Maps preserving general means of positive operators

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Abstract. Under some mild conditions, the general form of bijective transformations of the set of all positive linear operators on a Hilbert space which preserve a symmetric mean in the sense of Kubo-Ando theory is described.

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(symmetry) and some mild regularity assumption (e.g. a sort of continuity) on the transformations. As for the main idea of the proof, it is based on the knowledge of the structure of the order automorphisms of $B(H)^+$ that was determined in our paper [3] (alternatively, see Section 2.5 in the book [4]). Some recent applications of that result to other problems can be found in the paper [7].

Now, let us summarize the basic notions and results of Kubo-Ando theory that we shall need in our investigations. In what follows, $H$ denotes a complex Hilbert space with $\dim H \geq 2$. A binary operation $\sigma$ on $B(H)^+$ is said to be a connection if the following requirements are fulfilled (from (i) to (iii) all operators are supposed to belong to $B(H)^+$).

(i) If $A \leq C$ and $B \leq D$, then $A \sigma B \leq C \sigma D$.
(ii) $C(A \sigma B)C \leq (CAC) \sigma CBC$.
(iii) If $A_n \downarrow A$ and $B_n \downarrow B$ strongly, then $A_n \sigma B_n \downarrow A \sigma B$ strongly.

If $I \sigma I = I$ holds too, then the connection $\sigma$ is called a mean. A mean $\sigma$ is called symmetric if $A \sigma B = B \sigma A$ holds for all $A, B \in B(H)^+$. Operations like convex combination and order among connections are defined in a natural way.

By the fundamental result Theorem 3.2 in [2], there is an affine order-isomorphism from the class of connections $\sigma$ onto the class of operator monotone functions $f : ]0, \infty[ \to ]0, \infty[$ given by the formula $f(t) = I \sigma tI \ (t > 0)$. For invertible $A, B \in B(H)^+$, we have

$$A \sigma B = A^{1/2} f(A^{-1/2} BA^{-1/2}) A^{1/2}. \tag{1.1}$$

By property (iii), we obtain that the formula (1.1) extends to any invertible $A \in B(H)^+$ and arbitrary $B \in B(H)^+$. We shall need the following so-called transfer property of connections. By (ii), we easily deduce that $C(A \sigma B)C = (CAC) \sigma CBC$ holds for all invertible $C \in B(H)^+$. Now, using polar decomposition, the formula (1.1) and the continuity property (iii), it follows that we have

$$T(A \sigma B)T^* = (TAT^*)\sigma(TBT^*)$$

for all invertible bounded linear or conjugate-linear operator $T$ on $H$. This trivially implies that $(LA)\sigma(tB) = t(A \sigma B)$ for any $t > 0$ and $A, B \in B(H)^+$.

We shall also use Theorem 3.3 in [2] which says that for every mean $\sigma$, we have $A \sigma A = A$, $A \in B(H)^+$.

Relating to operator monotone functions, it is well-known that they have a certain
integral representation. Namely, by Lemma 3.1 in [2], the formula
\[
(1.2) \quad f(s) = \int_{[0, \infty]} \frac{s(1 + t)}{s + t} dm(t), \quad s > 0
\]
provides an affine isomorphism from the class of all positive Radon measures \( m \) on \([0, \infty]\) onto the set of all operator monotone functions \( f : [0, \infty] \to [0, \infty] \). Remark that in the above representation, we have \( f(0) = \lim_{s \to 0} f(s) = m(\{0\}) \) and \( \lim_{s \to \infty} f(s)/s = m(\{\infty\}) \).

Using the formula (1.2), we then obtain an integral representation of any connection \( \sigma \). In fact, Theorem 3.4 in [2] tells us that for each connection \( \sigma \) on \( B(H)^+ \), there exists a unique positive Radon measure on \([0, \infty]\) such that with \( a = m(\{0\}) \) and \( b = m(\{\infty\}) \) we have
\[
(1.3) \quad A\sigma B = aA + bB + \int_{[0, \infty]} \frac{1 + t}{t} \{(tA : B)\} dm(t)
\]
for all \( A, B \in B(H)^+ \) (here and below : stands for the parallel sum of positive operators which is the half of the harmonic mean). Moreover, the correspondence \( \sigma \mapsto m \) is an affine isomorphism from the class of all connections onto the class of all positive Radon measures on \([0, \infty]\).

Below whenever we write \( \sigma, f, m \) we always mean that \( \sigma \) is a connection, \( f \) is its representing operator monotone function and \( m \) is its representing positive Radon measure.

2. Results and proofs. In what follows, \( \sigma \) denotes a symmetric mean with the property that \( I \sigma 0 = 0 \sigma I = 0 \). This means that in (1.3) we have \( a = m(\{0\}) = f(0) = 0 \). Taking into account that symmetry of \( \sigma \) means that \( f(s) = sf(1/s) \) holds for all \( s > 0 \) (see Corollary 4.2. in [2]), we obtain that \( b = m(\{\infty\}) = \lim_{s \to \infty} f(s)/s = \lim_{s \to 0} f(s)/s = f(0) = 0 \). Consequently, the integral representation of \( \sigma \) is
\[
(2.1) \quad A\sigma B = \int_{[0, \infty]} \frac{1 + t}{t} \{(tA : B)\} dm(t), \quad A, B \in B(H)^+.
\]
There is still another fact that we shall need: for any operator \( A \in B(H)^+ \) we have \( I \sigma A = f(A) \) (see (3.7) in [2]).

\textbf{Theorem 2.1.} Let \( \phi : B(H)^+ \to B(H)^+ \) be a bijective map that satisfies
\[
(2.2) \quad \phi(A\sigma B) = \phi(A)\sigma\phi(B)
\]
for all \( A, B \in B(H)^+ \). If there exists an invertible operator \( A \in B(H)^+ \) such that either \( \phi \) is continuous on the set of scalar multiples \( tA \), \( t \geq 0 \) of \( A \), or it maps any
scalar multiple \( tA, t \geq 0 \) of \( A \) to a scalar multiple of \( \phi(A) \), then it follows that \( \phi \) is of the form

\[
\phi(A) = TAT^*, \quad A \in B(H)^+
\]

with some invertible bounded linear or conjugate-linear operator \( T : H \to H \).

Observe that any transformation \( \phi \) of the form \( \phi(A) = TAT^*, A \in B(H)^+ \) with an invertible bounded linear or conjugate-linear operator \( T : H \to H \) satisfies (2.2) and also has both of the regularity properties appearing in the theorem above as assumptions.

For the proof of the theorem, we need some auxiliary results.

**Lemma 2.2.** For any \( A \in B(H)^+ \), we have that \( A \) is a projection if and only if \( I_\sigma A = A \).

**Proof.** First we point out that Lemma 5.1 in [2] tells us that for \( f \) we have \( s < f(s) < 1 \) for \( 0 < s < 1 \) and \( 1 < f(s) < s \) for \( 1 < s < \infty \). Now, \( f(A) = A \) holds if and only if \( f(s) = s \) holds on the spectrum of \( A \). This is equivalent to the spectrum of \( A \) is in \( \{0, 1\} \), which means exactly that \( A \) is a projection. \( \Box \)

Theorem 3.7 in [2] tells us that for any projections \( P, Q \) on \( H \) we have \( P \sigma Q = P \wedge Q \). It follows that \( P \leq Q \) if and only if \( P \sigma Q = P \).

**Lemma 2.3.** We have that \( f \) is injective.

**Proof.** In fact, assuming on the contrary that \( f \) is non-injective, it follows that \( f \) is constant on some closed interval either before or after the point 1. In the latter case, there are two more possibilities: the interval can be of finite or infinite length. Using Lemma 5.1 in [2], the concavity of \( f(s) \) and the convexity of \( sf(s) \) (Lemma 5.2 in [2]) elementary considerations show that we would arrive at contradictions in all cases. This gives us the injectivity of \( f \). \( \Box \)

**Lemma 2.4.** Suppose that \( f \) is unbounded. Then \( A \in B(H)^+ \) is invertible if and only if the equation \( A \sigma X = Y \) has a solution \( X \in B(H)^+ \) for any given \( Y \in B(H)^+ \).

**Proof.** Since \( f \) is injective and unbounded, we have \( f^{-1} : [0, \infty] \to [0, \infty] \). Suppose that \( A \) is invertible. We have learnt from (1.1) that

\[
(2.3) \quad A \sigma X = A^{1/2} f(A^{-1/2} X A^{-1/2}) A^{1/2}
\]

holds for any \( X \in B(H)^+ \). It requires only easy computation to see that for a given \( Y \), defining \( X = A^{1/2} f^{-1}(A^{-1/2} Y A^{-1/2}) A^{1/2} \), gives a solution of the equation \( A \sigma X = Y \). Assume now that \( A \) is not invertible. In that case, for any invertible \( X \in B(H)^+ \), we have

\[
A \sigma X = X \sigma A = X^{1/2} f(X^{-1/2} AX^{-1/2}) X^{1/2}.
\]
Since \( f(0) = 0 \), by the spectral mapping theorem we deduce that the right-hand side of this equality and hence \( A\sigma X \), too, are non-invertible. If \( X \in B(H)^+ \) is arbitrary, it follows from \( A\sigma X \leq A\sigma(X + I) \) that \( A\sigma X \) is also not invertible. \( \square \)

**Lemma 2.5.** Suppose that \( f \) is bounded. The operator \( A \in B(H)^+ \) is invertible if and only if the set of all operators of the form

\[
(...((A\sigma T_1)\sigma T_2)...)\sigma T_n, \quad n \in \mathbb{N}, T_1, \ldots, T_n \in B(H)^+
\]

coincides with \( B(H)^+ \).

**Proof.** To prove the necessity, assume that \( A \) is invertible. By the transfer property it is easy to see that there is no serious loss of generality in assuming that \( A = I \). Let \( \lim_{s \to \infty} f(s) = r. \) Clearly, we have \( 1 < r < \infty. \) Pick an \( s \) with \( 1 < s \) and let \( Y \in B(H)^+ \). Pick \( n \in \mathbb{N} \) such that \( \|Y\|/f(s)^n < r. \) Choosing \( T_1 = sI, T_2 = f(s)sI, T_3 = f(s)^2sI, \ldots, T_n = f(s)^{n-1}sI, T_{n+1} = f(s)^nX, \) we see that

\[
(...((A\sigma T_1)\sigma T_2)...)\sigma T_{n+1} = f(s)^n f(X).
\]

The equation \( f(s)^n f(X) = Y \) clearly has solution \( X = f^{-1}(Y/f(s)^n) \). As for the sufficiency, if \( A \) is non-invertible then we obtain that all \( (...((A\sigma T_1)\sigma T_2)...)\sigma T_n \) are non-invertible. \( \square \)

In what follows, we compute \( A\sigma P \) for an arbitrary positive operator \( A \in B(H)^+ \) and rank-one projection \( P \) on \( H \). To do so, we recall the notion of the strength of a positive operator along a ray represented by a unit vector. This concept was introduced by Busch and Gudder in [1]. Let \( A \in B(H)^+ \) be a positive operator, consider a unit vector \( \varphi \) in \( H \) and denote by \( P_\varphi \) the rank-one projection onto the subspace generated by \( \varphi \). The quantity

\[
\lambda(A, P_\varphi) = \sup\{\lambda \in \mathbb{R}_+: \lambda P_\varphi \leq A\}
\]

is called the strength of \( A \) along the ray represented by \( \varphi \). According to [1, Theorem 4], we have the following formula for the strength:

\[
(2.4) \quad \lambda(A, P_\varphi) = \begin{cases} \|A^{-1/2}\varphi\|^{-2}, & \text{if } \varphi \in \text{rng}(A^{1/2}); \\ 0, & \text{else.} \end{cases}
\]

(The symbol \( \text{rng} \) denotes the range of operators and \( A^{-1/2} \) denotes the inverse of \( A^{1/2} \) on its range.)

**Lemma 2.6.** Let \( A \in B(H)^+ \) and \( P \) be a rank-one projection on \( H \). We have \( A\sigma P = P\sigma A = f(\lambda(A, P))P \).

**Proof.** First assume that \( \lambda(A, P) > 0 \). By (2.1), we have

\[
(2.5) \quad A\sigma P = \int_{[0, \infty]} \frac{1 + t}{t} \{(tA : P\})dm(t).
\]
The parallel sum is known to be the half of the harmonic mean. In Lemma 2, [6] we proved that for an arbitrary positive operator \( T \in B(H)^+ \) and rank-one projection \( P \) on \( H \), we have

\[
2(T : P) = T!P = \frac{2\lambda(T, P)}{\lambda(T, P) + 1} P.
\]

Therefore, denoting \( s = \lambda(A, P) \), we can continue (2.5) as follows:

\[
A\sigma P = \int_{[0,\infty]} \frac{1 + t}{t} \frac{\lambda(tA, P)}{\lambda(tA, P) + 1} P dm(t) = \int_{[0,\infty]} \frac{1 + t}{t} \frac{ts}{ts + 1} dm(t)P = s \int_{[0,\infty]} \frac{1}{(1/s)(t + 1)} dm(t)P = sf(1/s)P = f(s)P = f(\lambda(A, P))P.
\]

If \( \lambda(A, P) = 0 \), then in a similar fashion we see \( A\sigma P = 0 = f(\lambda(A, P))P \).

**Lemma 2.7.** For any \( A, B \in B(H)^+ \), we have \( A\sigma B \neq 0 \) if and only if \( \text{rng} A^{1/2} \cap \text{rng} B^{1/2} \neq \{0\} \).

**Proof.** To see the sufficiency, by the formula (2.4) it follows from \( \text{rng} A^{1/2} \cap \text{rng} B^{1/2} \neq \{0\} \) that there is a rank-one projection \( P \) on \( H \) and a number \( t > 0 \) such that \( tP \leq A, B \). This implies \( 0 \neq tP = (tP)\sigma(tP) \leq A\sigma B \).

Conversely, assume \( A\sigma B \neq 0 \). By (2.1), we deduce that \( (tA) : B \neq 0 \) and hence that \( (tA)B \neq 0 \) holds for some \( t > 0 \). It then follows that some positive scalar multiple of a rank-one projection \( P \) is less than or equal to \( (tA)B \). Since \( B \) is less than or equal to a scalar multiple of the identity, by the monotonicity property (i) of means this further implies that for some \( s > 1 \), we have \( P \leq (sA)! (sI) = s(AI) \).

Therefore, \( 2P/(I + P) = P \leq 2sA/(I + A) \). The inverse function of \( h : t \mapsto 2st/(1+t) \), \( t \geq 0 \) is \( k : r \mapsto r/(2s - r) \), \( 0 \leq r < 2s \), which is easily seen to be operator monotone on the interval \([0,2s] \). It follows that

\[
k(2P/(I + P)) \leq k(2sA/(I + A)) = A.
\]

One can check that the operator on the left hand side is \( k(2P/(I + P)) = (1/(2s - 1))P \). This gives us that a positive scalar multiple of \( P \) is less than or equal to \( A \) implying that the range of \( P \) is included in \( \text{rng} A^{1/2} \). We obtain in a similar fashion that \( \text{rng} P \subset \text{rng} B^{1/2} \) holds, too. This completes the proof of the lemma.

After these preliminaries, we are now in a position to present the proof of Theorem 2.1.

**Proof of Theorem 2.1.** By Lemmas 2.4 and 2.5, we see that our transformation \( \phi \) preserves the invertible operators in both directions. This means that for any \( A \in B(H)^+ \), we have \( A \) is invertible if and only if \( \phi(A) \) is invertible.
It follows that $\phi(I)$ is invertible. By the transfer property, the transformation $\phi(I)^{-1/2} \cdot \phi(.) \cdot \phi(I)^{-1/2}$ is a bijective map on $B(H)^+$ which fulfils (2.2) and sends $I$ to $I$. Therefore, we may and do assume that already our original map $\phi$ satisfies $\phi(I) = I$.

By the characterization of projections (Lemma 2.2) and the order among them (see the sentence before Lemma 2.3), it follows that $\phi$ preserves the projections in both directions as well as the order among them. Consequently, $\phi(0) = 0$.

Now, by Lemma 2.7, we see that for any $A, B \in B(H)^+$ we have $\text{rng } A^{1/2} \cap \text{rng } B^{1/2} \neq \{0\}$ if and only if $\text{rng } \phi(A)^{1/2} \cap \text{rng } \phi(B)^{1/2} \neq \{0\}$. One can easily verify that it implies

$$\text{rng } A^{1/2} \subset \text{rng } B^{1/2} \iff \text{rng } \phi(A)^{1/2} \subset \text{rng } \phi(B)^{1/2}.$$ 

From this property, we infer that $\phi$ preserves the (finite) rank of the elements of $B(H)^+$.

Let $P$ be a rank-one projection. For every $t > 0$, the operator $\phi(tP)$ is of rank one. Since $\phi(tP)\sigma \phi(P) = \phi((tP)\sigma P) \neq 0$, it follows that the range of $\phi(tP)$ has non-trivial intersection with the range of $\phi(P)$. This gives us that $\phi(tP)$ is a scalar multiple of $\phi(P)$. Next we deduce that there exists a bijective function $g_P : [0, \infty] \to [0, \infty]$ such that

$$\phi(tP) = g_P(t)\phi(P)$$

holds for every $t \geq 0$.

Assume now that $\phi$ is continuous on the scalar multiples of an invertible operator $A \in B(H)^+$. Considering the transformation

$$X \mapsto \phi(A)^{-1/2} \cdot \phi(A^{1/2}XA^{1/2}) \cdot \phi(A)^{-1/2},$$

we obtain a bijective map which satisfies (2.2), sends $I$ to $I$ and is continuous on the set of nonnegative scalar multiples of the identity. Hence, there is no real loss of generality in assuming that this particular operator $A$ equals $I$. In what follows, we do use this assumption.

For any $t \geq 0$, we have

$$\phi(tI)\sigma \phi(P) = \phi((tI)\sigma P) = \phi(f(t)P) = g_P(f(t))\phi(P).$$

By the formula (2.3), for an arbitrary $B \in B(H)^+$, the transformation $C \mapsto C\sigma B$ is norm continuous on the set of all invertible elements of $B(H)^+$. We deduce that $g_P(f(t)) \to g_P(f(t_0))$ whenever $t_0 > 0$ and $t \to t_0$. Applying (iii), the same follows for $t_0 = 0$, too. Therefore, we obtain that $g_P$ is continuous on the range of $f$. Since
$g_P$ is a bijection of the nonnegative reals and $g_P(0) = 0$, $g_P(1) = 1$ obviously hold, it follows that $g_P$ is strictly increasing on the range of $f$.

We show that $\phi$ preserves the order of positive operators in both directions. We compute on one hand

$$\phi(A\sigma P) = \phi(f(\lambda(A, P)) P) = g_P(f(\lambda(A, P)))\phi(P),$$

while on the other hand we have

$$\phi(A\sigma P) = \phi(A)\sigma\phi(P) = \lambda(\phi(A), \phi(P))\phi(P).$$

We obtain that $g_P(f(\lambda(A, P))) = \lambda(\phi(A), \phi(P))$ holds for any $A \in B(H)^+$ and rank-one projection $P$. Now, for given $A, B \in B(H)^+$ and arbitrary rank-one projection $P$ on $H$, we have

$$\lambda(A, P) \leq \lambda(B, P) \Leftrightarrow g_P(f(\lambda(A, P))) \leq g_P(f(\lambda(B, P))) \Leftrightarrow \lambda(\phi(A), \phi(P)) \leq \lambda(\phi(B), \phi(P)).$$

We learn from [1] that $A \leq B$ holds if and only if $\lambda(A, P) \leq \lambda(B, P)$ holds for every rank-one projection $P$ on $H$. Therefore, it follows that we have $A \leq B$ if and only if $\phi(A) \leq \phi(B)$. This means that $\phi$ is an order automorphism of $B(H)^+$. The structure of such transformations was described in [3]. We proved there that every such map is implemented by an invertible bounded linear or conjugate-linear operator. Consequently, it follows that $\phi$ is of the form

$$\phi(A) = TAT^*, \quad A \in B(H)^+$$

with an invertible bounded linear or conjugate-linear operator $T$ on $H$, and this completes the proof of the theorem in the present case.

Suppose now that $\phi$ maps the scalar multiples of an invertible operator $A$ into scalar multiples of $\phi(A)$. Considering the transformation

$$T \mapsto \phi(A)^{-1/2}\phi(A^{1/2}TA^{1/2})\phi(A)^{-1/2}$$

just as above, we see that there is no loss of generality in assuming that this particular operator $A$ equals the identity. In what follows, we assume that it is really the case. It means that there is an injective function $g : [0, \infty] \rightarrow [0, \infty]$ such that $\phi(tI) = g(t)I$, $t \geq 0$.

We compute

$$\phi(f(t)P) = \phi((tI)\sigma P) = (g(t)I)\sigma\phi(P) = f(g(t))\phi(P).$$
Now, for an arbitrary $B \in B(H)^+$, we compute $B\sigma P = f(\lambda(B, P))P$ implying $\phi(B\sigma P) = \phi(f(\lambda(B, P))P) = f(g(\lambda(B, P)))\phi(P)$. On the other hand, we have

$$\phi(B\sigma P) = \phi(B)\phi(P) = f(\lambda(\phi(B), \phi(P))).$$

Therefore, by the injectivity of $f$, it follows that

$$g(\lambda(B, P)) = \lambda(\phi(B), \phi(P))$$

holds for all $B \in B(H)^+$ and rank-one projection $P$ on $H$. Pick an invertible $B \in B(H)^+$ and consider the set of all $\lambda(B, P)$ where $P$ runs through the connected set of all rank-one projections. By the formula (2.4), it is easy to see that this set is an interval. Taking into account the equality (2.6), it follows that the injective function $g : [0, \infty[ \to [0, \infty[$ maps every interval $[\alpha, \beta[$ with $0 < \alpha < \beta < \infty$ onto an interval. It apparently yields that the restriction of $g$ onto the open interval $[0, \infty[$ is strictly monotone. Assume for a moment that this function is strictly monotone decreasing.

Let $A, B \in B(H)^+$ be invertible and $P$ an arbitrary rank-one projection on $H$. Then we infer

$$\lambda(A, P) \leq \lambda(B, P) \iff g(\lambda(B, P)) \leq g(\lambda(A, P))$$

$$\iff \lambda(\phi(B), \phi(P)) \leq \lambda(\phi(A), \phi(P)).$$

This means that $\phi$, when restricted onto the set of invertible elements of $B(H)^+$, is an order reversing automorphism. Considering the transformation $A \mapsto \phi(A)^{-1}$ we obviously obtain an order automorphism of that set. By a result in [7], the structure of those transformations is just the same as that of the order automorphisms of the whole set $B(H)^+$. That is, they are implemented by invertible bounded linear or conjugate-linear operators on $H$. Since here we also have that the identity is sent to the identity, it follows easily that $\phi$ is of the form $\phi(A) = U A^{-1} U^*$ with a unitary or antiunitary operator $U$ on $H$. It implies that the inverse operation satisfies (2.2), i.e., we have $(A\sigma B)^{-1} = A^{-1} \sigma B^{-1}$ for all invertible $A, B \in B(H)^+$. Putting $A = I$ and $B = sI$, this immediately gives us that $1/f(s) = f(1/s)$, $s > 0$. But the mean $\sigma$ is symmetric implying that $sf(1/s) = f(s)$ holds for all $s > 0$. It follows trivially that we necessarily have $f(s) = \sqrt{s}$, $s > 0$, i.e., $\sigma$ is the geometric mean. But the bijective maps on $B(H)^+$ preserving the geometric mean have been described in [5]. The result presented there shows that those maps just coincide with the order automorphisms of $B(H)^+$. But this contradicts the fact above that $\phi$ is an order reversing automorphism of the set of all invertible elements of $B(H)^+$. It yields that the case where the restriction of $g$ onto the open interval $[0, \infty[$ is strictly monotone decreasing is untenable. It remains that $g$ is strictly monotone increasing on $[0, \infty[$ and hence also on $[0, \infty[$. Using (2.6), we can see just as before that $\phi$ is an order automorphism of $B(H)^+$ and then complete the proof as in the first case. \[ \Box \]
We conclude the paper with a result describing the bijective transformations of $B(H)^+$ which preserve the norm of means of operators. The theorem below shows that every such map originates from an isometric linear or conjugate-linear *-algebra automorphism of $B(H)$.

**Theorem 2.8.** Let $\phi : B(H)^+ \to B(H)^+$ be a bijective map with the property that

$$\|\phi(A)\sigma \phi(B)\| = \|A\sigma B\|$$

holds for every $A, B \in B(H)^+$. Then there exists either a unitary or an antiunitary operator $U$ on $H$ such that $\phi$ is of the form

$$\phi(A) = UAU^*, \quad A \in B(H)^+.$$

**Proof.** First observe that $\phi(0) = 0$. Indeed, it follows from

$$\|\phi(A)\| = \|\phi(A)\sigma \phi(A)\| = \|A\sigma A\| = \|A\|$$

meaning that $\phi$ is norm-preserving.

The assumptions in the theorem imply that for any $A, B \in B(H)^+$, we have $A\sigma B \neq 0$ if and only if $\phi(A)\sigma \phi(B) \neq 0$. Just as in the proof of Theorem 2.1, we infer that $\text{rng } A^{1/2} \cap \text{rng } B^{1/2} \neq \{0\}$ holds if and only if $\text{rng } \phi(A)^{1/2} \cap \text{rng } \phi(B)^{1/2} \neq \{0\}$ which then implies that

$$\text{rng } A^{1/2} \subset \text{rng } B^{1/2} \iff \text{rng } \phi(A)^{1/2} \subset \text{rng } \phi(B)^{1/2}.$$ 

We can proceed showing that $\phi$ necessarily preserves the rank-one operators in both directions. As $\phi$ preserves the norm, too, it follows that $\phi$ preserves the rank-one projections in both directions. For any rank-one projection $P$ on $H$, we have

$$f(\lambda(\phi(A), \phi(P))) = \|\phi(A)\sigma \phi(P)\| = \|A\sigma P\| = f(\lambda(A, P))$$

implying $\lambda(\phi(A), \phi(P)) = \lambda(A, P)$. Just as in the proof of Theorem 2.1, this gives us that $\phi$ is an order automorphism of $B(H)^+$ and hence it is of the form

$$\phi(A) = TAT^*, \quad A \in B(H)^+$$

with an invertible bounded linear or conjugate-linear operator $T$ on $H$. Referring again to the property that $\phi$ preserves the norm, one can easily deduce that $T$ is in fact either a unitary or an antiunitary operator. This completes the proof of the theorem. ☐
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