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Yanbo Li

Zhankui Xiao

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ADDITIVITY OF MAPS ON GENERALIZED MATRIX ALGEBRAS*

YANBO LI† AND ZHANKUI XIAO‡

Abstract. In this paper, it is proven that every multiplicative bijective map, Jordan bijective map, Jordan triple bijective map on generalized matrix algebras is additive.

Key words. Multiplicative map, Jordan map, Jordan triple map, Generalized matrix algebras.

AMS subject classifications. 16W99, 47B47, 47L35.

1. Introduction. Let $\mathcal{R}$ be a commutative ring with identity, A and B be two algebras (maybe without identity) over $\mathcal{R}$. Let M be an (A,B)-bimodule and N a (B,A)-bimodule. Assume that there are two bimodule homomorphisms $\varphi : M \otimes_B N \to A$ and $\psi : N \otimes_A M \to B$ satisfying the following associativity conditions: $(mn)m' = m(nm')$ and $(nm)n' = n(mn')$ for all $m, m' \in M$ and $n, n' \in N$, where we put $mn = \varphi(m \otimes n)$ and $nm = \psi(n \otimes m)$. A *generalized matrix algebra* $\mathrm{Mat}(A, M, N, B)$ is an algebra of the form

$$\mathrm{Mat}(A, M, N, B) := \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} : a \in A, m \in M, n \in N, b \in B \right\}$$

under the usual matrix-like addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Obviously, the *triangular algebras* studied in [1] are a kind of generalized matrix algebras. We refer the reader to [11, Section 2] for some classical examples of generalized matrix algebras.

We now introduce some needed definitions. Let ϕ be a map from A to B and a, b, c be arbitrary elements of A.

(1) ϕ is said to be *multiplicative* if

$$\phi(ab) = \phi(a)\phi(b).$$

(2) ϕ is called a *Jordan map* if

$$\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a).$$

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†Department of Information and Computing Sciences, Northeastern University at Qinhuangdao, Qinhuangdao, 066004, P.R. China (liyanbo707@163.com).
‡School of Mathematical Sciences, Huaqiao University, Quanzhou, 362021, P.R. China (zhkxiao@bit.edu.cn).

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(3) ϕ is called a *Jordan triple map* if

$$\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a).$$

It is of interest to study the relationship between the multiplication and the addition of an algebra. Martindale in [9] found that every multiplicative bijective map from A to an arbitrary algebra is additive if A contains a nontrivial idempotent. We would like to point out that the algebra on which Martindale worked is in fact a kind of generalized matrix algebra (see [11, (2.1)]). Following the standard argument of [9], several mathematicians have investigated the additivity of maps on rings as well as operator algebras (see [3, 4, 7, 8, 10]).

As far as we know, Cheung [2] first initiated the study of linear maps on triangular algebras. He determined the class of triangular algebras on which every commuting linear map is proper. Motivated by the aforementioned results, Ling and Lu in [6] studied Jordan maps of nest algebras, a kind of triangular algebras coming from operator theory. They showed that every Jordan bijective map on a standard subalgebra of a nest algebra is additive. This result was extended by Ji [5] to Jordan surjective map pair of triangular algebras. Recently, Cheng and Jing in [1] proved that every multiplicative bijective map, Jordan bijective map, Jordan triple bijective map and elementary surjective map on triangular algebras is additive. On the other hand, Wei and Xiao in [11] extended the main results of [2] to generalized matrix algebras. Therefore, it is natural to consider the additivity of the above defined maps on generalized matrix algebras.

Throughout this paper, let $\mathcal{G}$ be a generalized matrix algebra $\mathrm{Mat}(A, M, N, B)$. We always assume that M is faithful as a left A-module and also as a right B-module, but no further conditions on N. For convenience, we set

$$\mathcal{G}_{11} := \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in A \right\}; \quad \mathcal{G}_{12} := \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} : m \in M \right\};$$

$$\mathcal{G}_{21} := \left\{ \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} : n \in N \right\}; \quad \mathcal{G}_{22} := \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} : b \in B \right\}.$$

Then every element $a \in \mathcal{G}$ could be written as $a = a_{11} + a_{12} + a_{21} + a_{22}$, where $a_{ij} \in \mathcal{G}_{ij}$. This special structure allows us to borrow the argument in [9] even without the existence of nontrivial idempotents.

2. Multiplicative maps. In this section, we shall study the additivity of multiplicative maps on generalized matrix algebras. Firstly, we state the main theorem of this section.

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THEOREM 2.1. *Let A and B be two algebras over a commutative ring $\mathcal{R}$. Let M be faithful as a left A-module and also as a right B-module. Let $\mathcal{G}$ be the generalized matrix algebra* $\mathrm{Mat}(A, M, N, B)$ *which satisfies*

(1) *for $a \in A$, $aA = 0$ or $Aa = 0$ implies that $a = 0$,*

(2) *for $b \in B$, $bB = 0$ or $Bb = 0$ implies that $b = 0$,*

(3) *for $m \in M$, $Am = 0$ or $mB = 0$ implies that $m = 0$,*

(4) *for $n \in N$, $Bn = 0$ or $nA = 0$ implies that $n = 0$.*

Then a multiplicative bijective map ϕ from $\mathcal{G}$ onto an arbitrary ring $\mathcal{R}'$ is additive.

We need the following lemmas.

LEMMA 2.2. $\phi(0) = 0$.

Proof. Since ϕ is surjective, there exists $a \in \mathcal{G}$ such that $\phi(a) = 0$. Therefore, $\phi(0) = \phi(0 \cdot a) = \phi(0)\phi(a) = 0$. □

LEMMA 2.3. *For $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

$$\phi(a_{11} + b_{12} + d_{22}) = \phi(a_{11}) + \phi(b_{12} + d_{22}) = \phi(a_{11} + b_{12}) + \phi(d_{22}).$$

Proof. Choose $c \in \mathcal{G}$ such that $\phi(c) = \phi(a_{11}) + \phi(b_{12} + d_{22})$. For arbitrary $t_{22} \in \mathcal{G}_{22}$, it is clear that

$$\begin{aligned}\phi(t_{22}c) &= \phi(t_{22})\phi(c) = \phi(t_{22})(\phi(a_{11}) + \phi(b_{12} + d_{22})) \\ &= \phi(t_{22})\phi(a_{11}) + \phi(t_{22})\phi(b_{12} + d_{22}) = \phi(t_{22}d_{22}).\end{aligned}$$

Thus, $t_{22}c = t_{22}d_{22}$, and hence, $c_{21} = 0$ and $c_{22} = d_{22}$. For $s_{11} \in \mathcal{G}_{11}$, we have

$$\begin{aligned}\phi(cs_{11}) &= \phi(c)\phi(s_{11}) = (\phi(a_{11}) + \phi(b_{12} + d_{22}))\phi(s_{11}) \\ &= \phi(a_{11})\phi(s_{11}) + \phi(b_{12} + d_{22})\phi(s_{11}) = \phi(a_{11}s_{11}).\end{aligned}$$

Then $cs_{11} = a_{11}s_{11}$, and hence, $c_{11} = a_{11}$. In order to determine c_{12}, we get

$$\begin{aligned}\phi(s_{11}ct_{22}) &= \phi(s_{11})\phi(c)\phi(t_{22}) = \phi(s_{11})(\phi(a_{11}) + \phi(b_{12} + d_{22}))\phi(t_{22}) \\ &= \phi(s_{11})\phi(a_{11})\phi(t_{22}) + \phi(s_{11}(b_{12} + d_{22})t_{22}) \\ &= \phi(s_{11}b_{12}t_{22})\end{aligned}$$

for all $s_{11} \in \mathcal{G}_{11}$ and $t_{22} \in \mathcal{G}_{22}$. It follows that $s_{11}ct_{22} = s_{11}b_{12}t_{22}$, which implies that $c_{12} = b_{12}$. This shows that

$$\phi(a_{11} + b_{12} + d_{22}) = \phi(a_{11}) + \phi(b_{12} + d_{22}).$$

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The second claim $\phi(a_{11}+b_{12}+d_{22}) = \phi(a_{11}+b_{12}) + \phi(d_{22})$ is proven similarly. ∎

Corollary 2.4. *For $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

(1) $\phi(a_{11}+b_{12}) = \phi(a_{11}) + \phi(b_{12})$,

(2) $\phi(b_{12}+d_{22}) = \phi(b_{12}) + \phi(d_{22})$.

Using a similar method as in Lemma 2.3, we obtain the following lemma.

Lemma 2.5. *For $a_{11} \in \mathcal{G}_{11}$, $e_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

$$\phi(a_{11}+e_{21}+d_{22}) = \phi(a_{11}) + \phi(e_{21}+d_{22}) = \phi(a_{11}+e_{21}) + \phi(d_{22}).$$

Corollary 2.6. *For $a_{11} \in \mathcal{G}_{11}$, $e_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

(1) $\phi(a_{11}+e_{21}) = \phi(a_{11}) + \phi(e_{21})$,

(2) $\phi(e_{21}+d_{22}) = \phi(e_{21}) + \phi(d_{22})$.

Lemma 2.7. *For $a_{11} \in \mathcal{G}_{11}$, $b_{12}, c_{12} \in \mathcal{G}_{12}$, $e_{21}, f_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$, the following two identities hold:*

(1) $\phi(a_{11}b_{12} + c_{12}d_{22}) = \phi(a_{11}b_{12}) + \phi(c_{12}d_{22})$,

(2) $\phi(d_{22}e_{21} + f_{21}a_{11}) = \phi(d_{22}e_{21}) + \phi(f_{21}a_{11})$.

Proof. We first prove (1). By Corollary 2.4, we have

$$\begin{aligned}
\phi(a_{11}b_{12} + c_{12}d_{22}) &= \phi((a_{11}+c_{12})(b_{12}+d_{22})) \\
&= \phi(a_{11}+c_{12})\phi(b_{12}+d_{22}) \\
&= (\phi(a_{11}) + \phi(c_{12}))(\phi(b_{12}) + \phi(d_{22})) \\
&= \phi(a_{11}b_{12}) + \phi(c_{12}d_{22}).
\end{aligned}$$

The claim (2) can be proved similarly by Corollary 2.6. ∎

Proof of Theorem 2.1.

Step 1. ϕ is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.

For arbitrary $a_{12}, b_{12} \in \mathcal{G}_{12}$, we need to prove $\phi(a_{12}+b_{12}) = \phi(a_{12}) + \phi(b_{12})$. Choose an element $c \in \mathcal{G}$ such that $\phi(c) = \phi(a_{12}) + \phi(b_{12})$. For $s_{11} \in \mathcal{G}_{11}$ and $t_{22} \in \mathcal{G}_{22}$, we have

$$\begin{aligned}
\phi(s_{11}ct_{22}) &= \phi(s_{11})\phi(c)\phi(t_{22}) = \phi(s_{11})(\phi(a_{12}) + \phi(b_{12}))\phi(t_{22}) \\
&= \phi(s_{11}a_{12}t_{22}) + \phi(s_{11}b_{12}t_{22}) \\
&= \phi(s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22}).
\end{aligned}$$

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Note that the last equality follows from Lemma 2.7 (1). Hence, $s_{11}ct_{22} = s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22}$. It follows that $c_{12} = a_{12} + b_{12}$. Moreover, $\phi(cs_{11}) = \phi(c)\phi(s_{11}) = (\phi(a_{12}) + \phi(b_{12}))\phi(s_{11}) = 0$. This implies that $c_{11} = 0$ and $c_{21} = 0$. By considering $\phi(t_{22}c)$, we obtain $c_{22} = 0$. Hence, ϕ is additive on $\mathcal{G}_{12}$. By Lemma 2.7 (2), we can similarly prove that ϕ is additive on $\mathcal{G}_{21}$.

Step 2. ϕ is additive on $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$.

For arbitrary $a_{11}, b_{11} \in \mathcal{G}_{11}$, we need to show $\phi(a_{11} + b_{11}) = \phi(a_{11}) + \phi(b_{11})$. Choose an element $c \in \mathcal{G}$ such that $\phi(c) = \phi(a_{11}) + \phi(b_{11})$. Take $s_{11} \in \mathcal{G}_{11}$ and $t_{22} \in \mathcal{G}_{22}$, it is easy to know that $\phi(t_{22}c) = 0$ and $\phi(s_{11}ct_{22}) = 0$. Hence, $c_{21} = 0$, $c_{22} = 0$, $c_{12} = 0$.

On the other hand, for $u_{12} \in \mathcal{G}_{12}$, it follows from Step 1 that

$$\begin{aligned}
\phi(cu_{12}) &= \phi(c)\phi(u_{12}) = (\phi(a_{11}) + \phi(b_{11}))\phi(u_{12}) \\
&= \phi(a_{11}u_{12}) + \phi(b_{11}u_{12}) \\
&= \phi(a_{11}u_{12} + b_{11}u_{12}).
\end{aligned}$$

Therefore, $cu_{12} = a_{11}u_{12} + b_{11}u_{12}$. Since M is faithful as a left A-module, we now get that $c_{11} = a_{11} + b_{11}$. Hence, ϕ is additive on $\mathcal{G}_{11}$. The additivity of ϕ on $\mathcal{G}_{22}$ can be proved similarly.

Step 3. $\phi(a + b) = \phi(a) + \phi(b)$ for arbitrary $a, b \in \mathcal{G}$.

Firstly, we show that $\phi(a_{11} + b_{12} + c_{21} + d_{22}) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$ for $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, $c_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$. In fact, if $\phi(f) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$ for $f \in \mathcal{G}$, then for $s_{11}, t_{11} \in \mathcal{G}_{11}$, we have $\phi(s_{11}ft_{11}) = \phi(s_{11})\phi(a_{11})\phi(t_{11}) = \phi(s_{11}a_{11}t_{11})$. Hence, $f_{11} = a_{11}$. Similarly $f_{12} = b_{12}$, $f_{21} = c_{21}$, and $f_{22} = d_{22}$. Now for arbitrary $a, b \in \mathcal{G}$, we have

$$\begin{aligned}
\phi(a + b) &= \phi(a_{11} + a_{12} + a_{21} + a_{22} + b_{11} + b_{12} + b_{21} + b_{22}) \\
&= \phi((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22})) \\
&= \phi(a_{11} + b_{11}) + \phi(a_{12} + b_{12}) + \phi(a_{21} + b_{21}) + \phi(a_{22} + b_{22}) \\
&= \phi(a_{11}) + \phi(b_{11}) + \phi(a_{12}) + \phi(b_{12}) + \phi(a_{21}) + \phi(b_{21}) + \phi(a_{22}) + \phi(b_{22}) \\
&= \phi(a_{11} + a_{12} + a_{21} + a_{22}) + \phi(b_{11} + b_{12} + b_{21} + b_{22}) \\
&= \phi(a) + \phi(b).
\end{aligned}$$

This completes the proof of the main theorem. □

Clearly when $N = 0$, our Theorem 2.1 degenerates to [1, Theorem 2.1].

We end this section by a corollary as follows.

COROLLARY 2.8. *Let A and B be two unital algebras over a commutative ring*

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$\mathcal{R}$. Let M be faithful as a left A-module and also as a right B-module, and $\mathcal{G}$ be the generalized matrix algebra $\mathrm{Mat}(A,M,N,B)$. *Then a multiplicative bijective map ϕ from $\mathcal{G}$ onto an arbitrary ring $\mathcal{R}'$ is additive.*

3. Jordan maps. In this section, we shall study the additivity of Jordan maps on generalized matrix algebras. However, for $\mathrm{Mat}(A,M,N,B)$, we have to assume that N is faithful as a left B-module and also as a right A-module.

THEOREM 3.1. *Let A and B be two algebras over a commutative ring $\mathcal{R}$. Let M be faithful as a left A-module and also as a right B-module and N be faithful as a left B-module and also as a right A-module. Let $\mathcal{G}$ be the generalized matrix algebra* $\mathrm{Mat}(A,M,N,B)$ *which satisfies*

(1) *for $a \in A$, if $ax + xa = 0$ for all $x \in A$, then $a = 0$,*

(2) *for $b \in B$, if $by + yb = 0$ for all $y \in B$, then $b = 0$,*

(3) *for $m \in M$, $Am = 0$ or $mB = 0$ implies that $m = 0$,*

(4) *for $n \in N$, $Bn = 0$ or $nA = 0$ implies that $n = 0$.*

Then a Jordan bijective map ϕ from $\mathcal{G}$ onto an arbitrary ring $\mathcal{R}'$ is additive.

We first prove some lemmas.

LEMMA 3.2. $\phi(0) = 0$.

Proof. Let $\phi(a) = 0$, where $a \in \mathcal{G}$. Then $\phi(0) = \phi(a \cdot 0 + 0 \cdot a) = \phi(a)\phi(0) + \phi(0)\phi(a) = 0$. ☐

LEMMA 3.3. *For $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

$$\phi(a_{11} + b_{12} + d_{22}) = \phi(a_{11}) + \phi(b_{12} + d_{22}) = \phi(a_{11} + b_{12}) + \phi(d_{22}).$$

Proof. We only prove $\phi(a_{11} + b_{12} + d_{22}) = \phi(a_{11}) + \phi(b_{12} + d_{22})$, the other equality $\phi(a_{11} + b_{12} + d_{22}) = \phi(a_{11} + b_{12}) + \phi(d_{22})$ can be proved similarly. Let $\phi(c) = \phi(a_{11}) + \phi(b_{12} + d_{22})$ with $c \in \mathcal{G}$. Then for arbitrary $t_{22} \in \mathcal{G}_{22}$, we have

$$\begin{aligned}
\phi(ct_{22} + t_{22}c) &= \phi(c)\phi(t_{22}) + \phi(t_{22})\phi(c) \\
&= (\phi(a_{11}) + \phi(b_{12} + d_{22}))\phi(t_{22}) + \phi(t_{22})(\phi(a_{11}) + \phi(b_{12} + d_{22})) \\
&= \phi(a_{11}t_{22} + t_{22}a_{11}) + \phi((b_{12} + d_{22})t_{22} + t_{22}(b_{12} + d_{22})) \\
&= \phi((b_{12} + d_{22})t_{22} + t_{22}d_{22}).
\end{aligned}$$

This implies that $ct_{22}+t_{22}c = (b_{12}+d_{22})t_{22}+t_{22}d_{22}$. Hence, $c_{12}t_{22} = b_{12}t_{22}$, $t_{22}c_{21} = 0$, and $c_{22}t_{22} + t_{22}c_{22} = d_{22}t_{22} + t_{22}d_{22}$. So $c_{12} = b_{12}$, $c_{21} = 0$, and $c_{22} = d_{22}$. At the

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same time, for all $s_{11} \in \mathcal{G}_{11}$ and $u_{12} \in \mathcal{G}_{12}$,

$$\begin{aligned}\phi(cs_{11} + s_{11}c) &= \phi(a_{11}s_{11} + s_{11}a_{11}) + \phi(s_{11}(b_{12} + d_{22}) + (b_{12} + d_{22})s_{11}) \\ &= \phi(a_{11}s_{11} + s_{11}a_{11}) + \phi(s_{11}b_{12}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\phi((cs_{11} + s_{11}c)u_{12} + u_{12}(cs_{11} + s_{11}c)) \\ =&\phi((a_{11}s_{11} + s_{11}a_{11})u_{12} + u_{12}(a_{11}s_{11} + s_{11}a_{11})) + \phi(s_{11}b_{12}u_{12} + u_{12}s_{11}b_{12}) \\ =&\phi((a_{11}s_{11} + s_{11}a_{11})u_{12}).\end{aligned}$$

Thus, $(cs_{11} + s_{11}c)u_{12} + u_{12}(cs_{11} + s_{11}c) = (a_{11}s_{11} + s_{11}a_{11})u_{12}$, that is,

$$cs_{11}u_{12} + s_{11}cu_{12} + u_{12}cs_{11} + u_{12}s_{11}c = (a_{11}s_{11} + s_{11}a_{11})u_{12}.$$

It follows from $c_{21} = 0$ that $c_{11}s_{11}u_{12} + s_{11}c_{11}u_{12} = (a_{11}s_{11} + s_{11}a_{11})u_{12}$. Since M is faithful as a left A-module, then

$$c_{11}s_{11} + s_{11}c_{11} = a_{11}s_{11} + s_{11}a_{11}$$

and hence $c_{11} = a_{11}$. □

Corollary 3.4. *For $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

(1) $\phi(a_{11} + b_{12}) = \phi(a_{11}) + \phi(b_{12})$,

(2) $\phi(b_{12} + d_{22}) = \phi(b_{12}) + \phi(d_{22})$.

The following lemma is an analogue of Lemma 3.3.

Lemma 3.5. *For $a_{11} \in \mathcal{G}_{11}$, $e_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

$$\phi(a_{11} + e_{21} + d_{22}) = \phi(a_{11}) + \phi(e_{21} + d_{22}) = \phi(a_{11} + e_{21}) + \phi(d_{22}).$$

Corollary 3.6. *For $a_{11} \in \mathcal{G}_{11}$, $e_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$, we have*

(1) $\phi(a_{11} + e_{21}) = \phi(a_{11}) + \phi(e_{21})$,

(2) $\phi(e_{21} + d_{22}) = \phi(e_{21}) + \phi(d_{22})$.

Lemma 3.7. *For $a_{11} \in \mathcal{G}_{11}$, $b_{12}, c_{12} \in \mathcal{G}_{12}$, $e_{21}, f_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$, the following two equalities hold.*

(1) $\phi(a_{11}b_{12} + c_{12}d_{22}) = \phi(a_{11}b_{12}) + \phi(c_{12}d_{22})$,

(2) $\phi(d_{22}e_{21} + f_{21}a_{11}) = \phi(d_{22}e_{21}) + \phi(f_{21}a_{11})$.

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Proof. We first prove (2). By Corollary 3.6 (2), we have

$$\begin{aligned}
&\phi(d_{22}e_{21}+f_{21}a_{11})\\
=&\phi((d_{22}+f_{21})(e_{21}+a_{11})+(e_{21}+a_{11})(d_{22}+f_{21}))\\
=&\phi(d_{22}+f_{21})\phi(e_{21}+a_{11})+\phi(e_{21}+a_{11})\phi(d_{22}+f_{21})\\
=&(\phi(d_{22})+\phi(f_{21}))(\phi(e_{21})+\phi(a_{11}))+(\phi(e_{21})+\phi(a_{11}))(\phi(d_{22})+\phi(f_{21}))\\
=&\phi(d_{22}e_{21}+e_{21}d_{22})+\phi(d_{22}a_{11}+a_{11}d_{22})\\
&+\phi(f_{21}e_{21}+e_{21}f_{21})+\phi(f_{21}a_{11}+a_{11}f_{21})\\
=&\phi(d_{22}e_{21})+\phi(f_{21}a_{11}).
\end{aligned}$$

The equality (1) is proven similarly. ◻

LEMMA 3.8. *ϕ is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.*

Proof. For arbitrary $a_{21}, b_{21} \in \mathcal{G}_{21}$, we prove $\phi(a_{21}+b_{21}) = \phi(a_{21})+\phi(b_{21})$. Let $\phi(c) = \phi(a_{21})+\phi(b_{21})$ with $c \in \mathcal{G}$. Then for $s_{11} \in \mathcal{G}_{11}$ and $t_{22} \in \mathcal{G}_{22}$, we have

$$\phi(cs_{11}+s_{11}c) = \phi(a_{21}s_{11}+s_{11}a_{21})+\phi(b_{21}s_{11}+s_{11}b_{21}) = \phi(a_{21}s_{11})+\phi(b_{21}s_{11}),$$

and from Lemma 3.7 (2) that

$$\begin{aligned}
&\phi(t_{22}(cs_{11}+s_{11}c)+(cs_{11}+s_{11}c)t_{22})\\
=&\phi(t_{22}a_{21}s_{11}+a_{21}s_{11}t_{22})+\phi(t_{22}b_{21}s_{11}+b_{21}s_{11}t_{22})\\
=&\phi(t_{22}a_{21}s_{11})+\phi(t_{22}b_{21}s_{11}) = \phi(t_{22}a_{21}s_{11}+t_{22}b_{21}s_{11}).
\end{aligned}$$

It follows that $t_{22}c_{21}s_{11}+s_{11}c_{12}t_{22} = t_{22}a_{21}s_{11}+t_{22}b_{21}s_{11}$. Then we get $c_{12}=0$ and $c_{21} = a_{21}+b_{21}$. Moreover, for $s_{11} \in \mathcal{G}_{11}$ and $u_{21} \in \mathcal{G}_{21}$,

$$\begin{aligned}
&\phi(u_{21}(cs_{11}+s_{11}c)+(cs_{11}+s_{11}c)u_{21})\\
=&\phi(u_{21}s_{11}a_{21}+s_{11}a_{21}u_{21})+\phi(u_{21}s_{11}b_{21}+s_{11}b_{21}u_{21}) = 0.
\end{aligned}$$

Then $u_{21}cs_{11}+u_{21}s_{11}c+s_{11}cu_{21} = 0$. It follows from $c_{12}=0$ that $u_{21}c_{11}s_{11}+u_{21}s_{11}c_{11} = 0$. Since N is faithful as a right A-module, then $c_{11}s_{11}+s_{11}c_{11} = 0$, and hence, $c_{11} = 0$. Note that

$$\phi(ct_{22}+t_{22}c) = \phi(a_{21}t_{22}+t_{22}a_{21})+\phi(b_{21}t_{22}+t_{22}b_{21}) = \phi(t_{22}a_{21})+\phi(t_{22}b_{21})$$

for all $t_{22} \in \mathcal{G}_{22}$. Therefore,

$$\begin{aligned}
&\phi(u_{21}(ct_{22}+t_{22}c)+(ct_{22}+t_{22}c)u_{21})\\
=&\phi(u_{21}t_{22}a_{21}+t_{22}a_{21}u_{21})+\phi(u_{21}t_{22}b_{21}+t_{22}b_{21}u_{21}) = 0.
\end{aligned}$$

Thus, $u_{21}ct_{22}+ct_{22}u_{21}+t_{22}cu_{21} = 0$. Then we have from $c_{12}=0$ that $c_{22}t_{22}u_{21}+t_{22}c_{22}u_{21} = 0$. Note that N is faithful as a left B-module, we get $c_{22}t_{22}+t_{22}c_{22} = 0$,

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and hence, $c_{22} = 0$. This shows that ϕ is additive on $\mathcal{G}_{21}$. The additivity of ϕ on $\mathcal{G}_{12}$ is proven similarly. ☐

LEMMA 3.9. *ϕ is additive on $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$.*

Proof. For $a_{11}, b_{11} \in \mathcal{G}_{11}$, let $\phi(c) = \phi(a_{11}) + \phi(b_{11})$ with $c \in \mathcal{G}$. Then for all $t_{22} \in \mathcal{G}_{22}$,

$$\phi(ct_{22} + t_{22}c) = \phi(a_{11}t_{22} + t_{22}a_{11}) + \phi(b_{11}t_{22} + t_{22}b_{11}) = 0.$$

This implies that $ct_{22} + t_{22}c = 0$. Then we have $c_{12} = 0$, $c_{21} = 0$, and $c_{22} = 0$. At the same time, for $u_{12} \in \mathcal{G}_{12}$, by Lemma 3.8, we have

$$\begin{aligned}\phi(u_{12}c + cu_{12}) &= \phi(u_{12}a_{11} + a_{11}u_{12}) + \phi(u_{12}b_{11} + b_{11}u_{12}) \\ &= \phi(a_{11}u_{12}) + \phi(b_{11}u_{12}) = \phi(a_{11}u_{12} + b_{11}u_{12}).\end{aligned}$$

Hence, $u_{12}c + cu_{12} = a_{11}u_{12} + b_{11}u_{12}$. Since $c_{12} = 0$, $c_{21} = 0$, and $c_{22} = 0$, we have $c_{11}u_{12} = a_{11}u_{12} + b_{11}u_{12}$. Then $c_{11} = a_{11} + b_{11}$ follows from the fact that M is faithful as a left A-module. The additivity of ϕ on $\mathcal{G}_{22}$ is proven similarly. ☐

LEMMA 3.10. *For $x \in \mathcal{G}$, $a_{11} \in \mathcal{G}_{11}$, and $b_{22} \in \mathcal{G}_{22}$, we have*

(1) $\phi(x + a_{11}) = \phi(x) + \phi(a_{11})$,

(2) $\phi(x + b_{22}) = \phi(x) + \phi(b_{22})$.

Proof. We only prove the claim (1). Let $\phi(c) = \phi(x) + \phi(a_{11})$ for some $c \in \mathcal{G}$. Then for all $t_{22} \in \mathcal{G}_{22}$,

$$\phi(ct_{22} + t_{22}c) = \phi(xt_{22} + t_{22}x) + \phi(t_{22}a_{11} + a_{11}t_{22}) = \phi(xt_{22} + t_{22}x).$$

So $ct_{22} + t_{22}c = xt_{22} + t_{22}x$. This implies that $c_{12} = x_{12}$, $c_{21} = x_{21}$, and $c_{22} = x_{22}$. Moreover, for all $t_{11} \in \mathcal{G}_{11}$,

$$\phi(ct_{11} + t_{11}c) = \phi(xt_{11} + t_{11}x) + \phi(t_{11}a_{11} + a_{11}t_{11}).$$

Therefore, for $u_{12} \in \mathcal{G}_{12}$, we have

$$\begin{aligned}&\phi(u_{12}(ct_{11} + t_{11}c) + (ct_{11} + t_{11}c)u_{12}) \\ =&\phi(u_{12}xt_{11} + xt_{11}u_{12} + t_{11}xu_{12}) + \phi(a_{11}t_{11}u_{12} + t_{11}a_{11}u_{12}) \\ =&\phi(u_{12}xt_{11} + x_{11}t_{11}u_{12} + x_{21}t_{11}u_{12} + t_{11}xu_{12}) + \phi(a_{11}t_{11}u_{12} + t_{11}a_{11}u_{12}) \\ =&\phi(u_{12}xt_{11} + x_{11}t_{11}u_{12} + x_{21}t_{11}u_{12} + t_{11}xu_{12} + a_{11}t_{11}u_{12} + t_{11}a_{11}u_{12}).\end{aligned}$$

Note that in the last equality we applied Lemma 3.3. It follows that

$$c_{11}t_{11}u_{12} + t_{11}c_{11}u_{12} = x_{11}t_{11}u_{12} + t_{11}x_{11}u_{12} + a_{11}t_{11}u_{12} + t_{11}a_{11}u_{12}.$$

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Note that M is faithful as a left A-module, then

$$c_{11}t_{11} + t_{11}c_{11} = x_{11}t_{11} + t_{11}x_{11} + a_{11}t_{11} + t_{11}a_{11}.$$

This implies that $c_{11} = x_{11} + a_{11}$. $\square$

LEMMA 3.11. *The equality* $\phi(a_{11}b_{12} + c_{21}d_{11}) = \phi(a_{11}b_{12}) + \phi(c_{21}d_{11})$ *holds for all* $a_{11}, d_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, *and* $c_{21} \in \mathcal{G}_{21}$.

Proof. On one hand, by Lemmas 3.9 and 3.10, we have

$$\begin{aligned}
&\phi(((a_{11}b_{12} + c_{21}d_{11}) + a_{11}d_{11} + d_{11}a_{11} + b_{12}c_{21}) + c_{21}b_{12})\\
=&\phi((a_{11}b_{12} + c_{21}d_{11}) + a_{11}d_{11} + d_{11}a_{11} + b_{12}c_{21}) + \phi(c_{21}b_{12})\\
=&\phi(a_{11}b_{12} + c_{21}d_{11}) + \phi(a_{11}d_{11} + d_{11}a_{11} + b_{12}c_{21}) + \phi(c_{21}b_{12})\\
=&\phi(a_{11}b_{12} + c_{21}d_{11}) + \phi(a_{11}d_{11} + d_{11}a_{11}) + \phi(b_{12}c_{21}) + \phi(c_{21}b_{12})\\
=&\phi(a_{11}b_{12} + c_{21}d_{11}) + \phi(a_{11}d_{11}) + \phi(d_{11}a_{11}) + \phi(b_{12}c_{21}) + \phi(c_{21}b_{12}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\phi(a_{11}b_{12} + c_{21}d_{11} + a_{11}d_{11} + d_{11}a_{11} + b_{12}c_{21} + c_{21}b_{12})\\
=&\phi((a_{11} + c_{21})(b_{12} + d_{11}) + (b_{12} + d_{11})(a_{11} + c_{21}))\\
=&\phi(a_{11} + c_{21})\phi(b_{12} + d_{11}) + \phi(b_{12} + d_{11})\phi(a_{11} + c_{21})\\
=&(\phi(a_{11}) + \phi(c_{21}))(\phi(b_{12}) + \phi(d_{11})) + (\phi(b_{12}) + \phi(d_{11}))(\phi(a_{11}) + \phi(c_{21}))\\
=&\phi(a_{11})\phi(b_{12}) + \phi(a_{11})\phi(d_{11}) + \phi(c_{21})\phi(b_{12}) + \phi(c_{21})\phi(d_{11}) +\\
&\phi(b_{12})\phi(a_{11}) + \phi(d_{11})\phi(a_{11}) + \phi(b_{12})\phi(c_{21}) + \phi(d_{11})\phi(c_{21})\\
=&\phi(a_{11}b_{12} + b_{12}a_{11}) + \phi(c_{21}d_{11} + d_{11}c_{21}) +\\
&\phi(a_{11}d_{11}) + \phi(d_{11}a_{11}) + \phi(b_{12}c_{21}) + \phi(c_{21}b_{12})\\
=&\phi(a_{11}b_{12}) + \phi(c_{21}d_{11}) + \phi(a_{11}d_{11}) + \phi(d_{11}a_{11}) + \phi(b_{12}c_{21}) + \phi(c_{21}b_{12}).
\end{aligned}$$

Therefore, we obtain $\phi(a_{11}b_{12} + c_{21}d_{11}) = \phi(a_{11}b_{12}) + \phi(c_{21}d_{11})$. $\square$

Proof of Theorem 3.1.

Step 1. $\phi(a_{11} + b_{12} + c_{21} + d_{22}) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$ for $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, $c_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$.

Let $\phi(f) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$ for some $f \in \mathcal{G}$. By Corollaries 3.4 and 3.6, $\phi(f) = \phi(a_{11} + b_{12}) + \phi(c_{21} + d_{22})$. Then for $s_{11} \in \mathcal{G}_{11}$, we have

$$\begin{aligned}
&\phi(fs_{11} + s_{11}f)\\
=&\ \phi(s_{11}(a_{11} + b_{12}) + (a_{11} + b_{12})s_{11}) + \phi(s_{11}(c_{21} + d_{22}) + (c_{21} + d_{22})s_{11})\\
=&\ \phi(s_{11}a_{11} + s_{11}b_{12} + a_{11}s_{11}) + \phi(c_{21}s_{11}).
\end{aligned}$$

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Hence, for $t_{22} \in \mathcal{G}_{22}$, we obtain

$$\begin{aligned}\phi(t_{22}(fs_{11} + s_{11}f) + (fs_{11} + s_{11}f)t_{22}) &= \phi(s_{11}b_{12}t_{22}) + \phi(t_{22}c_{21}s_{11})\\ &= \phi(s_{11}b_{12}t_{22} + t_{22}c_{21}s_{11}).\end{aligned}$$

Note that we applied Lemma 3.11 in the last equality. It follows that

$$t_{22}fs_{11} + s_{11}ft_{22} = s_{11}b_{12}t_{22} + t_{22}c_{21}s_{11}.$$

This implies that $t_{22}fs_{11} = t_{22}c_{21}s_{11}$ and $s_{11}ft_{22} = s_{11}b_{12}t_{22}$. That means, $f_{21} = c_{21}$ and $f_{12} = b_{12}$.

Furthermore, by Lemma 3.3, we have

$$\begin{aligned}&\phi(u_{12}(fs_{11} + s_{11}f) + (fs_{11} + s_{11}f)u_{12})\\ &= \phi(s_{11}a_{11}u_{12} + a_{11}s_{11}u_{12}) + \phi(c_{21}s_{11}u_{12} + u_{12}c_{21}s_{11})\\ &= \phi(s_{11}a_{11}u_{12} + a_{11}s_{11}u_{12} + c_{21}s_{11}u_{12} + u_{12}c_{21}s_{11})\end{aligned}$$

for all $s_{11} \in \mathcal{G}_{11}, u_{12} \in \mathcal{G}_{12}$. It follows that

$$u_{12}fs_{11} + fs_{11}u_{12} + s_{11}fu_{12} = s_{11}a_{11}u_{12} + a_{11}s_{11}u_{12} + c_{21}s_{11}u_{12} + u_{12}c_{21}s_{11}.$$

Hence,

$$f_{11}s_{11}u_{12} + s_{11}f_{11}u_{12} = s_{11}a_{11}u_{12} + a_{11}s_{11}u_{12}.$$

Since M is faithful as a left A-module, we get $f_{11}s_{11} + s_{11}f_{11} = s_{11}a_{11} + a_{11}s_{11}$, and hence, $f_{11} = a_{11}$. Similarly, $f_{22} = d_{22}$.

Step 2. $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in \mathcal{G}$.

This follows directly from Lemmas 3.8 and 3.9, and Step 1. □

4. Jordan triple maps. In this section, we shall study the additivity of Jordan triple maps on generalized matrix algebras. However, for generalized matrix algebra $\mathrm{Mat}(A, M, N, B)$, we also have to assume that N is faithful as a left B-module and also as a right A-module.

THEOREM 4.1. *Let A and B be two algebras over a commutative ring $\mathcal{R}$. Let M be faithful as a left A-module and also as a right B-module and N be faithful as a left B-module and also as a right A-module. Let $\mathcal{G}$ be the generalized matrix algebra $\mathrm{Mat}(A, M, N, B)$ which satisfies*

(1) *for $a \in A$, $aA = 0$ or $Aa = 0$ implies that $a = 0$,*

(2) *for $b \in B$, $bB = 0$ or $Bb = 0$ implies that $b = 0$,*

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(3) *for* $m \in M$, $Am = 0$ *or* $mB = 0$ *implies that* $m = 0$,

(4) *for* $n \in N$, $Bn = 0$ *or* $nA = 0$ *implies that* $n = 0$.

Then a Jordan triple bijective map ϕ *from* $\mathcal{G}$ *onto an arbitrary ring* $\mathcal{R}'$ *is additive.*

The following lemmas are needed.

LEMMA 4.2. $\phi(0) = 0$.

Proof. Let $\phi(a) = 0$, where $a \in \mathcal{G}$. Then $\phi(0) = \phi(a \cdot 0 \cdot 0 + 0 \cdot 0 \cdot a) = \phi(a)\phi(0)\phi(0) + \phi(0)\phi(0)\phi(a) = 0$. □

LEMMA 4.3. *For* $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, $e_{21} \in \mathcal{G}_{21}$, *and* $d_{22} \in \mathcal{G}_{22}$, *we have*

(1) $\phi(a_{11} + b_{12} + d_{22}) = \phi(a_{11}) + \phi(b_{12} + d_{22}) = \phi(a_{11} + b_{12}) + \phi(d_{22})$,

(2) $\phi(a_{11} + e_{21} + d_{22}) = \phi(a_{11}) + \phi(e_{21} + d_{22}) = \phi(a_{11} + e_{21}) + \phi(d_{22})$.

Proof. We only prove $\phi(a_{11}+b_{12}+d_{22}) = \phi(a_{11}+b_{12})+\phi(d_{22})$, the other equations are proved similarly. Let $\phi(c) = \phi(a_{11} + b_{12}) + \phi(d_{22})$ for some $c \in \mathcal{G}$. It is easy to know that $\phi(stc + cts) = \phi(sta_{11} + stb_{12} + a_{11}ts + b_{12}ts) + \phi(std_{22} + d_{22}ts)$ for all $s_{11} \in \mathcal{G}_{11}$ and $t_{12} \in \mathcal{G}_{12}$. Then $\phi(s_{11}t_{12}c) = \phi(s_{11}t_{12}d_{22})$, that is, $s_{11}t_{12}c = s_{11}t_{12}d_{22}$. This implies that $c_{22} = d_{22}$. Similarly, by taking $s_{22} \in \mathcal{G}_{22}$ and $t_{12} \in \mathcal{G}_{12}$, we get $c_{11} = a_{11}$. Now for $s_{11}, t_{11} \in \mathcal{G}_{11}$,

$$\begin{aligned} &\phi(s_{11}t_{11}c + ct_{11}s_{11}) \\ &= \phi(s_{11}t_{11}a_{11} + s_{11}t_{11}b_{12} + a_{11}t_{11}s_{11} + b_{12}t_{11}s_{11}) + \phi(s_{11}t_{11}d_{22} + d_{22}t_{11}s_{11}) \\ &= \phi(s_{11}t_{11}a_{11} + s_{11}t_{11}b_{12} + a_{11}t_{11}s_{11}). \end{aligned}$$

Thus,

$$s_{11}t_{11}c_{11} + s_{11}t_{11}c_{12} + c_{11}t_{11}s_{11} + c_{21}t_{11}s_{11} = s_{11}t_{11}a_{11} + s_{11}t_{11}b_{12} + a_{11}t_{11}s_{11}.$$

So $c_{21} = 0$ and $c_{12} = b_{12}$. □

COROLLARY 4.4. *For* $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, $e_{21} \in \mathcal{G}_{21}$, *and* $d_{22} \in \mathcal{G}_{22}$, *we have*

(1) $\phi(a_{11} + b_{12}) = \phi(a_{11}) + \phi(b_{12})$,

(2) $\phi(b_{12} + d_{22}) = \phi(b_{12}) + \phi(d_{22})$,

(3) $\phi(a_{11} + e_{21}) = \phi(a_{11}) + \phi(e_{21})$,

(4) $\phi(e_{21} + d_{22}) = \phi(e_{21}) + \phi(d_{22})$,

(5) $\phi(a_{11} + d_{22}) = \phi(a_{11}) + \phi(d_{22})$.

LEMMA 4.5. *For* $t_{11}, a_{11} \in \mathcal{G}_{11}$, $b_{12}, c_{12} \in \mathcal{G}_{12}$, $e_{21}, f_{21} \in \mathcal{G}_{21}$, *and* $d_{22} \in \mathcal{G}_{22}$, *the following four equations hold.*

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(1) $\phi(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) = \phi(t_{11}a_{11}b_{12}) + \phi(t_{11}c_{12}d_{22})$,

(2) $\phi(d_{22}e_{21}t_{11} + f_{21}a_{11}t_{11}) = \phi(d_{22}e_{21}t_{11}) + \phi(f_{21}a_{11}t_{11})$,

(3) $\phi(a_{11}b_{12}s_{22} + c_{12}d_{22}s_{22}) = \phi(a_{11}b_{12})s_{22} + \phi(c_{12}d_{22}s_{22})$,

(4) $\phi(s_{22}d_{22}e_{21} + s_{22}f_{21}a_{11}) = \phi(s_{22}d_{22}e_{21}) + \phi(s_{22}f_{21}a_{11})$.

Proof. We first prove (2). By Corollary 4.4, we have

$$\begin{aligned}
&\phi(d_{22}e_{21}t_{11} + f_{21}a_{11}t_{11})\\
&= \phi((d_{22} + f_{21})(e_{21} + a_{11})t_{11} + t_{11}(e_{21} + a_{11})(d_{22} + f_{21}))\\
&= \phi(d_{22} + f_{21})\phi(e_{21} + a_{11})\phi(t_{11}) + \phi(t_{11})\phi(e_{21} + a_{11})\phi(d_{22} + f_{21})\\
&= (\phi(d_{22}) + \phi(f_{21}))(\phi(e_{21}) + \phi(a_{11}))\phi(t_{11}) +\\
&\quad \phi(t_{11})(\phi(e_{21}) + \phi(a_{11}))(\phi(d_{22}) + \phi(f_{21}))\\
&= \phi(d_{22}e_{21}t_{11} + t_{11}e_{21}d_{22}) + \phi(d_{22}a_{11}t_{11} + t_{11}a_{11}d_{22}) +\\
&\quad \phi(f_{21}e_{21}t_{11} + t_{11}e_{21}f_{21}) + \phi(f_{21}a_{11}t_{11} + t_{11}a_{11}f_{21})\\
&= \phi(d_{22}e_{21}t_{11}) + \phi(f_{21}a_{11}t_{11}).
\end{aligned}$$

Other equalities are proven similarly. □

Lemma 4.6. *ϕ is additive on $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$.*

Proof. For arbitrary $a_{21}, b_{21} \in \mathcal{G}_{21}$, we prove $\phi(a_{21} + b_{21}) = \phi(a_{21}) + \phi(b_{21})$. Let $\phi(c) = \phi(a_{21}) + \phi(b_{21})$ for some $c \in \mathcal{G}$. For $s_{22}, t_{22} \in \mathcal{G}_{22}$,

$$\begin{aligned}
\phi(s_{22}t_{22}c + ct_{22}s_{22}) &= \phi(s_{22}t_{22}a_{21} + a_{21}t_{22}s_{22}) + \phi(s_{22}t_{22}b_{21} + b_{21}t_{22}s_{22})\\
&= \phi(s_{22}t_{22}a_{21}) + \phi(s_{22}t_{22}b_{21}).
\end{aligned}$$

Then for $u_{11}, v_{11} \in \mathcal{G}_{11}$, by Lemma 4.5,

$$\begin{aligned}
\phi(s_{22}t_{22}cv_{11}u_{11} + u_{11}v_{11}ct_{22}s_{22}) &= \phi(s_{22}t_{22}a_{21}v_{11}u_{11}) + \phi(s_{22}t_{22}b_{21}v_{11}u_{11})\\
&= \phi(s_{22}t_{22}a_{21}v_{11}u_{11} + s_{22}t_{22}b_{21}v_{11}u_{11}).
\end{aligned}$$

Hence, $u_{11}v_{11}c_{12}t_{22}s_{22} = 0$ and $s_{22}t_{22}cv_{11}u_{11} = s_{22}t_{22}(a_{21} + b_{21})v_{11}u_{11}$; that is, $c_{12} = 0$ and $c_{21} = a_{21} + b_{21}$. Taking $s_{22} \in \mathcal{G}_{22}$ and $t_{21} \in \mathcal{G}_{21}$, it follows that $\phi(s_{22}t_{21}c) = 0$. This implies that $c_{11} = 0$ since N is faithful as a right A-module. Similarly, we can get $c_{22} = 0$. Then ϕ is additive on $\mathcal{G}_{21}$.

It is proven similarly that ϕ is additive on $\mathcal{G}_{12}$. □

Lemma 4.7. *ϕ is additive on $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$.*

Proof. For $a_{11}, b_{11} \in \mathcal{G}_{11}$, let $\phi(c) = \phi(a_{11}) + \phi(b_{11})$ for some $c \in \mathcal{G}$. For $u_{11}, v_{11} \in \mathcal{G}_{11}$ and $s_{22}, t_{22} \in \mathcal{G}_{22}$, computing as in Lemma 4.6, we get $c_{21} = 0$ and

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$c_{12}=0$. Taking $s_{11}\in\mathcal{G}_{11}$ and $t_{21}\in\mathcal{G}_{21}$, one can obtain $c_{22}=0$ easily. Finally, considering $s_{22}\in\mathcal{G}_{22}$ and $t_{12}\in\mathcal{G}_{12}$, it follows from Lemma 4.6 that $c_{11}=a_{11}+b_{11}$. Then we get $c=a_{11}+b_{11}$. Similarly, we can prove ϕ is additive on $\mathcal{G}_{22}$. $\square$

Lemma 4.8. *For $x\in\mathcal{G}$, $a_{11}\in\mathcal{G}_{11}$, and $b_{22}\in\mathcal{G}_{22}$, we have*

(1) $\phi(x+a_{11})=\phi(x)+\phi(a_{11})$,

(2) $\phi(x+b_{22})=\phi(x)+\phi(b_{22})$.

Proof. (1) Let $\phi(c)=\phi(x)+\phi(a_{11})$ for some $c\in\mathcal{G}$. For $u_{11},v_{11}\in\mathcal{G}_{11}$ and $s_{22},t_{22}\in\mathcal{G}_{22}$, computing as in Lemma 4.6, we have $c_{12}=x_{12}$ and $c_{21}=x_{21}$. Furthermore, for $u_{12}\in\mathcal{G}_{12}$ and $s_{22}\in\mathcal{G}_{22}$, it follows that

$$\begin{aligned}\phi(cu_{12}s_{22}) &= \phi(xu_{12}s_{22})+\phi(a_{11}u_{12}s_{22})\\ &= \phi(x_{11}u_{12}s_{22}+x_{21}u_{12}s_{22})+\phi(a_{11}u_{12}s_{22})\\ &= \phi(x_{11}u_{12}s_{22}+x_{21}u_{12}s_{22}+a_{11}u_{12}s_{22}).\end{aligned}$$

Note that we apply Corollary 4.4 in the last equality. Thus, $c_{11}=x_{11}+a_{11}$. We can get $c_{22}=x_{22}$ similarly.

(2) is proven similarly. $\square$

Lemma 4.9. *The equality $\phi(a_{11}b_{12}e_{22}+c_{21}d_{11}a_{11})=\phi(a_{11}b_{12}e_{22})+\phi(c_{21}d_{11}a_{11})$ holds for all $a_{11},d_{11}\in\mathcal{G}_{11}$, $b_{12}\in\mathcal{G}_{12}$, $c_{21}\in\mathcal{G}_{21}$, and $e_{22}\in\mathcal{G}_{22}$.*

Proof. On one hand, by Lemmas 4.7 and 4.8 we have

$$\begin{aligned}&\phi(a_{11}b_{12}e_{22}+c_{21}d_{11}a_{11}+a_{11}d_{11}a_{11}+a_{11}b_{12}c_{21}+a_{11}d_{11}a_{11}+c_{21}b_{12}e_{22})\\ &=\phi(a_{11}b_{12}e_{22}+c_{21}d_{11}a_{11}+a_{11}d_{11}a_{11}+a_{11}b_{12}c_{21}+a_{11}d_{11}a_{11})+\phi(c_{21}b_{12}e_{22})\\ &=\phi(a_{11}b_{12}e_{22}+c_{21}d_{11}a_{11})+\\ &\quad\phi(a_{11}d_{11}a_{11}+a_{11}b_{12}c_{21}+a_{11}d_{11}a_{11})+\phi(c_{21}b_{12}e_{22})\\ &=\phi(a_{11}b_{12}e_{22}+c_{21}d_{11}a_{11})+\\ &\quad\phi(a_{11}d_{11}a_{11})+\phi(a_{11}b_{12}c_{21})+\phi(a_{11}d_{11}a_{11})+\phi(c_{21}b_{12}e_{22}).\end{aligned}$$

On the other hand, by Lemma 4.7,

$$\begin{aligned}&\phi(a_{11}b_{12}e_{22}+c_{21}d_{11}a_{11}+a_{11}d_{11}a_{11}+a_{11}b_{12}c_{21}+a_{11}d_{11}a_{11}+c_{21}b_{12}e_{22})\\ &=\phi((a_{11}+c_{21})(b_{12}+d_{11})(a_{11}+e_{22})+(a_{11}+e_{22})(b_{12}+d_{11})(a_{11}+c_{21}))\\ &=\phi(a_{11}+c_{21})\phi(b_{12}+d_{11})\phi(a_{11}+e_{22})+\phi(a_{11}+e_{22})\phi(b_{12}+d_{11})\phi(a_{11}+c_{21})\\ &=(\phi(a_{11})+\phi(c_{21}))(\phi(b_{12})+\phi(d_{11}))(\phi(a_{11})+\phi(e_{22}))+\\ &\quad(\phi(a_{11})+\phi(e_{22}))(\phi(b_{12})+\phi(d_{11}))(\phi(a_{11})+\phi(c_{21}))\\ &=\phi(a_{11}b_{12}e_{22})+\phi(c_{21}d_{11}a_{11})+\\ &\quad\phi(a_{11}d_{11}a_{11})+\phi(a_{11}b_{12}c_{21})+\phi(a_{11}d_{11}a_{11})+\phi(c_{21}b_{12}e_{22}).\end{aligned}$$

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Therefore, $\phi(a_{11}b_{12}e_{22} + c_{21}d_{11}a_{11}) = \phi(a_{11}b_{12}e_{22}) + \phi(c_{21}d_{11}a_{11})$. □

Proof of Theorem 4.1.

Step 1. Let $a_{11} \in \mathcal{G}_{11}$, $b_{12} \in \mathcal{G}_{12}$, $c_{21} \in \mathcal{G}_{21}$, and $d_{22} \in \mathcal{G}_{22}$. Then $\phi(a_{11} + b_{12} + c_{21} + d_{22}) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$.

Let $\phi(f) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{21}) + \phi(d_{22})$ with $f \in \mathcal{G}$. For $u_{11}, v_{11} \in \mathcal{G}_{11}$ and $s_{22}, t_{22} \in \mathcal{G}_{22}$, it follows by computing as in Lemma 4.6 that,

$$\phi(u_{11}v_{11}ft_{22}s_{22} + s_{22}t_{22}fv_{11}u_{11}) = \phi(u_{11}v_{11}b_{12}t_{22}s_{22}) + \phi(s_{22}t_{22}c_{21}v_{11}u_{11}).$$

Now by Lemma 4.9, we get

$$u_{11}v_{11}f_{12}t_{22}s_{22} + s_{22}t_{22}f_{21}v_{11}u_{11} = u_{11}v_{11}b_{12}t_{22}s_{22} + s_{22}t_{22}c_{21}v_{11}u_{11}.$$

This implies that $f_{12} = b_{12}$ and $f_{21} = c_{21}$. By considering $u_{11} \in \mathcal{G}_{11}$ and $w_{21} \in \mathcal{G}_{21}$, it follows from Corollary 4.4 that $f_{22} = d_{22}$. Similarly, $f_{11} = a_{11}$.

Step 2. $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in \mathcal{G}$.

This follows directly from Lemmas 4.6, 4.7, and Step 1. □

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