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INVARINACE PROPERTIES OF AN OPERATOR PRODUCT INVOLVING GENERALIZED INVERSES∗

ZHIPPING XIONG† AND YINGYING QIN‡

Abstract. Given bounded linear operators $T_1$, $T_2$ and $T_3$, this paper investigates certain invariance properties of the operator product $T_1XT_3$ with respect to the choice of bounded linear operator $X$, where $X$ is a generalized inverse of $T_2$. Different types of generalized inverses are taken into account.

Key words. Bounded linear operators, Generalized inverse, Moore-Penrose inverse, Operator product, Invariance property.

AMS subject classifications. 15A09, 15A24, 47A05.

1. Introduction. Throughout this paper, $H$, $K$, and $L$ denote arbitrary Hilbert spaces. We use $L(H, K)$ to denote the set of all bounded linear operators from $H$ to $K$. Also, $L(H)=L(H, H)$. $I$ denotes the identity operator on Hilbert spaces and $O$ is the zero operator on Hilbert spaces. For $T \in L(H, K)$, the symbols $T^*$, $R(T)$, and $N(T)$ will stand for the adjoint operator, the range, and the kernel of $T$, respectively.

Let $T \in L(H, K)$. If there exists an operator $X \in L(K, H)$ satisfying the following four operator equations:

(1) $TXT = T$,  (2) $XTX = X$,  (3) $(TX)^* = TX$,  (4) $(XT)^* = XT$,

then $X$ is called a Moore-Penrose inverse of $T$ and denoted by $T^\dagger$. As we know, $T$ has a Moore-Penrose inverse if and only if $R(T)$ is closed and the Moore-Penrose inverse of $T$ is unique (see, for example, [5, 7, 14, 17, 18, 19, 20, 21, 22, 23]). For a subset $\eta \subseteq \{1, 2, 3, 4\}$, the set of operators satisfying the equations contained in $\eta$ is denoted by $T\eta$. An operator from $T\eta$ is called an $\eta$-inverse of $T$. For example, an operator $X$ of the set $T\{1\}$ is called a $\{1\}$-inverse of $T$ and denoted by $T^{(1)}$ or $T^{-1}$. One usually denotes any $\{1, 3\}$-inverse of $T$ as $T^{(1, 3)}$ and any $\{1, 4\}$-inverse of $T$ is denoted by $T^{(1, 4)}$. The unique $\{1, 2, 3, 4\}$-inverse of $T$ is the Moore-Penrose inverse of $T$. We
refer the reader to [5, 6, 9, 10, 11, 15, 16, 23, 24] for basic results on the generalized inverses of operators.

Invariance properties of operator product involving generalized inverses are fundamental in the theory of operators. They have attracted considerable attention and many interesting results have been obtained (see, for example, [1, 2, 3, 4, 12, 13]). In this paper, given bounded linear operators $T_1 \in L(L, H)$, $T_2 \in L(L, K)$ and $T_3 \in L(H, K)$, we investigate properties of the operator product $T_1 T_2^{-} T_3$ for various types of generalized inverses $T_2^{-}$ of $T_2$, where our interest is focused on invariance properties with respect to the choice of $T_2^{-}$ concerning the value and range of $T_1 T_2^{-} T_3$.

We first mention the following three results, which will be used in this paper.

**Lemma 1.1.** [9]. Let $T \in L(H, K)$ have a closed range. Then

$$T \{1\} = \{T^\dagger + Y - T^\dagger TYT^\dagger : Y \in L(K, H)\}.$$  

**Lemma 1.2.** [8, 9]. Let $T \in L(H, K)$ have a closed range and $X \in L(K, H)$. Then the following statements are equivalent:

1. $TXT = T$ and $(TX)^* = TX$;
2. there exists some $Y \in L(K, H)$ such that $X = T^\dagger + (I - T^\dagger T)Y$.

**Lemma 1.3.** [8, 9]. Let $T \in L(H, K)$ have a closed range and $X \in L(K, H)$. Then the following statements are equivalent:

1. $TXT = T$ and $(XT)^* = XT$;
2. there exists some $Y \in L(K, H)$ such that $X = T^\dagger + Y(I - TT^\dagger)$.

2. **Invariance properties of operator product $T_1 T_2^{(1)} T_3$.** Let $T_1 \in L(L, H)$, $T_2 \in L(L, K)$ and $T_3 \in L(H, K)$ be such that $T_1$, $T_2$ and $T_3$ have closed ranges. In this section, we will study several invariance properties of the operator product $T_1 T_2^{(1)} T_3$ with respect to the choice of $T_2^{(1)} \in T_2\{1\}$. The main result is the following theorem.

**Theorem 2.1.** Let $T_1 \in L(L, H)$, $T_2 \in L(L, K)$ and $T_3 \in L(H, K)$ be such that $T_1$, $T_2$, $T_3$ have closed ranges. Suppose that $T_1$, $T_2$, and $T_3$ are not zero operators on Hilbert spaces. Then the following statements are equivalent:

1. The operator product $T_1 T_2^{(1)} T_3$ does not depend on the choice of $T_2^{(1)} \in T_2\{1\}$;
(2) \( R(T_1^*) \subseteq R(T_2^*) \) and \( R(T_3) \subseteq R(T_2) \).

Proof. According to the technique of block operator matrices in [7], we know that the operator \( T_2 \) has the following matrix form with respect to the orthogonal sum of subspaces:

\[
T_2 = \begin{pmatrix}
T_{21}^{11} & O \\
O & O
\end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix},
\]

where \( T_{21}^{11} \) is invertible in \( L(R(T_2^*), R(T_2)) \), and

\[
T_2^* = \begin{pmatrix}
(T_{21}^{11})^{-1} & O \\
O & O
\end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}.
\]

Also we have that the operator \( T_1 \) has the following form:

\[
T_1 = \begin{pmatrix}
T_{11}^{11} & T_{12}^{11} \\
O & O
\end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_1^*) \\ N(T_1^*) \end{pmatrix},
\]

and

\[
T_1^* = \begin{pmatrix}
(T_{11}^{11})^* & O \\
(T_{12}^{11})^* & O
\end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_1^*) \\ N(T_2) \end{pmatrix},
\]

and

\[
T_1 T_1^* = \begin{pmatrix}
D & O \\
O & O
\end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix},
\]

where \( D = T_{11}^{11} (T_{11}^{11})^* + T_{12}^{11} (T_{12}^{11})^* \) is positive and invertible in \( L(R(T_1)) \). In particular

\[
T_1^* = T_1^* (T_1 T_1^*)^\dagger = \begin{pmatrix}
(T_{11}^{11})^* D^{-1} & O \\
(T_{12}^{11})^* D^{-1} & O
\end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}.
\]

Furthermore, we obtain that the operator \( T_3 \) has the following matrix form with respect to the orthogonal sum of subspaces:

\[
T_3 = \begin{pmatrix}
T_{31}^{11} & O \\
T_{31}^{21} & O
\end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix},
\]

and

\[
T_3^* = \begin{pmatrix}
(T_{31}^{11})^* & (T_{32}^{21})^* \\
O & O
\end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix},
\]

and

\[
T_3^* T_3 = \begin{pmatrix}
S & O \\
O & O
\end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix},
\]

where \( S = T_{31}^{11} (T_{31}^{11})^* + T_{32}^{21} (T_{32}^{21})^* \) is invertible in \( L(R(T_3^*)) \).
Next, we will prove the facts that (1) and (2) in Theorem 2.1 are equivalent.

(2)⇒(1): The inclusion \( R(T_1) \subseteq R(T'_2) \) is equivalent to \( T_1T'_2T_2 = T_1 \). Then from (2.1)-(2.6), we have

\[
T_1T'_2T_2 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11} & O \\ O & O \end{pmatrix}.
\]

Hence, according to (2.3) and (2.7), the equality \( T_1T'_2T_2 = T_1 \) is equivalent to \( T_1^{12} = O \), that is

\[
R(T_1) \subseteq R(T'_2) \iff T_1^{12} = O.
\]

In the same manner, we can prove that the inclusion \( R(T_3) \subseteq R(T_2) \) is equivalent to

\[
R(T_3) \subseteq R(T_2) \iff T_3^{21} = O.
\]

On the other hand, from Lemma 1.1 it follows that arbitrary \( T_2^{(1)} \in T_2\{1\} \) has the form

\[
T_2^{(1)} = \begin{pmatrix} (T_2^{11})^{-1} & U \\ V & W \end{pmatrix},
\]

where \( U, V \) and \( W \) are bounded linear operators on appropriate spaces. Hence, from (2.2), (2.3), (2.5), (2.8), (2.9), (2.10), we have that

\[
T_1T_2^{(1)}T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix} = T_1T_2^{(1)}T_3.
\]

Combining (2.10) with (2.11), we have the result (2)⇒(1).

(1)⇒(2): Since the Moore-Penrose inverse of a bounded linear operator is unique and belongs to the set of \( \{1\}\)-inverse, it is clear that \( T_1T_2^{(1)}T_3 \) doesn’t depend on the choice of \( T_2^{(1)} \in T_2\{1\} \) if and only if the equality \( T_1T_2^{(1)}T_3 = T_1T_2^{(1)}T_3 \) holds for every \( T_2^{(1)} \in T_2\{1\} \). By Lemma 1.1, it follows that both

\[
M = \begin{pmatrix} (T_2^{11})^{-1} & (T_2^{11})^* \\ O & (T_2^{11})^* \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} (T_2^{11})^{-1} & O \\ (T_2^{11})^* & (T_2^{21})^* \end{pmatrix}
\]

are \( \{1\}\)-inverses of \( T_2 \). Since

\[
T_1MT_3 = T_1NT_3 = T_1T_2^{(1)}T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix},
\]
from (2.2), (2.3), (2.5), (2.12), (2.13), we get

\[ T_1^{11}(T_1^{11})^*T_3^{21} + T_1^{12}(T_1^{12})^*T_3^{21} = O \]  

and

\[ T_1^{12}(T_3^{11})^*T_3^{11} + T_1^{12}(T_3^{21})^*(T_3^{21}) = O. \]

Combining (2.14), (2.15) with the equalities (2.3) and (2.5), we have

\[ DT_3^{21} = O \quad \text{and} \quad T_1^{12}S = O. \]

Since \( D \) and \( S \) are invertible, from (2.16) we obtain

\[ T_3^{21} = O \quad \text{and} \quad T_1^{12} = O, \]

which are respectively equivalent to \( R(T_3) \subseteq R(T_2) \) and \( R(T_1^*) \subseteq R(T_2^*). \]  

**Corollary 2.2.** Let \( T_1 \in L(\mathbb{L}, \mathbb{H}), T_2 \in L(\mathbb{L}, \mathbb{K}), \) and \( T_3 \in L(\mathbb{H}, \mathbb{K}) \) be such that \( T_1, T_2, T_3 \) have closed ranges. Suppose that \( T_1, T_2, \) and \( T_3 \) are not zero operators on Hilbert spaces. Then the identity \( T_1T_2T_3^* = O \) holds for every \( T_2(1) \in T_2(1) \) if and only if \( R(T_1^*) \subseteq R(T_2^*), R(T_2) \subseteq R(T_3), \) and \( R(T_3) \subseteq N(T_1T_2^*). \) 

**Corollary 2.3.** Let \( T_1 \in L(\mathbb{L}, \mathbb{H}), T_2 \in L(\mathbb{L}, \mathbb{K}), \) and \( T_3 \in L(\mathbb{H}, \mathbb{K}) \) be such that \( T_1, T_2, T_3 \) have closed ranges. If \( R(T_1^*) \subseteq R(T_2^*) \) and \( R(T_3) \subseteq R(T_2), \) then \( R(T_1T_2^*T_3^*) \) is the same for every \( T_2(1) \in T_2(1). \) 

**3. Invariance properties of products** \( T_1T_2^{(1,3)}T_3 \) and \( T_1T_2^{(1,4)}T_3. \) Let \( T_1 \in L(\mathbb{L}, \mathbb{H}), T_2 \in L(\mathbb{L}, \mathbb{K}), \) and \( T_3 \in L(\mathbb{H}, \mathbb{K}) \). In this section, we will investigate the invariance properties of the operator products \( T_1T_2^{(1,3)}T_3 \) and \( T_1T_2^{(1,4)}T_3 \) with respect to any \( T_2^{(1,3)} \in T_2(1,3) \) and \( T_2^{(1,4)} \in T_2(1,4). \)

**Theorem 3.1.** Let \( T_1 \in L(\mathbb{L}, \mathbb{H}), T_2 \in L(\mathbb{L}, \mathbb{K}), \) and \( T_3 \in L(\mathbb{H}, \mathbb{K}) \) be such that \( T_1, T_2, T_3 \) have closed ranges. Suppose that \( T_1, T_2, \) and \( T_3 \) are not zero operators on Hilbert spaces. Then the following statements are equivalent:

1. The equality \( T_1T_2^{(1,3)}T_3 = T_1T_2^{*}T_3 \) holds for every \( T_2^{(1,3)} \in T_2(1,3); \)
2. \( R(T_1^*) \subseteq R(T_2^*). \)

**Proof.** By Lemma 1.2, we have that arbitrary \( T_2^{(1,3)} \in T_2(1,3) \) has the form

\[ T_2^{(1,3)} = \begin{pmatrix} (T_2^{11})^{-1} & O \\ W_{21} & W_{22} \end{pmatrix}, \]

where \( W_{21} \) and \( W_{22} \) are bounded linear operators on appropriate subspaces. Then from the equalities (2.2), (2.3), (2.5) and (3.1), we know that for any \( T_2^{(1,3)} \in T_2(1,3) \)

\[ T_1T_2^{(1,3)}T_3 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} T_3^{11} \\ T_3^{21} \end{pmatrix} \]
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\[
\begin{pmatrix}
    T_{11}^{-1}(T_{11}^*)^{-1}T_{3}^{-1} + T_{12}W_{21}T_{3}^{-1} + T_{12}W_{22}T_{3}^{-1}
    \end{pmatrix}
\]

and

\[
(3.2) \quad T_1T_2^*T_3 = \begin{pmatrix}
    T_{11}^{-1}T_{3}^{-1} & O \\
    O & O
\end{pmatrix}.
\]

We now prove that (1) and (2) in Theorem 3.1 are equivalent.

(2)⇒(1): Since

\[
R(T_1^*) \subseteq R(T_2^*) \iff T_1 = O,
\]

we have that for arbitrary \( T_2(1,3) \in T_2\{1,3\} \),

\[
T_1T_2^{(1,3)}T_3 = \begin{pmatrix}
    T_{11}^{-1}(T_{12}^{-1})^{-1}T_{3}^{-1} + T_{12}W_{21}T_{3}^{-1} + T_{12}W_{22}T_{3}^{-1} & O \\
    O & O
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    T_{11}^{-1}T_{3}^{-1} & O \\
    O & O
\end{pmatrix}
\]

\[
= T_1T_2^*T_3.
\]

(1)⇒(2): From Lemma 1.2, it follows that

\[
M_1 = \begin{pmatrix}
    (T_{21}^*)^{-1} & O \\
    (T_{31}^*) & (T_{32}^*)
\end{pmatrix}
\]

is a \( \{1,3\}\)-inverses of \( T_2 \). Then

\[
(3.3) \quad T_1M_1T_3 = \begin{pmatrix}
    T_{11} & T_{12} \\
    O & O
\end{pmatrix}
\begin{pmatrix}
    (T_{21}^*)^{-1} & O \\
    (T_{31}^*) & (T_{32}^*)
\end{pmatrix}
\begin{pmatrix}
    T_{31}^{-1} & O \\
    T_{32}^{-1} & O
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    T_{11}^{-1}(T_{12}^{-1})^{-1}T_{3}^{-1} + T_{12}W_{21}T_{3}^{-1} + T_{12}W_{22}T_{3}^{-1} & O \\
    O & O
\end{pmatrix}
\]

and

\[
(3.4) \quad T_1M_1T_3 = T_1T_2^*T_3 = \begin{pmatrix}
    T_{11}^{-1}T_{3}^{-1} & O \\
    O & O
\end{pmatrix}.
\]

Hence, from (3.3) and (3.4), we have

\[
(3.5) \quad T_{12}^{(1,3)}T_3 + T_{12}^{(2,1)}T_3 = O.
\]

Combining (3.5) with the equality (2.5), we have \( T_{12}^{(1,2)} = O \). Since \( S \) is invertible, \( T_1 = O \), that is \( R(T_1^*) \subseteq R(T_2^*) \).
Corollary 3.2. Let \( T_1 \in L(\mathbb{L}, \mathbb{K}) \), \( T_2 \in L(\mathbb{L}, \mathbb{K}) \), and \( T_3 \in L(\mathbb{K}, \mathbb{K}) \) be such that \( T_1 \), \( T_2 \), \( T_3 \) have closed ranges. Suppose that \( T_1 \), \( T_2 \), and \( T_3 \) are not zero operators on Hilbert spaces. Then the identity \( T_1 T_2^{(1,3)} T_3 = O \) holds for every \( T_2^{(1,3)} \in T_2 \{1, 3\} \) if and only if \( R(T_1^*) \subseteq R(T_2^*) \) and \( R(T_3) \subseteq N(T_1 T_2^*) \).

Next, we will investigate the invariance of the range of the operator product \( T_1 T_2^{(1,3)} T_3 \) with respect to the choices of \( T_2^{(1,3)} \in T_2 \{1, 3\} \).

Theorem 3.3. Let \( T_1 \in L(\mathbb{L}, \mathbb{K}) \), \( T_2 \in L(\mathbb{L}, \mathbb{K}) \), and \( T_3 \in L(\mathbb{K}, \mathbb{K}) \) be such that \( T_1 \), \( T_2 \), \( T_3 \) have closed ranges. Suppose that \( T_1 \), \( T_2 \), and \( T_3 \) are not zero operators on Hilbert spaces and \( T_1 T_2^* T_3 \neq O \). Then the following statements are equivalent:

1. \( R(T_1 T_2^{(1,3)} T_3) \) is the same for every \( T_2^{(1,3)} \in T_2 \{1, 3\} \);
2. \( R(T_1^*) \subseteq R(T_2^*) \).

Proof. According to the above proof in Theorem 3.1, it follows that for any \( T_2^{(1,3)} \in T_2 \{1, 3\} \)

\[
T_1 T_2^{(1,3)} T_3 = \begin{pmatrix} T_1^{11} (T_2^{11})^{-1} T_3^{11} + T_1^{12} W_{21} T_3^{11} + T_1^{12} W_{22} T_3^{21} & O \\ O & O \end{pmatrix}
\]

and

\[
(T_1 T_2^{(1,3)} T_3)^\dagger = \begin{pmatrix} (T_1^{11} (T_2^{11})^{-1} T_3^{11} + T_1^{12} W_{21} T_3^{11} + T_1^{12} W_{22} T_3^{21})^\dagger & O \\ O & O \end{pmatrix}
\]

where \( W_{21} \) and \( W_{22} \) are bounded linear operators on appropriate subspaces. Furthermore, from (3.2) we have

\[
(T_1 T_2^{(1,3)} T_3)^\dagger = \begin{pmatrix} (T_1^{11} (T_2^{11})^{-1} T_3^{11})^\dagger & O \\ O & O \end{pmatrix}
\]

(2) \(\Rightarrow\) (1): Clearly the invariance of the product \( T_1 T_2^{(1,3)} T_3 \) with respect to \( T_2^{(1,3)} \in T_2 \{1, 3\} \) is sufficient for the invariance of \( R(T_1 T_2^{(1,3)} T_3) \) with respect to the choices of \( T_2^{(1,3)} \in T_2 \{1, 3\} \). Then from Theorem 3.1, we have the result “(2) \(\Rightarrow\) (1)” in Theorem 3.3.

(1) \(\Rightarrow\) (2): From (1) in Theorem 3.3, we know that the equality \( R(T_1 T_2^{(1,3)} T_3) = R(T_1 T_2^* T_3) \) holds for any \( T_2^{(1,3)} \in T_2 \{1, 3\} \). Under the definition of the range of operators, it follows that the equality \( R(T_1 T_2^{(1,3)} T_3) = R(T_1 T_2^* T_3) \) holds for any \( T_2^{(1,3)} \) if and only if the following two inclusions:

\[
R(T_1 T_2^{(1,3)} T_3) \subseteq R(T_1 T_2^* T_3)
\]

and

\[
R(T_1 T_2^* T_3) \subseteq R(T_1 T_2^{(1,3)} T_3)
\]
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hold for any $T_{2}^{(1,3)} \in T_{2}\{1, 3\}$, which are respectively equivalent to the following two identities:

\[(3.9) \quad T_{1}T_{2}^{1}T_{3}(T_{1}T_{2}^{1}T_{3})^{\dagger}T_{1}T_{2}^{(1,3)}T_{3} = T_{1}T_{2}^{(1,3)}T_{3} \]

and

\[(3.10) \quad T_{1}T_{2}^{(1,3)}T_{3}(T_{1}T_{2}^{(1,3)}T_{3})^{\dagger}T_{1}T_{2}^{1}T_{3} = T_{1}T_{2}^{1}T_{3} \]

valid for any $T_{2}^{(1,3)} \in T_{2}\{1, 3\}$. Hence, from the equalities (3.2), (3.6), (3.7), (3.8) (3.9), (3.10), we have

\[(3.11) \quad (T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11})(T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11})^{\dagger} \mu = \mu \]

and

\[(3.12) \quad \mu \mu^{\dagger}T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11} = T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11}, \]

where $W_{21}$ and $W_{22}$ are bounded linear operators on appropriate subspaces and

\[(3.13) \quad \mu = T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11} + T_{1}^{12}W_{21}T_{3}^{11} + T_{1}^{12}W_{22}T_{3}^{21}. \]

Combining (3.11), (3.12), with (3.13), we get the following equality:

\[(3.14) \quad (T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11})(T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11})^{\dagger} = \mu \mu^{\dagger} \]

is valid for arbitrary bounded linear operators $W_{21}$ and $W_{22}$.

On the other hand, since $T_{1}T_{2}^{1}T_{3} \neq O$, it follows that

\[T_{1}T_{2}^{1}T_{3} = \begin{pmatrix} T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11} & O \\ O & O \end{pmatrix} \neq O, \]

that is $T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11} \neq O$, which also implies that $T_{1}^{11} \neq O$, $T_{2}^{11} \neq O$ and $T_{3}^{11} \neq O$. Let $W_{22} = O$, then from (3.14), we obtain that the following equality:

\[(T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11})(T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11})^{\dagger} = (T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11} + T_{1}^{12}W_{21}T_{3}^{11})(T_{1}^{11}(T_{2}^{11})^{-1}T_{3}^{11} + T_{1}^{12}W_{21}T_{3}^{11})^{\dagger} \]

holds for any $W_{21}$. This implies $T_{1}^{12} = O$. According to the fact: $R(T_{1}^{1}) \subseteq R(T_{2}^{1}) \Leftrightarrow T_{1}^{12} = O$, it follows that (1) $\Rightarrow$ (2). $\Box$

By Lemma 1.2 and Lemma 1.3, we know that for a bounded linear operator $T_{2}$, $X \in T_{2}\{1, 4\}$ if and only if $X^{\dagger} \in T_{2}\{1, 3\}$. So results for the operator product $T_{1}T_{2}^{(1,4)}T_{3}$ involving $(1, 4)$-inverses of $T_{2}$ follow from the previous theorems in this section.
Theorem 3.4. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that $T_1$, $T_2$, $T_3$ have closed ranges. Suppose that $T_1$, $T_2$, and $T_3$ are not zero operators on Hilbert spaces. Then the following statements are equivalent:

1. The equality $T_1 T_2^{(1,4)} T_3 = T_1 T_2 T_3^\dagger$ holds for every $T_2^{(1,4)} \in T_2 \{1, 4\}$; 

2. $R(T_3) \subseteq R(T_2)$.

Corollary 3.5. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that $T_1$, $T_2$, $T_3$ have closed ranges. Suppose that $T_1$, $T_2$, and $T_3$ are not zero operators on Hilbert spaces. Then the identity $T_1 T_2^{(1,4)} T_3 = O$ holds for every $T_2^{(1,4)} \in T_2 \{1, 4\}$ if and only if $R(T_3) \subseteq R(T_2)$ and $R(T_3) \subseteq N(T_1 T_2^{(1,4)})$.

Theorem 3.6. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that $T_1$, $T_2$, $T_3$ have closed ranges. Suppose that $T_1$, $T_2$, and $T_3$ are not zero operators on Hilbert spaces and $T_1 T_2^{(1,4)} T_3 \neq O$. Then the following statements are equivalent:

1. $R(T_1 T_2^{(1,4)} T_3)$ is the same for every $T_2^{(1,4)} \in B\{1, 4\}$; 

2. $R(T_3) \subseteq R(T_2)$.

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REFERENCES

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