2011

Linear maps preserving the idempotency of Jordan products of operators

Li Fang
LINEAR MAPS PRESERVING THE IDEMPOTENCY OF JORDAN PRODUCTS OF OPERATORS∗

LI FANG†

Abstract. Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space $X$ and let $I^*(X)$ be the set of non-zero idempotent operators in $B(X)$. A surjective map $\varphi : B(X) \to B(X)$ preserves nonzero idempotency of the Jordan products of two operators if for every pair $A, B \in B(X)$, the relation $AB + BA \in I^*(X)$ implies $\varphi(A)\varphi(B) + \varphi(B)\varphi(A) \in I^*(X)$. In this paper, the structures of linear surjective maps on $B(X)$ preserving the nonzero idempotency of Jordan products of two operators are given.

Key words. Banach space, Preserver, Idempotent, Jordan product.

AMS subject classifications. 47B49.

1. Introduction. This paper is a continuation of our recent work on preserver problems concerning certain properties of products or triple Jordan products of operators [3, 4, 13], and the related works in [2] and [7].

Let $X$ be a complex Banach space, and let $B(X)$ be the algebra of all bounded linear operators on $X$. The dual of $X$ is denoted by $X'$ and the adjoint of $T \in B(X)$ by $T'$. Let $I^*(X)$, $I_1(X)$ and $N_1(X)$ be the set of nonzero idempotent operators, the set of rank-one idempotent operators and the set of rank-one nilpotent operators in $B(X)$, respectively. If $X$ has dimension $n$ with $2 \leq n < \infty$, then $B(X)$ is identified with the algebra $M_n$ of $n \times n$ complex matrices and $I_n(X)$ refers to the set of idempotent matrices in $M_n$. For an operator $T \in B(X)$, the range, the kernel and the rank of $T$ are denoted by $R(T)$, $N(T)$ and rank $T$, respectively. Let $F(X)$ and $F_1(X)$ denote the set of finite rank operators and the set of rank-one operators in $B(X)$, respectively.

For a non-zero vector $x \in X$ and a non-zero $f \in X'$, we denote by $x \otimes f$ the rank one operator defined by $(x \otimes f)y = f(y)x$, $y \in X$. Note that every bounded linear rank one operator on $X$ can be written in this form. The rank-one operator $x \otimes f$ is an idempotent operator if and only if $f(x) = 1$, and $x \otimes f$ is a nilpotent operator if and only if $f(x) = 0$. Given $P, Q \in I(X)$, $P \leq Q$ if $PQ = QP = P$ and $P < Q$ if $P \leq Q$ with $P \neq Q$.

∗Received by the editors on November 7, 2010. Accepted for publication on July 2, 2011. Handling Editor: Bit-Shun Tam.

†Department of Mathematics, Northwest University, Xi’an 710127, China (fangli@nwu.edu.cn). Supported by the National Natural Science Foundation of China (No. 11026134) and the Special Research Project of Educational Department of Shaanxi Province (No. 09JK741).
In this paper, we are interested in determining the structure of linear surjective maps \( \varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X}) \) for every pair \( A, B \in \mathcal{B}(\mathcal{X}) \) having the property that

\[
AB + BA \in \mathcal{I}^*(\mathcal{X}) \Rightarrow \varphi(A)\varphi(B) + \varphi(B)\varphi(A) \in \mathcal{I}^*(\mathcal{X}).
\]

We will derive the following two theorems regarding the structure.

**Theorem 1.1.** Let \( \varphi \) be a linear map on \( \mathcal{M}_n \) with \( n \geq 3 \). Then \( \varphi \) preserves the nonzero idempotency of Jordan products of two operators if and only if there exist an invertible matrix \( A \in \mathcal{M}_n \) and a constant \( \lambda \in \{1, -1\} \) such that one of the following holds.

1. \( \varphi(X) = \lambda AXA^{-1} \) for all \( X \in \mathcal{M}_n \);
2. \( \varphi(X) = \lambda AX^tA^{-1} \) for all \( X \in \mathcal{M}_n \), where \( X^t \) is the transpose of \( X \).

**Theorem 1.2.** Let \( \mathcal{X} \) be a complex infinite dimensional Banach space and let \( \varphi \) be a linear surjective map on \( \mathcal{B}(\mathcal{X}) \). Then \( \varphi \) preserves the nonzero idempotency of Jordan products of two operators if and only if there exist a bounded invertible linear or conjugate-linear operator \( A : \mathcal{X} \to \mathcal{X} \) and a constant \( \lambda \in \{1, -1\} \) such that

\[
\varphi(X) = \lambda AXA^{-1} \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{X}),
\]

or, only if \( \mathcal{X} \) is reflexive, there exist a bounded invertible linear or conjugate-linear operator \( A : \mathcal{X}^\prime \to \mathcal{X} \) and a constant \( \lambda \in \{1, -1\} \) such that

\[
\varphi(X) = \lambda AX^tA^{-1} \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{X}).
\]

**2. Preliminary results.** Assume that \( \mathcal{X} \) is a complex Banach space with dimension at least 3. In this section, we introduce some elementary results that will be used in the proofs of main theorems.

**Definition 2.1.** [11] Let \( U \) and \( V \) be vector spaces over a filed \( F \). Linear operators \( T_1, \ldots, T_n : U \to V \) are locally linearly dependent if \( T_1u, \ldots, T_nu \) are linearly dependent for every \( u \in U \).

**Lemma 2.2.** [11] Assume that the operators \( A, B, I \) are locally linearly dependent. Then there exist scalars \( \lambda \) and \( \mu \) such that \( (A - \lambda)(B - \mu) = 0 \) and either \( (A - \lambda)^2 = 0 \) or \( (B - \mu)^2 = 0 \).

**Lemma 2.3.** Let \( P, Q \in \mathcal{B}(\mathcal{X}) \) be two idempotent operators. Then \( zP + (1 - z)Q \in \mathcal{I}^*(\mathcal{X}) \) for any \( z \in \mathbb{C} \setminus \{0, 1\} \) if and only if \( P + Q = PQ + QP \).

**Proof.** We obtain the desired result by direct calculation. \( \Box \)
Lemma 2.4. Let \( A \in \mathcal{B}(\mathcal{X}) \) and \( x \otimes f \in \mathcal{B}(\mathcal{X}) \) be with \( f(x) \neq 0 \). Then \( Ax \otimes f + x \otimes fA \in \mathcal{I}^*(\mathcal{X}) \) if and only if \( Ax = \frac{1}{2f(x)}x \) or \( A'f = \frac{1}{2f(x)}f \).

Proof. Note that \( Ax \otimes f + x \otimes fA \in \mathcal{I}^*(\mathcal{X}) \) implies that \( Ax \otimes [f(Ax)f + f(x)A'f - f] + x \otimes [f(A^2x)f + f(Ax)A'f - A'f] = 0 \). If \( Ax \) and \( x \) as well as \( A'f \) and \( f \) are linearly independent, then \( f(Ax) = 1, f(x) = 0 \) and \( f(A^2x) = 0 \). But this contradicts the hypothesis \( f(x) \neq 0 \). Hence, either \( Ax \) and \( x \) are linearly dependent or \( A'f \) and \( f \) are linearly dependent. Then we can compute that \( Ax = \frac{1}{2f(x)}x \) or \( A'f = \frac{1}{2f(x)}f \).

The idea of the following lemma comes from [10].

Lemma 2.5. Let \( A \in \mathcal{B}(\mathcal{X}) \) be any non-scalar operator and let \( \alpha \) be any fixed complex number. Then there exists an idempotent operator \( P \in \mathcal{B}(\mathcal{X}) \) of rank one such that \( \alpha \) is an eigenvalue of \( A + 2P \).

Proof. As \( A \) is a non-scalar operator, we can find \( x \in \mathcal{X} \) such that \( x \) and \( Ax \) are linearly independent. Define \( P \in \mathcal{B}(\mathcal{X}) \) by

\[
P x = \frac{\alpha}{2}x - \frac{1}{2}Ax, \quad PAx = \alpha\left(\frac{\alpha}{2} - 1\right)x - \left(1 - \frac{\alpha}{2}\right)Ax \quad \text{and} \quad Pz = 0
\]

for every \( z \in \mathcal{X} \otimes [x, Ax] \). Clearly, \( P \) is an idempotent operator of rank one and \( (A + 2P)x = \alpha x \).

3. Main results. Assume that \( \mathcal{X} \) is a complex Banach space with dimension at least 3 and we consider a linear surjective map \( \varphi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X}) \) preserving the nonzero idempotency of Jordan products of operators, that is, \( \varphi(A)\varphi(B) + \varphi(B)\varphi(A) \in \mathcal{I}^*(\mathcal{X}) \) whenever \( AB + BA \in \mathcal{I}^*(\mathcal{X}) \) for every pair \( A, B \in \mathcal{B}(\mathcal{X}) \).

Lemma 3.1. Let \( \varphi \) be as above. Then \( \varphi \) is injective.

Proof. Assume that \( \varphi(A) = 0 \) for some non-zero operator \( A \in \mathcal{B}(\mathcal{X}) \). Suppose that there exists an \( x \in \mathcal{X} \) such that \( x \), \( Ax \) and \( A^2x \) are linearly independent. Then there is an \( f \in \mathcal{X}' \) such that \( f(x) = f(A^2x) = 0 \) and \( f(Ax) = 1 \). It follows that \( Ax \otimes f + x \otimes fA \in \mathcal{I}^*(\mathcal{X}) \). But, \( \varphi(A)\varphi(x \otimes f) + \varphi(x \otimes f)\varphi(A) = 0 \notin \mathcal{I}^*(\mathcal{X}) \). This contradicts implies that \( x \), \( Ax \) and \( A^2x \) are linearly dependent for every \( x \in \mathcal{X} \).

By Lemma 2.2, there exist scalars \( \lambda \) and \( \mu \) such that \( (A - \lambda)(A^2 - \mu) = 0 \) and either \( (A - \lambda)^2 = 0 \) or \( (A^2 - \mu)^2 = 0 \).

If \( (A - \lambda)(A^2 - \mu) = 0 \) and \( (A - \lambda)^2 = 0 \), then \( (\mu - \lambda^2)A = \lambda(\mu - \lambda^2) \). When \( \mu - \lambda^2 \neq 0 \), we get that \( A = \lambda I \). For any pair \( x_1 \in \mathcal{X} \) and \( f_1 \in \mathcal{X}' \) with \( f_1(x_1) = \frac{1}{x_1} \), we have \( Ax_1 \otimes f_1 + x_1 \otimes f_1A \in \mathcal{I}^*(\mathcal{X}) \) and then \( \varphi(A)\varphi(x_1 \otimes f_1) + \varphi(x_1 \otimes f_1)\varphi(A) \in \mathcal{I}^*(\mathcal{X}) \). However, this contradicts \( \varphi(A)\varphi(x_1 \otimes f_1) + \varphi(x_1 \otimes f_1)\varphi(A) = 0 \). When \( \mu - \lambda^2 = 0 \) and \( \lambda \neq 0 \), we know that \( A - \lambda I \) is a nilpotent operator and there is a non-zero vector \( x_2 \in \mathcal{X} \) such that \( Ax_2 = \lambda x_2 \). Selecting \( f_2 \in \mathcal{X}' \) with \( f_2(x_2) = \frac{1}{x_2} \), we get \( Ax_2 \otimes f_2 + x_2 \otimes f_2A \in \mathcal{I}^*(\mathcal{X}) \), which implies \( \varphi(A)\varphi(x_2 \otimes f_2) + \varphi(x_2 \otimes f_2)\varphi(A) \in \mathcal{I}^*(\mathcal{X}) \).
But $\varphi(A)\varphi(x_2 \otimes f_2) + \varphi(x_2 \otimes f_2)\varphi(A) = 0$ and we get a contradiction. When $\mu - \lambda^2 = 0$ and $\lambda = 0$, we know that $A^2 = 0$ and there exists a non-zero vector $x_3 \in \mathcal{X}$ such that $x_3$ and $Ax_3$ are linearly independent. Then there is an $f_3 \in \mathcal{X}'$ with $f_3(x_3) = 0$ and $f_3(Ax_3) = 1$. Hence, $Ax_3 \otimes f_3 + x_3 \otimes f_3A \in I'(\mathcal{X})$ and so $\varphi(A)\varphi(x_3 \otimes f_3) + \varphi(x_3 \otimes f_3)\varphi(A) \in I'(\mathcal{X})$. However, this contradicts $\varphi(A)\varphi(x_3 \otimes f_3) + \varphi(x_3 \otimes f_3)\varphi(A) = 0$.

If $(A - \lambda)(A^2 - \mu) = 0$ and $(A^2 - \mu)^2 = 0$, then $(\mu - \lambda^2)A^2 = \lambda(\mu - \lambda^2)$. When $\mu - \lambda^2 \neq 0$, we know that $A^2 = \mu I$. Suppose $\mu \neq 0$. First, we assume that $A$ is a non-scalar operator. Then there is an $x_4 \in \mathcal{X}$ such that $x_4$ and $Ax_4$ are linearly independent. We can find $f_4 \in \mathcal{X}'$ with $f_4(x_4) = \frac{1}{2\mu}$ and $f_4(Ax_4) = 0$. It follows $A(Ax_4) \otimes f_4 + (Ax_4) \otimes f_4A \in I'(\mathcal{X})$, which implies $\varphi(A)\varphi(Ax_4 \otimes f_4) + \varphi(Ax_4 \otimes f_4)\varphi(A) = 0 \in I'(\mathcal{X})$. This gives a contradiction. Second, we assume $A = \beta I$ for some non-zero number $\beta$. For any pair $x_5 \in \mathcal{X}$ and $f_5 \in \mathcal{X}'$ with $f_5(x_5) = \frac{1}{2\beta}$, we have $Ax_5 \otimes f_5 + x_5 \otimes f_5A \in I'(\mathcal{X})$ and so $\varphi(A)\varphi(x_5 \otimes f_5) + \varphi(x_5 \otimes f_5)\varphi(A) = 0 \in I'(\mathcal{X})$. This also gives a contradiction. Hence, $\mu = 0$ and then $A^2 = 0$. Thus, there exists a non-zero vector $x_6 \in \mathcal{X}$ such that $x_6$ and $Ax_6$ are linearly independent. So, there is $f_6 \in \mathcal{X}'$ with $f_6(x_6) = 0$ and $f_6(Ax_6) = 0$. Obviously, $Ax_6 \otimes f_6 + x_6 \otimes f_6A \in I'(\mathcal{X})$ and then $\varphi(A)\varphi(x_6 \otimes f_6) + \varphi(x_6 \otimes f_6)\varphi(A) \in I'(\mathcal{X})$. However, $\varphi(A)\varphi(x_6 \otimes f_6) + \varphi(x_6 \otimes f_6)\varphi(A) = 0$ and this is also a contradiction. When $\mu - \lambda^2 = 0$, we know that $A^3 = -\lambda A^2 - \lambda^2 A + \lambda^3 = 0$. Suppose $\lambda \neq 0$, we get $\frac{1}{2\lambda^2}((A^2 - \lambda A - \lambda^2 I)A + A(A^2 - \lambda A - \lambda^2 I)) = I \in I'(\mathcal{X})$ implies $\frac{1}{2\lambda^2}(\varphi(A^2 - \lambda A - \lambda^2 I)\varphi(A) + \varphi(A)\varphi(A^2 - \lambda A - \lambda^2 I)) = 0 \in I'(\mathcal{X})$. This is also a contradiction. Suppose $\lambda = 0$, we know $A^3 = 0$. If $A^2 \neq 0$, then there is an $x_7 \in \mathcal{X}$ such that $Ax_7$ and $A^2x_7$ are linearly independent. So, there exists an $f_7 \in \mathcal{X}'$ such that $f_7(Ax_7) = 0$ and $f_7(A^2x_7) = 1$. Hence, $AAx_7 \otimes f_7 + Ax_7 \otimes f_7A \in I'(\mathcal{X})$ and then $\varphi(A)\varphi(Ax_7 \otimes f_7) + \varphi(x_7 \otimes f_7)\varphi(A) = 0 \in I'(\mathcal{X})$. This is also a contradiction. If $A^2 = 0$, then we can derive another contradiction by using a routine argument demonstrated above.

Thus, $A = 0$. Therefore, $\varphi$ is injective. []

**Lemma 3.2.** Let $N \in \mathcal{B}(\mathcal{X})$ be of finite rank and $N^2 = 0$. Then $\varphi(N)$ is a nilpotent operator.

**Proof.** Note that $R(N)$ is finite dimensional. It is known that finite dimensional subspaces of a Banach space are complemented, and so $\mathcal{X} = R(N) + \mathcal{M}$ for some closed subspace $\mathcal{M}$ of $\mathcal{X}$ where $\mathcal{M}$ is a complementary subspace of $R(N)$. Then $N$ has the following operator matrix

$$N = \begin{pmatrix} 0 & N_1 \\ 0 & 0 \end{pmatrix}.$$ 

Putting $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, we have that $(\frac{1}{2}(P + zN))(P + zN) + (P + zN)(\frac{1}{2}(P + zN)) = P + zN \in I'(\mathcal{X})$ for all $z \in \mathbb{C}$. Thus, $(\frac{1}{2}\varphi(P + zN))\varphi(P + zN) + \varphi(P + zN)(\frac{1}{2}\varphi(P + zN)) = \frac{1}{2}\varphi(P + zN)$ for all $z \in \mathbb{C}$.
zN)) = (ϕ(P) + zϕ(N))² ∈ ℐ'(X) for all z ∈ ℂ. Hence, (ϕ(P) + zϕ(N))² = (ϕ(P) + zϕ(N))⁴ for all z ∈ ℂ. That is,

(ϕ(P))² + z(ϕ(P)ϕ(N) + ϕ(N)ϕ(P)) + z²(ϕ(N))² = (ϕ(P))⁴ + ⋯ + z⁴(ϕ(N))⁴

for all z ∈ ℂ. So, (ϕ(N))⁴ = 0. □

**Lemma 3.3.** If ϕ is surjective, then ϕ(I) = λI for some constant λ ∈ {1, −1}.

**Proof.** Since ϕ is a surjective map, there exists a non-zero operator A ∈ ℬ(X) such that ϕ(A) = I. Assume that A is a non-scalar operator. If there exists an x ∈ X such that x, Ax and A²x are linearly independent, then there is an f ∈ ℐ' such that f(x) = f(A²x) = 0 and f(Ax) = 1. So, Ax ⊗ f + x ⊗ fA ∈ ℐ'(X), which implies that ϕ(A)ϕ(x ⊗ f) + ϕ(x ⊗ f)ϕ(A) = 2ϕ(x ⊗ f) ∈ ℐ'(X). But 2ϕ(x ⊗ f) is a nilpotent operator by Lemma 3.2 and this contradiction implies that x, Ax and A²x are linearly independent for every x ∈ X. Then there exist scalars λ and µ such that (A − λ)(A² − µ) = 0 and either (A − λ)² = 0 or (A² − µ)² = 0 by Lemma 2.2.

If (A - λ)(A² - µ) = 0 and (A - λ)² = 0, then (µ - λ²)A = λ(µ - λ²). Since A is a non-scalar operator, we know that µ = λ². When λ = 0, we get that A² = 0 and there exist x₁ ∈ X and f₁ ∈ ℐ such that f₁(x₁) = 0 and f₁(Ax₁) = 1. So, A(x₁ ⊗ f₁) + (x₁ ⊗ f₁)A ∈ ℐ'(X) implies ϕ(A)ϕ(x₁ ⊗ f₁) + ϕ(x₁ ⊗ f₁)ϕ(A) = 2ϕ(x₁ ⊗ f₁) ∈ ℐ'(X). However, 2ϕ(x₁ ⊗ f₁) is a nilpotent operator by Lemma 3.2. This contradiction implies that λ ≠ 0 and A − λ is a nilpotent operator. Then there is a non-zero vector x₂ ∈ X such that (A − λ)x₂ = 0. For every f₂ ∈ ℐ with f₂(x₂) = 1, we know that Ax₂ ⊗ f₂ + x₂ ⊗ f₂A ∈ ℐ'(X), which implies that ϕ(A)ϕ(x₂ ⊗ f₂) + ϕ(x₂ ⊗ f₂)ϕ(A) = 2ϕ(x₂ ⊗ f₂) ∈ ℐ'(X). Moreover, 2λϕ(x₂ ⊗ f₂) ∈ ℐ'(X) follows from 2λx₂ ⊗ f₂ ∈ ℐ'(X). Hence, λ² = 1. When λ = 1, we know that A² = 2A + I = 0 and so (2I − A)² + (2I − A) = I ∈ ℐ'(X). Then, ϕ(2I − A)ϕ(2I − A) = 2ϕ(I) = 1 ∈ ℐ'(X). Setting 2ϕ(I) − I = R ∈ ℐ'(X), we get that ϕ(I) = \( \frac{1 - R}{2} \). However, \( I + \frac{1}{2}I = I \) ∈ ℐ'(X) implies ϕ(I)² = ϕ(I)X. Thus, ϕ(I)² = \( \frac{1 - 2R}{4} \) and so R = I. It follows that ϕ(I) = I = ϕ(A) and then A = I by the injection of ϕ. This is a contradiction. Similarly we can also obtain a contradiction when λ = -1.

If (A - λ)(A² - µ) = 0 and (A² - µ)² = 0, then (µ - λ²)A² = λ(µ - λ²). When µ - λ² ≠ 0, we get A² = µI. When λ ≠ 0, there is a vector x₄ ∈ X such that x₄ and Ax₄ are linearly independent. Selecting f₄ ∈ ℐ' with f₄(x₁) = 1 and f₄(Ax₄) = 0, we know that AAx₄ ⊗ f₄ + Ax₄ ⊗ f₄A ∈ ℐ'(X) and so 2ϕ(Ax₄ ⊗ f₄) ∈ ℐ'(X). However, ϕ(Ax₄ ⊗ f₄) is a nilpotent operator. This contradiction shows that ϕ(Ax₄ ⊗ f₄) = 0 and then A² = 0. Since x₄ and Ax₄ are linearly independent, there is an f₅ ∈ ℐ' with f₅(x₁) = 0 and f₅(Ax₄) = 1. So, Ax₄ ⊗ f₅ + x₄ ⊗ f₅A ∈ ℐ'(X) and then 2ϕ(x₄ ⊗ f₅) ∈ ℐ'(X). But this contradicts with that ϕ(x₄ ⊗ f₅) is a nilpotent operator. When µ - λ² = 0 and λ = 0, we know A² = 0. For the case that A² = 0, we also get
a contradiction using a similar argument above. For the case that $A^2 \neq 0$, there is an $x_6 \in \mathcal{X}$ such that $Ax_6$ and $A^2x_6$ are linearly independent. Selecting $f_6 \in \mathcal{X}$ such that $f_6(Ax_6) = 0$ and $f_6(A^2x_6) = 1$, we know $A Ax_6 \otimes f_6 + Ax_6 \otimes f_6 A \in \mathcal{T}^*(\mathcal{X})$ and then $2\phi(A Ax_6 \otimes f_6) \in \mathcal{T}^*(\mathcal{X})$. This contradicts the fact that $\phi(A Ax_6 \otimes f_6)$ is a nilpotent operator. So, $\lambda \neq 0$ and $A^2 - \lambda^2$ is a nilpotent operator. Hence, there is an $x_7 \in \mathcal{X}$ such that $A^2x_7 = \lambda^2x_7$. If $x_7$ and $Ax_7$ are linearly independent, then there is an $f_7 \in \mathcal{X}$ such that $f_7(x_7) = \frac{1}{\lambda}$ and $f_7(Ax_7) = 0$. It follows that $\phi(Ax_7 \otimes f_7) \in \mathcal{T}^*(\mathcal{X})$ which gives that $2\phi(Ax_7 \otimes f_7)$ is a nilpotent operator. While $\phi(Ax_7 \otimes f_7)$ is a nilpotent operator, we get a contradiction. If $Ax_7 = \alpha x_7$ for some $\alpha \in \mathbb{C}$, then $\alpha^2 = \lambda^2$. Selecting $f_5 \in \mathcal{X}$ such that $f_5(x_7) = \frac{1}{\alpha}$, we get that $f_5(Ax_7) = \frac{1}{\alpha x_7}$. By direct calculation, we know that $(2\lambda^2(Ax_7 \otimes f_5))(Ax_7 \otimes f_5) + (Ax_7 \otimes f_5)(2\lambda^2(Ax_7 \otimes f_5)) \in \mathcal{T}^*(\mathcal{X})$ and so $4\lambda^2\phi(Ax_7 \otimes f_5)^2 \in \mathcal{T}^*(\mathcal{X})$. Moreover, $A Ax_7 \otimes f_5 + Ax_7 \otimes f_5 A \in \mathcal{T}^*(\mathcal{X})$ implies that $2\phi(Ax_7 \otimes f_5) \in \mathcal{T}^*(\mathcal{X})$. It follows that $\lambda^2 = 1$. Since $I = 2A^2 - A^4 = (2I - A^2)(\frac{A^2}{2}) + (\frac{A^2}{2})(2I - A^2) = (2A^2 - A^4)(\frac{A^2}{2}) + (\frac{A^2}{2})(2A - A^4)$, we have that $\phi(2I - A^2)\phi(\frac{A^2}{2}) + \phi(\frac{A^2}{2})\phi(2I - A^2) \in \mathcal{T}^*(\mathcal{X})$ and $2I - \phi(A^3) \in \mathcal{T}^*(\mathcal{X})$. On the one hand, $\phi(2I - A^2)\phi(\frac{A^2}{2}) + \phi(\frac{A^2}{2})\phi(2I - A^2) = \phi(I)\phi(A^2) + \phi(A^2)\phi(I) - \phi(A^2)^2 \in \mathcal{T}^*(\mathcal{X})$. On the other hand, $A^3 - A + \lambda A^2 + \lambda = 0$ implies $I = \frac{1}{A}A + A^2 - \frac{1}{A}A^3 = (\frac{1}{A}I + A - \frac{1}{A}A^2)(\frac{A^2}{2}) + (\frac{A^2}{2})(\frac{1}{A}I + A - \frac{1}{A}A^2)$. Hence, $\frac{1}{A}[\phi(A^2) - \phi(I)] = I - E$ for some $E \in \mathcal{T}^*(\mathcal{X})$. It gives $\phi(A)^3 - \phi(A^2)\phi(I)\phi(A) + \phi(A^2)\phi(I) = \phi(I)^2 = I - E$. Furthermore, $\phi(A^3) - \phi(I)\phi(A^2) + \phi(A^2)\phi(I) = \phi(I)^2 = I - E$ and $\phi(A^3) - 2\phi(A^3) = -E$. However, $(2I - \phi(A^3))^2 = 4I - 4\phi(A^3) + \phi(A^3)^2 = 2I - \phi(A^3)$ implies $\phi(A^3) = 2I - E$. So, $\frac{1}{A}[\phi(A^2) - \phi(I)] = I - E = I + \phi(A^3)$ and then $\phi(\lambda A^2 + \lambda A - A^2 + I) = 0$. Using the fact that $\phi$ is a bijection, we get $\lambda A^3 + \lambda A - A^2 + E = 0$. Noting that $A^3 - A - \lambda A^2 + \lambda = 0$, we know $\lambda = 0$ and this is a contradiction.

Therefore, $A = \mu I$ for some non-zero complex number $\mu$. We thus have that $\phi(I) = \mu I$ for some constant $\mu \in \mathbb{C}$ and $\mu^2 = 1$. 

Next we assume that $\phi$ is surjective and $\phi(I) = I$. We may replace $\phi$ by $-\phi$ if $\phi(I) = -I$.

**Lemma 3.4.** $\phi$ has the following properties:

1. $\phi(\mathcal{T}^*(\mathcal{X})) \subseteq \mathcal{T}^*(\mathcal{X})$;
2. $\phi$ preserves the orthogonality of idempotents;
3. $\phi$ preserves the order of idempotents.

**Proof.** (1) It follows directly from $\phi(I) = I$, $\phi(0) = 0$ and the fact that $(\frac{1}{2}P)I + I(\frac{1}{2}P) = P$ for any $P \in \mathcal{T}^*(\mathcal{X})$.

(2) If $P, Q \in \mathcal{T}^*(\mathcal{X})$ and $P \perp Q$, then $P + Q \in \mathcal{T}^*(\mathcal{X})$. So, $\phi(P + Q) = \phi(P) + \phi(Q) \in \mathcal{T}^*(\mathcal{X})$ by (1). Since $\phi(P), \phi(Q) \in \mathcal{T}^*(\mathcal{X})$, we know that $\phi(P) \perp \phi(Q)$. 


(3) Let \( P, Q \in \mathcal{I}^*(\mathcal{X}) \) and \( P < Q \). Then \( PQ = QP = P \) and \( Q = P + (Q - P) \). Clearly, \( P \in \mathcal{I}^*(\mathcal{X}) \) and \( Q - P \in \mathcal{I}^*(\mathcal{X}) \). Thus, \( \varphi(Q) - \varphi(P) \in \mathcal{I}^*(\mathcal{X}) \), and we get \( 2\varphi(P) = \varphi(Q)\varphi(P) + \varphi(P)\varphi(Q) \). This implies that \( \varphi(Q)\varphi(P) = \varphi(P)\varphi(Q) = \varphi(P) \). \( \Box 

\text{Lemma 3.5.} \, \varphi(\mathcal{I}_1(\mathcal{X})) \subseteq \mathcal{I}_1(\mathcal{X}). 

\text{Proof.} \, \text{Let} \, P = x \otimes f \in \mathcal{I}_1(\mathcal{X}) \, \text{for some} \, x \in \mathcal{X} \, \text{and} \, f \in \mathcal{X}' \, \text{with} \, f(x) = 1. \, \text{Then} \, \varphi(P) \in \mathcal{I}^*(\mathcal{X}) \, \text{by Lemma 3.4.} \, \text{Assume that} \, \text{rank} \, \varphi(P) \geq 2. \, \text{Then there exists a} \, R \in \mathcal{I}_1(\mathcal{X}) \, \text{such that} \, R < \varphi(P) \, \text{and so} \, \varphi(P) - R \in \mathcal{I}^*(\mathcal{X}). \, \text{Since} \, \varphi \, \text{is bijective, there is a non-zero operator} \, B \in \mathcal{B}(\mathcal{X}) \, \text{such that} \, R = \varphi(B).

\text{If} \, x, \, Bx \, \text{and} \, B^2x \, \text{are linearly independent, there is} \, g \in \mathcal{X}' \, \text{such that} \, g(Bx) = 1 - f(Bx), \, g(B^2x) = 1 - f(B^2x) \, \text{and} \, g(x) = -1. \, \text{Then} \, Bx \otimes (f+g) + x \otimes (f+g)B \in \mathcal{I}^*(\mathcal{X}) \, \text{implies} \, \varphi(B)\varphi((x \otimes (f+g)) + \varphi((x \otimes (f+g)))\varphi(B) = 2R + R\varphi(x \otimes g) + \varphi(x \otimes g)R \in \mathcal{I}^*(\mathcal{X}). \, \text{On the one hand,} \, -x \otimes g \in \mathcal{I}^*(\mathcal{X}) \, \text{implies} \, -\varphi(x \otimes g) \in \mathcal{I}^*(\mathcal{X}), \, \text{and} \, zx \otimes f(-z)(-z)g \cdot x \otimes f \in \mathcal{I}^*(\mathcal{X}) \, \text{implies} \, z\varphi((x \otimes f)\varphi((-z)(-z)g) + (1-z)\varphi((-z)(-z)g)\varphi(x \otimes f) \in \mathcal{I}^*(\mathcal{X}) \, \text{for all} \, z \in \mathbb{C}. \, \text{On the other hand,} \, zx \otimes f \cdot (-z)(-z)g + (1-z)(-z)(-z)g \cdot x \otimes f = z(x \otimes g) + (1-z)x \otimes f \in \mathcal{I}^*(\mathcal{X}) \, \text{implies} \, z\varphi((x \otimes g) + (1-z)\varphi((x \otimes f) \in \mathcal{I}^*(\mathcal{X}) \, \text{for all} \, z \in \mathbb{C}. \, \text{Hence,} \, \varphi((x \otimes g)\varphi(x \otimes f), \, \varphi(x \otimes f)\varphi((x \otimes g) \in \mathcal{I}^*(\mathcal{X}) \, \text{and} \\

\varphi(x \otimes f)\varphi((-x)(-z)g)\varphi(x \otimes f) + \varphi((-x) \otimes g)\varphi(x \otimes f)\varphi((-x) \otimes g) \\
= \varphi(x \otimes f)\varphi((-x) \otimes g) + \varphi((-x) \otimes g)\varphi(x \otimes f) \\
= \varphi(x \otimes f) + \varphi((-x) \otimes g). \\

\text{Under the decomposition} \, \mathcal{X} = R(R) + (R(P) \otimes N(R)) + N(P), \, R, \, P \, \text{and} \, -\varphi(x \otimes g) \, \text{have the following operator matrices} \\
R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \, \text{and} \, \varphi(-x \otimes g) = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}, \\
\text{respectively. By direct calculation, we know} \\
\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ Q_{31} & Q_{32} & 0 \end{pmatrix} = \begin{pmatrix} 1 + Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & 1 + Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}. \\
\text{So,} \, Q_{11} = 1, \, Q_{12} = 0, \, Q_{21} = 0, \, Q_{33} = 0 \, \text{and} \, Q_{22} = 1. \, \text{It follows that} \\
2R + R\varphi(x \otimes g) + \varphi(x \otimes g)R = \begin{pmatrix} 0 & 0 & -Q_{13} \\ 0 & 0 & 0 \\ -Q_{31} & 0 & 0 \end{pmatrix}. \\
\text{Hence, we get a contradiction since} \, 2R + R\varphi(x \otimes g) + \varphi(x \otimes g)R \in \mathcal{I}^*(\mathcal{X}).
If \( x, Bx \) and \( B^2x \) are linearly dependant and \( x \) and \( Bx \) are linearly independent, then \( B^2x = \lambda x + \mu Bx \) for some \( \lambda, \mu \in \mathbb{C} \). When \( \mu = 0 \), there is \( h \in \mathcal{X}' \) such that 
\[
h(Bx) = 1 - f(Bx) \text{ and } h(x) = -1.
\]
Then we know that \( Bx \otimes (f + h) + x \otimes (f + h)B \in \mathcal{I}^*(\mathcal{X}) \). \( \varphi \) means \( \varphi(B) \varphi(z) \). Note that \( \varphi(B) \varphi(z) = 2R \varphi(x)R + \varphi(x)R \in \mathcal{I}^*(\mathcal{X}) \). Let us assume that \( \varphi \) exists satisfying \( \varphi(B) = 1 - f(B) \) and \( \varphi(x) = -1 \). By direct calculation, we know that \( \varphi(B) \varphi(x) = 2R \varphi(x)R + \varphi(x)R \in \mathcal{I}^*(\mathcal{X}) \). Note that \( -x \otimes h \in \mathcal{I}^*(\mathcal{X}) \) and \( z \otimes f \cdot h = (1 - z)(-x) \otimes h \cdot x \otimes f = z(-x) \otimes h + (1 - z)x \otimes f \in \mathcal{I}^*(\mathcal{X}) \). So, we can get another contradiction using a similar argument demonstrated above. Now, we assume that \( \mu \neq 0 \). If \( \lambda = 0 \), then \( \lambda = \mu \text{ and } x \otimes f \cdot h = (1 - z)(-x) \otimes h \cdot x \otimes f = z(-x) \otimes h + (1 - z)x \otimes f \in \mathcal{I}^*(\mathcal{X}) \). We get that \( \varphi(x) \otimes h_1 \in \mathcal{I}^*(\mathcal{X}) \), \( \varphi(x) \otimes f \cdot \varphi(-x) \otimes h_1 \) + \( (1 - z)\varphi(-x) \otimes h_1 \varphi(x) \otimes f \in \mathcal{I}^*(\mathcal{X}) \) for all \( x \in \mathbb{C} \) and \( z \in \mathbb{C} \). Under the decomposition \( \mathcal{X} = R(R) + (R(P) \otimes N(R)) + N(P) \), \( R \) and \( -\varphi(x) \otimes h_1 \) have the following operator matrices 
\[
R = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
and \( \varphi(-x) \otimes h_1 = \begin{pmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{pmatrix} \).

So, \( Q_{11} = 1 \), \( Q_{12} = 0 \), \( Q_{21} = 0 \), \( Q_{33} = 0 \) and \( Q_{22} = 1 \). It follows that 
\[
\varphi(I - \frac{2}{\mu}B)\varphi(x) \varphi(f + h_1) + \varphi(x) \varphi(f + h_1)\varphi(I - \frac{2}{\mu}B) = 2P - \frac{2}{\mu}R - 2\varphi(-x) \otimes h_1 + \frac{2}{\mu}\varphi(-x \otimes h_1)R + \varphi(-x) \otimes h_1 \in \mathcal{I}^*(\mathcal{X})
\]
\[
= \begin{pmatrix}
0 & 0 & (-2 + \frac{2}{\mu})Q_{13} \\
0 & 0 & -2Q_{23} \\
(-2 + \frac{2}{\mu})Q_{31} & -2Q_{32} & 0
\end{pmatrix} \in \mathcal{I}^*(\mathcal{X}).
\]

Clearly, this is a contradiction. If \( \lambda \neq 0 \), then there is \( h_2 \in \mathcal{X}' \) such that \( h_2(Bx) = -\frac{2}{\mu}f(Bx) \) and \( h_2(x) = -f(x) \). Hence, \( (I - \frac{2}{\mu}B)x \otimes (f + h_2) + x \otimes (f + h_2)(I - \frac{2}{\mu}B) \in \mathcal{I}^*(\mathcal{X}) \), and thus, 
\[
\varphi(I - \frac{2}{\mu}B)\varphi(x) \varphi(f + h_2) + \varphi(x) \varphi(f + h_2)\varphi(I - \frac{2}{\mu}B) = 2P - \frac{2}{\mu}R - 2\varphi(-x) \otimes h_2 + \frac{2}{\mu}\varphi(-x \otimes h_2)R + \varphi(-x) \otimes h_2 \in \mathcal{I}^*(\mathcal{X}).
\]

Li Fang
By an argument similar to that above, we also get a contradiction.

If $x$, $Bx$ and $B^2x$ are linearly dependent and $x$ and $Bx$ are linearly dependent, then $Bx = \beta x$ for some $\beta \in \mathbb{C}$. When $\beta = 0$, we know that $\frac{1}{2}((I + zB)x \otimes f + x \otimes f(I + zB)) \in \mathcal{T}^*(X)$ for all $z \in \mathbb{C}\setminus\{0\}$ implies that $\frac{1}{2}((I + zR)P + P(I + zR)) = P + zR \in \mathcal{T}^*(X)$ for all $z \in \mathbb{C}\setminus\{0\}$. Obviously, this is a contradiction. When $\beta \neq 0$, we know that $\frac{1}{2(1+\beta z)}((I + zB)x \otimes f + x \otimes f(I + zB)) \in \mathcal{T}^*(X)$ for all $z \in \mathbb{C}\setminus\{0\}$ with $1 + \beta z \neq 0$ implies that $\frac{1}{2(1+\beta z)}((I + zR)P + P(I + zR)) = \frac{1}{1+\beta z}(P + zR) \in \mathcal{T}^*(X)$ for all $z \in \mathbb{C}\setminus\{0\}$ with $1 + \beta z \neq 0$. This is also a contradiction.

Therefore, $\varphi(P) \in \mathcal{I}_1(X)$. □

The proof of the following lemma is similar to that of [13, Lemma 2.7].

**Lemma 3.6.** $\varphi(N_1(X)) \subseteq N_1(X)$.

*Proof.* Let $N = x \otimes f \in N_1(X)$ for some non-zero $x \in X$ and non-zero $f \in X'$ such that $f(x) = 0$. Then $\varphi(N) \in \mathcal{N}(X)$ by Lemma 3.2. Taking an $f_1 \in X'$ such that $f_1(x) = 1$ and setting $Q = x \otimes f_1$, we know that both $Q$ and $Q+N$ are in $\mathcal{I}_1(X)$. So are both $\varphi(Q)$ and $\varphi(Q+N)$ by Lemma 3.5. Then there exist $y_1$, $y_2 \in X$ and $g_1$, $g_2 \in X'$ such that $g_1(y_1) = g_2(y_2) = 1$, $\varphi(Q) = y_1 \otimes g_1$ and $\varphi(Q+N) = y_2 \otimes g_2$. Putting $P = \frac{1}{2}((Q+N) + Q) = \frac{1}{2}N + Q$, we get that $P \in \mathcal{I}_1(X)$ and then $\varphi(P) \in \mathcal{I}_1(X)$. However, $\varphi(P) = \frac{1}{2}(\varphi(Q+N) + \varphi(Q)) = \frac{1}{2}(y_1 \otimes g_1 + y_2 \otimes g_2)$. It follows that either $y_1$ and $y_2$ or $g_1$ and $g_2$ are linearly dependent. If $y_1$ and $y_2$ are linearly dependent, then we may assume that $y_1 = y_2$. Thus, $\varphi(P) = \frac{1}{2}y_1 \otimes (g_1 + g_2) \in \mathcal{I}_1(X)$ and then $g_1(y_1) + g_2(y_1) = 2$. Since $g_1(y_1) = 1$, we have $g_2(y_1) = 1$. Thus, $\varphi(N) = \varphi(N + Q) - \varphi(Q) = y_1 \otimes (g_1 - g_2)$ and $(g_1 - g_2)(y_1) = 0$. Hence, $\varphi(N) \in N_1(X)$. We can get that $\varphi(N) \in N_1(X)$ by similar discussion if $g_1$ and $g_2$ are linearly dependent. □

**Corollary 3.7.** $\varphi(F_1(X)) \subseteq F_1(X)$ and $\varphi(F(X)) \subseteq F(X)$.

*Proof.* Since every non-nilpotent rank-one operator is a non-zero scalar multiple of rank-one idempotent operator, we know that $\varphi(F_1(X)) \subseteq F_1(X)$ by Lemma 3.5 and Lemma 3.6 and the linearity of $\varphi$. Moreover, every finite-rank operator can be written as a linear combination of finitely many rank-one operators. It follows from the linearity of $\varphi$ that $\varphi(F(X)) \subseteq F(X)$. □

Note that a linear map $\varphi$ on $F(X)$ is rank non-increasing if rank $\varphi(X) \leq$ rank $X$ for any $X \in F(X)$. We start this section with [5, Corollary 2.1.5], which is restated in the following.

**Lemma 3.8.** Let $\varphi$ be a linear map on $F(X)$ which is rank non-increasing such that rank $\varphi(T_0) > 1$ for some $T_0 \in F(X)$. Then one of the following holds.
3.1 holds. \(\varphi\) injective by Lemma 3.1 and thus bijective. We now have \(B\) preserving the nonzero idempotency of Jordan product of two operators. Then \(\varphi(x \otimes f) = Af \otimes Cx\) for all \(x \in \mathcal{X}\) and \(f \in \mathcal{X}'\).

**(1)** There exist linear injective maps \(A : \mathcal{X} \to \mathcal{X}\) and \(C : \mathcal{X}' \to \mathcal{X}'\) such that \(\varphi(x \otimes f) = Ax \otimes Cf\) for all \(x \in \mathcal{X}\) and \(f \in \mathcal{X}'\).

**(2)** There exist linear injective maps \(A : \mathcal{X}' \to \mathcal{X}\) and \(C : \mathcal{X} \to \mathcal{X}'\) such that \(\varphi(x \otimes f) = Af \otimes Cx\) for all \(x \in \mathcal{X}\) and \(f \in \mathcal{X}'\).

**Proposition 3.9.** Let \(\varphi\) be a surjective linear map on \(\mathcal{B}(\mathcal{X})\) preserving the nonzero idempotency of Jordan products of two operators such that \(\varphi(I) = I\). Then one of the statements in Lemma 3.8 holds.

**Proof.** From Corollary 3.7, we know that the restriction \(\varphi|_{\mathcal{F}_t(\mathcal{X})}\) of \(\varphi\) on \(\mathcal{F}(\mathcal{X})\) is a linear map preserving rank non-increasing. Let \(Q\) be a rank-2 idempotent operator. Then there exists a \(P \in \mathcal{I}_2(\mathcal{X})\) such that \(P < Q\). By Lemma 3.4, we know that \(\varphi(P) < \varphi(Q)\) and \(\varphi(P) \in \mathcal{I}_1(\mathcal{X})\). So, rank \(\varphi(Q) \geq 2\). Then we get the desired results from Lemma 3.8. \(\square\)

**Proof of Theorem 1.1.** The sufficiency is clear. Let \(\varphi\) be a linear map on \(\mathcal{M}_n\) preserving the nonzero idempotency of Jordan product of two operators. Then \(\varphi\) is injective by Lemma 3.1 and thus bijective. We now have \(\varphi(I) = \lambda I\) for some constant \(\lambda \in \{1, -1\}\). we may assume \(\varphi(I) = I\). Then one of two statements in Proposition 3.1 holds.

If (1) holds, then we easily have that \(\varphi(X) = AXB\) for all \(X \in \mathcal{M}_n\). It is clear that \(B = A^{-1}\). We have that \(\varphi(X) = AXA^{-1}\) for all \(X \in \mathcal{M}_n\).

If (2) holds, then we similarly have that \(\varphi(X) = AX'A^{-1}\) for all \(X \in \mathcal{M}_n\). \(\square\)

We next consider the infinite dimensional case.

**Lemma 3.10.** Let \(\varphi\) be a surjective linear map on \(\mathcal{B}(\mathcal{X})\) preserving the nonzero idempotency of Jordan products of two operators such that \(\varphi(I) = I\). Then \(\varphi(\mathcal{F}_t(\mathcal{X})) = \mathcal{F}_t(\mathcal{X})\).

**Proof.** By Corollary 3.7, it is sufficient to prove that \(\mathcal{F}_t(\mathcal{X}) \subseteq \varphi(\mathcal{F}_t(\mathcal{X}))\). Let \(T \in \mathcal{B}(\mathcal{X})\) such that \(\varphi(T) = z \otimes h\) is of rank-one for some \(z \in \mathcal{X}\) and \(h \in \mathcal{X}'\). Clearly, \(T\) is a non-scalar operator. So, there is \(x \in \mathcal{X}\) such that \(x\) and \(Tx\) are linearly independent. If \(\text{rank } T > 1\), then there exists \(y \in \mathcal{X}\) such that \(Tx\) and \(Ty\) are linearly independent. It follows that \(x\) and \(y\) are linearly independent. Applying [9, Lemma 2.1], we know that \(x + \delta y\) and \(T(x + \delta y)\) are linearly independent for all but finite number \(\delta \in \mathbb{C}\). For any fixed \(\alpha \in \mathbb{C}\), there is a rank-one idempotent \(P(\delta, \alpha)\) such that \((T + P(\delta, \alpha))(x + \delta y) = \alpha(x + \delta y)\) by Lemma 2.4. Finding \(f(\delta, \alpha) \in \mathcal{X}'\) to satisfy \(f(\delta, \alpha)(x + \delta y) = \frac{1}{2\alpha}\), we get that \(\frac{1}{2\alpha}(T + P(\delta, \alpha))(x + \delta y) \otimes f(\delta, \alpha) + (x + \delta y) \otimes f(\delta, \alpha)(T + P(\delta, \alpha)) \in \mathcal{T}'(\mathcal{X})\), which implies that \(\frac{1}{2\alpha}(\varphi(T + P(\delta, \alpha))\varphi((x + \delta y) \otimes f(\delta, \alpha)) + \varphi((x + \delta y) \otimes f(\delta, \alpha))\varphi(T + P(\delta, \alpha))) \in \mathcal{T}'(\mathcal{X})\). Let \(\varphi(P(\delta, \alpha)) = e(\delta, \alpha) \otimes g(\delta, \alpha)\). If Proposition 3.9 (1) holds, then \(\frac{1}{2\alpha}(\varphi(T + P(\delta, \alpha))\varphi((x + \delta y) \otimes f(\delta, \alpha)) + \varphi((x + \delta y) \otimes f(\delta, \alpha))\varphi(T + P(\delta, \alpha))) = \)
Thus, $\varphi(T+P(\delta, \alpha))A(x+\delta y)\otimes Cf(\delta, \alpha)+A(x+\delta y)\otimes Cf(\delta, \alpha)\varphi(T+P(\delta, \alpha))_i \in T^*(\mathcal{X})$. This yields that $\varphi(T+P(\delta, \alpha))A(x+\delta y) = \alpha A(x+\delta y)$ or $\varphi(T+P(\delta, \alpha))fCf(\delta, \alpha) = \alpha Cf(\delta, \alpha)$. When $\varphi(T+P(\delta, \alpha))A(x+\delta y) = \alpha A(x+\delta y)$, we have that $h(A(x+\delta y))z + (\delta, \alpha)(A(x+\delta y))e(\delta, \alpha) = \alpha A(x+\delta y)$ for fixed $\delta \in \mathbb{C}\setminus\{0\}$ and any $\alpha \in \mathbb{C}\setminus\{0\}$. It follows that

$$g(\delta, \alpha_1)A(x+\delta y))e(\delta, \alpha_1) - g(\delta, \alpha_2)A(x+\delta y))e(\delta, \alpha_2) = (\alpha_1 - \alpha_2)A(x+\delta y)$$

for fixed $\delta \in \mathbb{C}$ and any $\alpha_1, \alpha_2 \in \mathbb{C}\setminus\{0\}$. Obviously,

$$(1 + \alpha_2 - \alpha_1)g(\delta, \alpha_1)A(x+\delta y)) = g(\delta, \alpha_2)A(x+\delta y))g(\delta, \alpha_1)(e(\delta, \alpha_2))$$

for fixed $\delta \in \mathbb{C}$ and any $\alpha_1, \alpha_2 \in \mathbb{C}\setminus\{0\}$ with $1 + \alpha_2 = \alpha_1$. So, $g(\delta, \alpha_2)(A(x+\delta y)) = 0$ or $g(\delta, \alpha_1)(e(\delta, \alpha_2)) = 0$. Hence, $g(\delta, \alpha_2)(A(x+\delta y)) = 0$ for fixed $\delta \in \mathbb{C}$ and any $\alpha_2 \in \mathbb{C}\setminus\{0\}$. This is a contradiction. When $\varphi(T+P(\delta, \alpha))f(\delta, \alpha) = \alpha f(\delta, \alpha)$, we get that $\alpha \in \sigma_p(z \otimes h + e(\delta, \alpha) \otimes g(\delta, \alpha))$ for fixed $\delta \in \mathbb{C}$ and any $\alpha \in \mathbb{C}\setminus\{0\}$, where $\sigma_p(z \otimes h + e(\delta, \alpha) \otimes g(\delta, \alpha))$ denotes that point spectrum of $z \otimes h + e(\delta, \alpha) \otimes g(\delta, \alpha)$. Note that $\mathcal{R}(z \otimes h + e(\delta, \alpha) \otimes g(\delta, \alpha)) \subseteq [z] + [A(x+\delta y), AT(x+\delta y)]$ for any $\alpha \in \mathbb{C}$. Here is a contradiction. If Proposition 3.9 (2) holds, then we get a contradiction again. Thus, $T$ is of rank-one.

The idea of the following proof comes from [9] and [13].

**Proof of Theorem 1.2.** The sufficiency is clear. Now, we prove the necessity and assume that $\varphi(I) = I$ by Lemma 3.3.

By Lemma 3.10, $\varphi$ maps the set of rank-one operators onto itself. This implies that the injective linear maps $A$ and $C$ mentioned in Proposition 3.9 are bijective. Suppose Proposition 3.9 (1) holds. By Lemma 3.4 and the linearity of $\varphi$, it can be shown that $Cf(Ax) = f(x)$ for all $x \in \mathcal{X}$ and $f \in \mathcal{X}'$, which implies that $C$ is the adjoint of $A^{-1}$, and hence $C$ is bounded. Thus, $A^{-1}$ and $A$ are bounded too. Furthermore, for any $y \in \mathcal{X}$,

$$\varphi(x \otimes f)y = (Ax \otimes Cf)y = (Cf(y)Ax = f(A^{-1}y)Ax = A(x \otimes f)A^{-1}y.$$ 

Thus, $\varphi(T) = ATA^{-1}$ for any rank-one operator $T$. Moreover, every finite-rank operator can be written as a linear combination of finitely many rank-one operators. With the linearity of $\varphi$, we know that $\varphi(T) = ATA^{-1}$ for any finite-rank operator $T$. Replacing $\varphi$ by $A^{-1}\varphi A$, we may assume that $\varphi(T) = T$ for every finite-rank operator $T$. We next prove that $\varphi(T) = T$ for every non-scalar and infinite-rank operator $T \in B(\mathcal{X})$. 


Let \( T \in \mathcal{B}(\mathcal{X}) \) be any non-scalar and infinite-rank operator. By the property of \( \varphi \), we know that

\[
TR + RT \in \mathcal{I}(\mathcal{X}) \setminus \{0\} \Rightarrow \varphi(T)\varphi(R) + \varphi(R)\varphi(T) = \varphi(T)R + R\varphi(T) \in \mathcal{I}(\mathcal{X}) \setminus \{0\}
\]

for every \( R \in \mathcal{F}_1(\mathcal{X}) \). Suppose there is \( x \in \mathcal{X} \) such that \((\varphi(A) + 2P_0)x\) and \((A + 2P_0)x\) are linearly independent for some rank-one idempotent \( P_0 \in \mathcal{I}^*(\mathcal{X}) \). If \( x \) and \((A + 2P_0)x\) are linearly independent, then there exists a rank-one idempotent \( Q \in \mathcal{I}(\mathcal{X}) \) such that \((A + 2P_0 + 2Q)x = x\) by Lemma 2.4. So, \((\varphi(A) + 2P_0 + 2Q)x \neq (A + 2P_0 + 2Q)x\) and there is \( f \in \mathcal{X}^\prime \) such that \( f((\varphi(A) + 2P_0 + 2Q)x) = 0 \) and \( f((A + 2P_0 + 2Q)x) = 1 \). It follows that \( \frac{1}{2}(A + 2P_0 + 2Q)x \otimes f + x \otimes f(A + 2P_0 + 2Q) \in \mathcal{I}^*(\mathcal{X}) \) and then \( \frac{1}{2}(\varphi(A) + 2P_0 + 2Q)x \otimes f + x \otimes f(\varphi(A) + 2P_0 + 2Q) \in \mathcal{I}^*(\mathcal{X}) \). Hence, \( f((\varphi(A) + 2P_0 + 2Q)x) = 2f(\varphi(A) + 2P_0 + 2Q)x \). Thus, \( f((\varphi(A) + 2P_0 + 2Q)x) = 0 \) and so \((\varphi(A) + 2P_0 + 2Q)x = 0 \). This is a contradiction. If \((A + 2P_0)x = \gamma x\) for some non-zero \( \gamma \in \mathbb{C} \), then there is \( g \in \mathcal{X}^\prime \) such that \( g((\varphi(A) + 2P_0)x) = 0 \) and \( g((A + 2P_0)x) = \gamma \). Hence, \( \frac{1}{\gamma^2}((A + 2P_0)x \otimes g + x \otimes g(2P_0 + \gamma)) \in \mathcal{I}^*(\mathcal{X}) \), which implies that \( \frac{1}{\gamma^2}(\varphi(A) + 2P_0)x \otimes g + x \otimes g(2P_0 + \gamma) \in \mathcal{I}^*(\mathcal{X}) \). By direct calculation, we get that \( g((\varphi(A) + 2P_0)x) = 2\gamma(\varphi(A) + 2P_0)x \) and so \( f((\varphi(A) + 2P_0 + 2Q)x) = 0 \). Thus, \((\varphi(A) + 2P_0 + 2Q)x = 0 \) and this is a contradiction. Thus, \( \varphi(A) + 2P \) and \( A + 2P \) are locally linearly dependent for any rank-one idempotent \( P \in \mathcal{I}^*(\mathcal{X}) \). By [10, Theorem 2.4], we know that there is \( \eta(P) \in \mathbb{C} \) such that \( \varphi(A) + 2P = \eta(P)(A + 2P) \) for any rank-one idempotent \( P \in \mathcal{I}^*(\mathcal{X}) \). Since \( A + 2P \) is a non-scalar operator, we can find \( x \in \mathcal{X} \) such that \( x \) and \((A + 2P)x\) are linearly independent. Then there exists an idempotent operator \( Q_1 \in \mathcal{B}(\mathcal{X}) \) of rank one such that \((A + 2P + 2Q_1)x = x\). There is \( h \in \mathcal{X}^\prime \) such that \( h(x) = 1 \). So, \( \frac{1}{2}((A + 2P + 2Q_1)x \otimes h + x \otimes h(A + 2P + 2Q_1)) \in \mathcal{I}^*(\mathcal{X}) \). It follows that \( \frac{1}{2}(\varphi(A + 2P + 2Q_1)x \otimes h + x \otimes h(\varphi(A + 2P + 2Q_1)) \in \mathcal{I}^*(\mathcal{X}) \). By direct calculation, we have that \( \varphi(A + 2P + 2Q_1)x = x \) or \( \varphi(A + 2P + 2Q_1)x = h \). Therefore, \( \eta(P + Q_1) = 1 \) and then \( \varphi(A) = A \).

Suppose Proposition 3.9 (2) holds. Then we have \((Cx)(Af) = f(x)\) for all \( x \in \mathcal{X} \) and \( f \in \mathcal{X}^\prime \) by a similar argument. So, \( C' = A^{-1}K^{-1} \), where \( K \) is the natural embedding of \( X \) into \( X'' \). Thus, \( A^{-1} \) is bounded and so as \( (A^{-1})' \) and \( C = (A^{-1})'K \). As \( C \) and \( (A^{-1})' \) are bijective, as so \( K \) and hence \( X \) is reflexive. Now by a similar argument, \( \varphi(T) = AT'A^{-1} \) for every finite-rank operator \( T \) and hence \( \varphi(T) = AT'A^{-1} \) for all every \( T \in \mathcal{B}(\mathcal{X}) \). \( \square \)

Acknowledgment. The author would like to thank the anonymous referees for their helpful comments and suggestions that greatly improved the original manuscript.
REFERENCES