}{1 + \sum_{j \neq i} m_{ji}} \right\}.$$
Proof. Since $T_{ji} = \min\{m_{ji}, n_{ji}\}$,

$$T_{ji} \leq m_{ji}, \ j \neq i, \ j \in N; \ T_i \leq m_i, \ i \in N.$$ 

Hence,

$$a_{ii} - T_i R_i \geq a_{ii} - m_i R_i, \quad \frac{1}{1 + \sum_{j \neq i} T_{ji}} \geq \frac{1}{1 + \sum_{j \neq i} m_{ji}}.$$ 

Therefore,

$$\min_{i} \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum_{j \neq i} T_{ji}} \right\} \geq \min_{i} \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j \neq i} m_{ji}} \right\}. \quad \square$$

Remark 3.7. Theorem 3.6 shows that the result of Theorem 3.2 is better than that of Theorem 3.2 in [10].

Theorem 3.8. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an M-matrix. Then

$$\min_{i} \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| T_{ji} \right\} \geq \min_{i} \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji} \right\}.$$ 

Proof. By the proof of Theorem 3.6, we have

$$T_{ji} \leq m_{ji}, \ j \neq i.$$ 

So

$$1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| T_{ji} \geq 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji}.$$ 

Thus,

$$\min_{i} \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| T_{ji} \right\} \geq \min_{i} \left\{ 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji} \right\}. \quad \square$$

Remark 3.9. Theorem 3.8 shows that the result of Theorem 3.4 is better than that of Theorem 3.4 in [10].
Theorem 3.10. Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be an irreducible M-matrix, and let \( A^{-1} = [b_{ij}] \) be a doubly stochastic matrix. Then

\[
\tau(A \circ A^{-1}) \geq \min_{i} \left\{ \frac{a_{ii} - s_{i} \sum_{j \neq i} \frac{|a_{ji}|s_{j}}{m_{ji}}}{1 + \sum_{j \neq i} m_{ji}} \right\}.
\]

Proof. Since \( A^{-1} \) is doubly stochastic, by Lemma 2.7, we have \( Ae = e \), \( A^T e = e \), so \( A \) is a strictly diagonally dominant M-matrix, and

\[
a_{ii} = \sum_{k \neq i} |a_{ik}| + 1 = \sum_{k \neq i} |a_{ki}| + 1, \quad a_{ii} > 1
\]

and

\[
d_{i} = \frac{\sum_{k \neq i} |a_{ik}|}{|a_{ii}|} < 1, \quad i \in N.
\]

For convenience, we denote

\[
\tilde{R}_{j} = \sum_{k \neq j} |a_{jk}|d_k, \quad j \in N.
\]

Then, for any \( j \in N \) with \( j \neq i \), we have

\[
\tilde{R}_{j} \leq |a_{ji}| + \sum_{k \neq j, i} |a_{jk}|d_k \leq R_{j} = \sum_{k \neq j} |a_{jk}| \leq a_{jj}.
\]

Therefore, there exists a real number \( \alpha_{ji} \) (\( 0 \leq \alpha_{ji} \leq 1 \)), such that

\[
|a_{ji}| + \sum_{k \neq j, i} |a_{jk}|d_k = \alpha_{ji}R_{j} + (1 - \alpha_{ji})\tilde{R}_{j}.
\]

Let \( \alpha_{j} = \max_{i \neq j} \{\alpha_{ji}\} \). Then \( 0 < \alpha_{j} \leq 1 \), (if \( \alpha_{j} = 0 \), then \( A \) is reducible, which is a contradiction). So, from the definition of \( s_{ij} \), we have

\[
s_{j} = \max_{i \neq j} \{s_{ji}\} = \frac{\alpha_{j}R_{j} + (1 - \alpha_{j})\tilde{R}_{j}}{a_{jj}}, \quad j \in N.
\]

Since \( 0 < \alpha_{j} \leq 1 \), we get \( 0 < s_{j} \leq 1 \).

Let \( \tau(A \circ A^{-1}) = \lambda \). By Lemma 2.8, there exists \( i_0 \in N \), such that

\[
|\lambda - a_{i_0i_0}b_{i_0i_0}| \leq s_{i_0} \sum_{j \neq i_0} \frac{1}{s_{j}}|a_{ji_0}b_{ji_0}|.
\]
Hence,

\[ |\lambda| \geq a_{i_0i_0}b_{i_0i_0} - s_{i_0} \sum_{j \neq i_0} \frac{1}{s_j} |a_{ji_0}b_{j_0i_0}| \]
\[ \geq a_{i_0i_0}b_{i_0i_0} - s_{i_0} \sum_{j \neq i_0} \frac{1}{s_j} |a_{ji_0}| + \sum_{k \neq j, i_0} \frac{|a_{jk}|r_k}{a_{jj}} b_{i_0i_0} \] (by Lemma 2.2 (a))
\[ = (a_{i_0i_0} - s_{i_0} \sum_{j \neq i_0} \frac{1}{s_j} |a_{ji_0}|n_{j_0i_0})b_{i_0i_0} \]
\[ \geq 1 + \sum_{j \neq i_0} m_{ji_0} \] (by Lemma 2.4)
\[ \geq a_{ii} - s_i \sum_{j \neq i} |a_{ji}|n_{ji} \]
\[ \geq \min_i \left\{ \frac{a_{ii} - s_i \sum_{j \neq i} |a_{ji}|n_{ji}}{1 + \sum_{j \neq i} m_{ji}} \right\}. \]

**Remark 3.11.** When \( A \) is reducible, without loss of generality, we can assume that \( A \) is a block upper triangular matrix of the form

\[
A = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1k} \\
A_{21} & A_{22} & \cdots & A_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & \cdots & A_{kk}
\end{bmatrix}
\]

with irreducible diagonal blocks \( A_{ii}, \ i \in K \). Then \( \tau(A \circ A^{-1}) = \min_{i \in K} \tau(A_{ii} \circ A_{ii}^{-1}) \).

Thus, the problem of the reducible matrix \( A \) is reduced to those of irreducible diagonal blocks \( A_{ii}, \ i \in K \). The result of Theorem 3.10 also holds.

By using Lemma 2.6, Lemma 2.10 and Theorem 3.10, we can get the following corollary.

**Corollary 3.12.** Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be an \( M \)-matrix. Then

\[
\tau(A \circ A^{-1}) \geq \min_i \left\{ \frac{a_{ii} - s_i \sum_{j \neq i} |a_{ji}|n_{ji}}{s_j} \right\}.
\]
4. Examples.

**Example 4.1.** (See also Example 3.1 in [9]) Let

\[
A = \begin{bmatrix}
4 & -1 & -1 & -1 \\
-2 & 5 & -1 & -1 \\
0 & -2 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{bmatrix}.
\]

By \( Ae = e \) and \( A^T e = e \), we know that \( A^{-1} \) is a doubly stochastic matrix. By calculating with Matlab 7.0, we have

\[
A^{-1} = \begin{bmatrix}
0.4 & 0.2 & 0.2 & 0.2 \\
0.2333 & 0.3667 & 0.2 & 0.2 \\
0.1667 & 0.2333 & 0.4 & 0.2 \\
0.2 & 0.2 & 0.2 & 4
\end{bmatrix}.
\]

If we apply the conjecture of Fiedler and Markham, we have

\[
\tau(A \circ A^{-1}) \geq \frac{2}{n} = 0.5;
\]

if we apply Theorem 3.1 of [9], we have

\[
\tau(A \circ A^{-1}) \geq 0.6624;
\]

if we apply Theorem 3.2 of [10], we have

\[
\tau(A \circ A^{-1}) \geq 0.7999.
\]

But, if we apply Theorem 3.2, we have

\[
\tau(A \circ A^{-1}) \geq 0.85;
\]

if we apply Theorem 3.10, we have

\[
\tau(A \circ A^{-1}) \geq 0.8602.
\]

In fact, \( \tau(A \circ A^{-1}) = 0.9755. \)

**Example 4.2.** Let

\[
A = \begin{bmatrix}
5 & -1 & -2 & -1 \\
-1 & 12 & -7 & -2 \\
-1 & -1 & 15 & -4 \\
-2 & -3 & 0 & 10
\end{bmatrix}.
\]
By calculating with Matlab 7.0, we have
\[ A^{-1} = \begin{bmatrix}
0.2372 & 0.0364 & 0.0486 & 0.0505 \\
0.0512 & 0.1043 & 0.0555 & 0.0482 \\
0.0360 & 0.0197 & 0.0806 & 0.0398 \\
0.0628 & 0.0386 & 0.0264 & 0.1245
\end{bmatrix}. \]

Therefore, \( A \) is a nonsingular \( M \)-matrix.

If we apply the conjecture of Fiedler and Markham, we have
\[ \tau(A \circ A^{-1}) \geq \frac{2}{n} = 0.5; \]
if we apply Theorem 3.5 of [9], we have
\[ \tau(A \circ A^{-1}) \geq 0.5689; \]
if we apply Theorem 3.4 of [10], we have
\[ \tau(A \circ A^{-1}) \geq 0.5422. \]

But, if we apply Theorem 3.4, we have
\[ \tau(A \circ A^{-1}) \geq 0.5959; \]
if we apply Corollary 3.12, we have
\[ \tau(A \circ A^{-1}) \geq 0.6021. \]

In fact, \( \tau(A \circ A^{-1}) = 0.9548. \)

**Remark 4.3.** The numerical examples show that in these cases the bounds of Theorem 3.2 and Theorem 3.10 are sharper than Theorem 3.1 in [9] and Theorem 3.2 in [10]; the bounds in Theorem 3.4 and Corollary 3.12 are sharper than Theorem 3.5 in [9] and Theorem 3.4 in [10].

**REFERENCES**


Some New Lower Bounds for the Minimum Eigenvalue of the Hadamard Product


